20 Quantifiers

Expressions of generality in a natural language like English behave in tangled ways. This chapter starts by considering just some of the complexities. We then outline how to design a formal language for exploring the logic of general propositions while avoiding these vagaries of ordinary language.

20.1 The multiplicity of quantifiers in ordinary language

(a) Consider the following English sentences, used as stand-alone claims:
   (1) Every play by Shakespeare is worth seeing.
   (2) Any play by Shakespeare is worth seeing.

   These two surely come to the same, expressing the same generalization about the plays.
   But now compare how the sentences respectively embed in wider contexts:
   (3) I don’t believe that every play by Shakespeare is worth seeing.
   (4) I don’t believe that any play by Shakespeare is worth seeing.
   (5) If every play by Shakespeare is worth seeing, I’ll eat my hat.
   (6) If any play by Shakespeare is worth seeing, I’ll eat my hat.

   The messages expressed within each pair are quite different. So the quantifiers ‘every’ and ‘any’ are certainly not always interchangeable. For another example to show this, consider the following pair (a ‘Monro’ is a Scottish mountain over 3000 feet):
   (7) If Jack can climb every Monro, he can climb every Monro.
   (8) If Jack can climb any Monro, he can climb any Monro.

   The first is a trivial tautology, the second could be false (at least on one natural reading, according to which (8) says that, if Jack can climb any one Monro, then he can climb them all, even the most challenging).

(b) What about ‘all’ and ‘each’? Take for example
   (9) All plays by Shakespeare are worth seeing.
   (10) Each play by Shakespeare is worth seeing.

   As stand-alone claims, these again seem to convey just the same message as (1) and (2).
   And now consider
   (11) I don’t believe that all plays by Shakespeare are worth seeing.
   (12) If all plays by Shakespeare are worth seeing, I’ll eat my hat.
§20.2 Quantifiers and scope

(11) is equivalent to (3), and (12) to (5). So, at least in these cases, sentences containing ‘all’ behave like sentences with ‘every’ rather than ‘any’. Similarly indeed for the corresponding ‘each’ sentences.

Can we therefore say at least this: the ‘all’, ‘each’ and ‘every’ constructions are just stylistic variants – the main difference being that ‘all’ goes with a plural verb while ‘each’ and ‘every’ goes with the corresponding singular form?

Well, ignoring the likes of ‘All snow is white’ (we can’t say ‘Each snow is white’), compare for example these two claims:

(13) You can fit each of Jane Austen’s novels into a single, ordinary-sized, book.
(14) You can fit all of Jane Austen’s novels into a single, ordinary-sized, book.

I take the first to be true, the second false – since ‘all’ is naturally read as ‘all, taken together’, while ‘each’ is more naturally read as ‘each, taken separately’. Similarly, compare

(15) Jill can legally marry each of Bill’s brothers.
(16) Jill can legally marry all of Bill’s brothers.

By my lights, the first may well be true, the second not (given laws against polygamy).

But if you disagree (perhaps in each pair you find both sentences ambiguous between the taken-together/taken-separately readings), then no matter: that just goes to reinforce the point that English expressions of generality do behave in quirkily complicated ways. Exercise: what other differences between ‘all’, ‘any’, ‘every’ and ‘each’ strike you?

(c) Given that we are interested in the logic of arguments involving expressions of generality like these, but are not especially interested in theorizing about the tangled complexities of English usage, what to do?

Let’s follow the strategy that we introduced to cope with the connectives. We will construct another artificial language (or rather, a family of artificial languages), with just one, simply-behaved, way of forming all/any/every/each generalizations. We will also replace other vernacular quantifiers, in particular ‘some’, with simply-behaved substitutes. Then we can again ‘divide and rule’ (see §8.6); in other words, we can tackle the assessment of quantificational inferences in two stages. First, we translate a given argument into a suitable formal language, enabling us to represent quantified messages in a uniform, tidy, and easily understood way. And then we assess the resulting formalized argument for validity.

20.2 Quantifiers and scope

Wanting to side-step the tricky multiplicities of English usage is one major motivation for going formal. There is another, deeper, reason for adopting an artificial language to regiment quantified propositions for logical purposes: we need to avoid the sorts of structural ambiguities that English quantified sentences are prone to.

(a) For a warm-up exercise, compare the following sentences:

(1) Some senior Republican senators took cocaine in the nineteen-nineties.
(2) Some women died from illegal abortions in the nineteen-nineties.
As a shock headline on the current political scene, (1) will be read as saying, of some present Republican senators, that they have a dark secret in their past (namely that they dabbled in illegal drugs in nineties). But (2) is not to be read, ludicrously, as saying of some present women that they have a secret in their past (namely that they died of an illegal abortion back then). The intended claim is of course that some women living in the nineties died then from illegal abortions.

With some mangling of tenses, and adding brackets for maximal clarity, we might represent the intended readings of the two sentences respectively as follows:

(3) \((\text{Some senior Republican senators are such that})(\text{in the nineteen-nineties it was true that}) \text{ they use cocaine.}\)

(4) \((\text{In the nineteen-nineties it was true that})(\text{some women are such that}) \text{ they die from illegal abortions.}\)

Here the final verbs are now deemed to be tenseless. And a compelling way of describing the situation is to say that the tense modifier and the quantifier are understood as having different scopes in the messages intended by (1) and (2) – compare §8.5 on the idea of scope. But note that the shared surface form of those two sentences doesn’t explicitly mark this difference. We implicitly rely on context and plausibility considerations in order to arrive at the natural readings.

Now consider

(5) \(\text{Some philosophers used to be enthusiastic logical positivists.}\)

Are we saying that some current philosophers went through a positivist phase in their brash younger days? Or are we saying that, once upon a time, some then philosophers were keen positivists? The intended message could be

(6) \((\text{Some philosophers are such that})(\text{it used to be true that}) \text{ they are enthusiastic logical positivists,}\)

or alternatively we could be claiming

(7) \((\text{It used to be true that})(\text{some philosophers are such that}) \text{ they are enthusiastic logical positivists.}\)

Without context to help us decode (5), the claim is simply ambiguous. And the ambiguity is structural (the individual words aren’t ambiguous). The issue is: which has wider scope, which governs more of the sentence, the generalizing operator or the tense modifier?

(b) Mixing ordinary expressions of generality and operators modifying tenses is therefore prone to produce scope ambiguities. So too is mixing vernacular quantifiers with what logicians call modal operators – i.e. with expressions of necessity and possibility.

Take, for example, that notorious philosophical claim

(8) \(\text{Every perceptual experience is possibly delusory.}\)

Does this mean that each and every perceptual experience is one which, taken separately, is open to doubt, i.e.

(9) \((\text{Every perceptual experience is such that})(\text{it is possible that}) \text{ it is delusory?}\)

Or is the thought that the whole lot could be delusory all at once, i.e.

(10) \((\text{It is possible that})(\text{every perceptual experience is such that}) \text{ it is delusory?}\)
§20.2 Quantifiers and scope

These claims are certainly different, and it is philosophically important which reading is meant e.g. in arguments for scepticism about the senses. Even if (9) is true, (10) doesn’t follow. Compare

(11) Every competitor might win a prize,
and the following two readings:

(12) (Every competitor is such that)(it is possible that) they win a prize,
(13) (It is possible that)(every competitor is such that) they win a prize.

It could be that (12) is true because it’s a fairly run competition of skill with very well matched entrants, while (13) is false because the rules governing the knock-out rounds ensure that only one ultimate winner can emerge.

Now, some very firmly say that (8) means (9); some equally firmly say it means (10). I take that as some evidence for my own view that in ordinary use it is dangerously ambiguous. In what order is the intended message built up using the quantifier and the modality? Surface grammar doesn’t fix this. (You don’t have to agree. Even if you think that (8) and likewise (11) is unambiguous, one way or the other, you must still acknowledge the key point, namely that the interpretation of such quantified sentences crucially involves paying attention to questions of scope.)

(c) Exploring the logic of tense operators and of modal operators is beyond the remit of this book. So, let’s turn to the kinds of cases that more immediately concern us. First, consider what happens when we mix everyday expressions of generality with negation.

Here is one kind of example. A statement of the kind ‘Some Fs are not Gs’ is usually heard as being entirely consistent with the corresponding ‘Some Fs are Gs’. For example, ‘Some students are not good at logic’ is no doubt sadly true; but it is consistent with the happier truth ‘Some students are good at logic’.

But now suppose Jack says ‘Some footballers deserve to earn five million a year’. Jill, in response, takes a high moral tone, expostulates about the evils of huge wages in a world of poverty and deprivation, and emphatically concludes: ‘So certainly, some footballers do not deserve to earn five million a year’. Jill plainly does not intend to be heard as saying something compatible with Jack’s original remark!

Thus, although vernacular English sentences of the form ‘Some Fs are not Gs’ are more usually construed as meaning

(14) (Some Fs are such that)(it is not the case that) they are Gs,
in some contexts the negation can be understood to have wide scope, i.e. to govern the whole sentence, with the resulting message being:

(15) (It is not the case that)(some Fs are such that) they are Gs.

Another example. In the Merchant of Venice, Portia says

(16) All that glisters is not gold,
which is naturally construed in context as meaning that not everything that glisters (i.e. glitters) is gold. So what she says is equivalent to

(17) (It is not the case that)(all that glisters is such that) it is gold.

But now compare the following sentence which in surface form is just like (16):
(18) All that perishes is not divine.
This is more naturally read with the quantifier and the negation scoped the other way around, i.e. as equivalent to
(19) (All that perishes is such that)(it is not the case that) it is divine.
In sum: like the logical operation of negation, quantifying may be thought of as an operation which can govern more or less of a sentence. And when an assertion mixes a negation and a quantifier, we may not be able to read off the intended relative scopes of the two logical operators simply from the surface form of the sentence being used – unless we set out to be very ploddingly explicit in the manner of e.g. (17) and (19). As in the tensed and modal cases, we can be left with a scope ambiguity – like here, perhaps:
(20) Jill didn’t finish every book.
(21) Everyone’s not yet arrived.
(22) I would not give that to anyone.
(d) Finally, let’s consider what can happen when a sentence contains more than one quantifier. Start with the following claims:
(23) On Mother’s Day, every mother will get some gift.
(24) At the Crack of Doom, the clouds will part and every human will look up to see some angry and jealous god.
The first is obviously to be read as saying that every mother will get her own present; the second is more naturally read as saying that some angry and jealous god will reveal himself to all alive that dread day. We can express the core parts of these claims respectively as follows:
(25) (Every mother is such that)(there is some gift such that) she will get it.
(26) (There is some angry and jealous god such that)(every human is such that) they will look up to see it.
So on the obvious reading of (23), the ‘every’ quantifier has the wider scope, i.e. governs more of the sentence, while we read (24) with the ‘some’ quantifier having wider scope.
Now consider another example of an ‘every/some’ sentence, this time one which can quite reasonably be read with the quantifiers scoped either way:
(27) Everyone loves a certain someone.
Is the claim that each person (perhaps in some contextually indicated group) has their particular beloved? Or is the claim that there is someone who is the apple of every eye?
Given an English sentence involving more than one quantifier, context and plausibility considerations may indeed often serve to privilege one particular reading and prevent misunderstanding – as is the case with (23) and (24). But there is always the potential for unresolved ambiguity, as perhaps in (27).
(e) Note, by the way, that these four kinds of scope ambiguity, arising when everyday quantifiers are mixed with tense, modality and negation and with other quantifiers, do not occur when proper names are used instead of the quantifier expressions. Consider, for example:
§20.3 Quantifier prefixes and ‘variables’ as pronouns

(28) Jack took cocaine in the nineteen-nineties.
(29) Jill might possibly win.
(30) Michael does not deserve five million a year.
(31) Romeo loves Juliet (or: Romeo loves someone).

We need to fix who is being talked about (and the unit of currency!). But none of these claims is structurally ambiguous – issues of scope do not arise. As it is sometimes put, genuine names are always scopeless.

20.3 Quantifier prefixes and ‘variables’ as pronouns

(a) When we turn to designing a formal language apt for regimenting arguments involving generality, a crucial requirement will be that we avoid all ambiguities. We therefore need to avoid the potential for scope ambiguities in particular. So we will need some way of perspicuously showing how a message involving a number of quantifiers and other logical operators is built up, making clear the relative scopes of the operators. How are we going to do this?

Well, we have in fact just seen how, in a variety of cases, we can do the trick in ordinary language by rather laboriously rephrasing using prefixed quantifier operators like ‘everyone is such that’, ‘some senior Republican senators are such that’, ‘every perceptual experience is such that’, which are linked to later pronouns such as ‘they’ and ‘it’. And then the scope of each operator is as plain as can be. We just apply the obvious rule – a prefixed quantifier operator governs the following clause with the linked pronoun(s). So now we have a model to emulate:

In our formal QL language(s), we will unambiguously render general propositions by using quantifier prefixes linked to later pronouns.

(b) That statement of strategy raises the obvious next question: how are we going to handle pronouns formally?

There are different uses of ordinary-language pronouns. Consider, for example, demonstrative uses – as when I point across the room and say ‘She is a great logician’ (I could have equivalently used a demonstrative and said ‘That woman is a great logician’). In this case, for my use of the pronoun to be in good order, I must successfully pick out a particular woman. Contrast the claim ‘No woman is such that she enjoys being harassed’. In this case, ‘she’ plainly doesn’t denote a particular person! Nor does it in ‘Every woman is such that she hates being harassed’. In these cases, the pronouns are not doing straightforwardly referential work but rather link back to, are bound to, the previous quantifier prefix. Pronouns like this are classed as one kind of anaphoric pronoun (literally, they ‘carry back’). But we will call them simply bound pronouns.

So let’s sharpen our question: what are our formal languages going to use as bound pronouns tied to quantifier prefixes?

(c) Go back to our ordinary language examples for a moment. Suppose we want to render the perhaps more natural reading of

(1) Everyone loves a certain someone,
by using prefixed operators linked to later pronouns. Then it won’t do to write e.g.

(2) (Everyone is such that)(there is someone such that) they love them.

(Remember: ‘they’ is our gender-neutral singular pronoun.) For (2) just introduces a new ambiguity – which pronoun is bound to which quantifier prefix? Is it, so to speak, everyone or someone who is doing the loving?

We need, therefore, some way to indicate which pronoun is bound to which quantifier prefix. One ordinary-language option would be to use something like

(3) (Everyone is such that)(there is someone such that) the former loves the latter.

But that’s not a very useful model to follow in a formal context – if only because who/what counts as ‘the former’ or ‘the latter’ is liable to keep changing as we manipulate propositions while working through an argument. So here’s a neat alternative trick:

We will henceforth adopt ‘x’, ‘y’, ‘z’ . . . as additional pronouns.

We then explicitly tag our quantifier prefixes with these new pronouns – as in ‘(Everyone x is such that)’, ‘(Someone y is such that)’ – in order to signal which quantifier goes with which pronoun and thereby make the cross-linkings perfectly clear.

Hence instead of (3) we can write

(4) (Everyone x is such that)(there is someone y such that) x loves y.

Similarly, the other reading of (1) – as in ‘Everyone loves a certain someone, namely Kylie’ – gets unambiguously rendered as

(5) (There is someone y such that)(everyone x is such that) x loves y.

Compare and contrast e.g.

(6) (There is someone x such that)(everyone y is such that) x loves y.

which unambiguously says that someone (Jesus?) loves us all.

(d) Of course, we have borrowed our new pronoun symbols from the mathematicians’ use of so-called variables.

Consider, for example, the arithmetical truism ‘for every number x, x + 1 = 1 + x’.

We here very naturally use a tagged quantifier prefix linked to a variable. But note that this just says ‘Every number is such that it plus one equals one plus it’. So we see that here too the ‘x’ is just doing the work of a bound pronoun like ‘it’. And while using ‘x’s etc. as pronouns tied to quantifier prefixes may be most familiar from the maths classroom, there is nothing deeply mathematical about this usage.

To repeat, these variables – as they are called in logic too, as well as in mathematics – just serve as a usefully terse form of pronoun.

20.4 Restricting quantifiers and fixing domains

(a) Natural language quantifiers can be simple and generic (as in ‘everyone’, ‘something’, ‘nothing’), or can be explicitly qualified so as to substantially restrict their
§20.4 Restricting quantifiers and fixing domains

coverage (as in ‘every philosopher’, ‘some play by Shakespeare’, ‘no prime number’).
Our formal QL languages will need a comparable way of making quantifications more or less restricted. How shall we do this?

But there’s a related issue. Take generic quantifiers for a moment. Suppose, for example, that at the beginning of class I ask ‘Is everyone here?’. Plainly I am not asking e.g. about everyone now living. I have in mind those who have signed up for the logic class, and I am asking if all of them are present. I continue: ‘I am impressed: someone solved that extremely tricky optional homework problem!’. Again, I am not talking about some student in another college or in another year, let alone just some random human: I mean that there is someone among the current logic class who solved the problem. So: although not explicitly restricted, ‘everyone’ and ‘someone’ here are naturally understood as ranging over a rather particular group of people; these people comprise, we say, the current range or domain of the quantifiers.

As in this example, we often leave it to context to fix the current domain, i.e. to determine who or what currently counts as ‘everyone’ or ‘everything’. For more examples, consider e.g. ‘Everyone enjoyed the concert’, ‘Make sure everyone has a glass of wine’, ‘Everything fitted into two suitcases’, ‘Can you finish everything this morning?’.

Moreover, we are adepts at following mid-conversation shifts in the domain of our quantifiers, using contextual cues and common sense to follow implicit changes in who counts as ‘everyone’ or ‘everything’. To continue with our logic class example, suppose I go on to say: ‘As we saw, everyone agrees that modus ponens is a good form of inference for the ordinary-language conditional’. You immediately pick up that I am this time generalizing not over the members of the class but over e.g. the mainstream logical authors whom we were explicitly discussing last week. And when I later remark, ‘Someone, however, has denied that modus ponens is always reliable’, you don’t take me to be contradicting myself; you charitably suppose that I am now moving on to cast the generalizing net rather wider, beyond the standard authors I’ve already mentioned, to include logical outliers and mavericks.

(b) Domain fixing in ordinary discourse is often left implicit. But when we go formal – with the aim of making everything ideally clear and determinate – we will obviously need some way of handling domains that does not rely on contextual clues or interpretative charity. Who or what do the generic quantifiers of a given formal language (the formal versions of ‘everyone’/‘everything’, ‘someone’/‘something’, etc.) range over?

The standard policy for QL languages is this:

For a particular formal language, we fix at the outset who or what all the generic quantifiers range over, i.e. we specify the domain of quantification for the language once and for all.

Then, when we want to focus on only some of this fixed domain, we quite explicitly restrict our generalizations.

Note: mathematicians using semi-formal language often introduce different styles or sorts of variables to run over different things (for example, an arithmetician might use
lower case variables for numbers, and upper case variables for sets of numbers). But QL languages keep things simple at the cost of occasional artificiality: for any language, there is just one style of variable ranging over one appropriately inclusive domain. And if we want to generalize over only some of the things in that domain, we have to restrict our inclusive quantifiers.

(c) But then, to return to the issue raised at the beginning of this section, how do we make explicit restrictions to generic quantifiers? There are various options here, but standard QL languages keep things simple by recruiting our familiar connectives to do the work. For example: suppose we fix that the unrestricted ‘(everyone $x$ is such that)’ quantifier prefix in a particular formal language runs inclusively over all people past and present. Then arguably we can render the restricted generalization

1. Every philosopher is wise.

with a conditional, using the formal version of

2. (Everyone $x$ is such that) if $x$ is a philosopher, then $x$ is wise.

Similarly, it seems we can render the restricted generalization

3. Someone in the class solved the hard problem,

with a conjunction, using the formal version of

4. (Someone $x$ is such that) $x$ is in the class and $x$ solved the hard problem.

Those renditions of restricted generalizations by using connectives look plausible attempts. But we will of course need to return to the question of how well they in fact capture the vernacular claims.

### 20.5 Quantifier symbols

(a) We now take one more step, moving from stilted-English-using-variables-as-pronouns towards something even closer to a standard logical language. We introduce the quantifier symbols ‘$\forall$’ and ‘$\exists$’, as follows:

Instead of writing ‘every(one) $x$ is such that’, we will simply use the very terse notation ‘($\forall x$)’, with the rotated ‘A’ reminding us that this can also informally be read as ‘for all $x$’. ‘($\forall x$)’ is said to be a universal quantifier.

And instead of ‘some(one) $y$ is such that’, we will use ‘($\exists y$)’. Here, the rotated ‘E’ reminds us that this can also informally be read as ‘there exists a $y$ such that’; hence ‘($\exists y$)’ is said to be an existential quantifier.

Assume that we are working in a context where the domain comprises, say, all people. Then our three examples from §20.3, there numbered

4. (Everyone $x$ is such that)(there is someone $y$ such that) $x$ loves $y$
5. (There is someone $y$ such that)(everyone $x$ is such that) $x$ loves $y$
6. (There is someone $x$ such that)(everyone $y$ is such that) $x$ loves $y$,

can now be very neatly abbreviated in turn as follows:
§20.6 Quantification further explained

(4') $(\forall x)(\exists y) \; x \text{ loves } y$
(5') $(\exists y)(\forall x) \; x \text{ loves } y$
(6') $(\exists x)(\forall y) \; x \text{ loves } y$.

(b) Keeping the domain of quantification as before, now consider how we can express restricted generalizations using our new notation. Take again

(1) Everyone in the class has arrived.

Then, following the suggestion made in the last section, we can half-symbolize this as

(2) $(\forall x)(x \text{ is in the class } \rightarrow x \text{ has arrived}),$

where ‘$(\forall x)$’ still ranges over everyone. Likewise,

(3) Some footballer deserves great riches.

can be half-symbolized as

(4) $(\exists x)(x \text{ is a footballer } \land x \text{ deserves great riches}).$

And what about the following sentence, which taken out of context is ambiguous?

(5) Some footballer does not deserve great riches.

We can half-symbolize the two possible readings along the following lines:

(6) $(\exists x)(x \text{ is a footballer } \land \neg x \text{ deserves great riches}),$

(7) $\neg (\exists x)(x \text{ is a footballer } \land x \text{ deserves great riches}).$

The quantifier operator and the negation are thus very perspicuously revealed as having different scopes in the two readings.

(c) The unholy mixture of English and formal logical symbolism that we have just fallen into using is often called Loglish. As we will see in the coming chapters, it is very handy for easing the transition between ordinary-language and our future official full-blown formal languages. We will make free use of it.

(We can indeed mix English and formal devices to various degrees, with some more mathematical uses coming close to pure QL. The fact that there is a smooth gradation of cases here makes it rather implausible to suppose – as some do – that there is a deep difference in the semantic nature of ordinary and formal languages.)

20.6 Quantification further explained

(a) Where we have got to? We have highlighted three desiderata for formal languages apt for exploring the logic of generality.

(1) We want to avoid the complicated multiplicity of ordinary-language ways of expressing general claims.

(2) We want to devise our notation so that the scope of a quantifying operation is always clear and there is no possibility of scope ambiguities (which means building in at ground level some way of marking the contrast between quantifiers and scopeless names).

(3) We want to clearly fix domains of quantification (and we want a clear way of explicitly restricting quantifiers).
We now know how to deal with the scope and domain issues:

(2') In our QL languages, following the Loglish model, we will use formal versions of ‘(∀x)’, ‘(∃y)’, etc., as quantifier prefixes linked to later formal variables ‘x’, ‘y’, which serve as pronouns. The order of quantifier prefixes and other logical operators in a wff will determine their relative scopes.

(3') For a particular QL language, we once and for all fix a common domain of quantification for all its quantifiers. When we want to generalize in that language over less than the whole domain, we will explicitly restrict quantifiers using connectives.

And as for the multiplicity issue

(1') We will see later just how far we can get using only ‘∀’ (for ‘every’) and ‘∃’ (for ‘some’) in our formal languages.

(b) Hopefully, given our explanations so far, these ideas should all seem, if not obvious, at least quite attractively straightforward. It is therefore an interesting historical question, why did the idea of a quantifier/variable notation for perspicuously representing general claims arise so late in the history of logic? A prototype was first introduced, using a different symbolism, by Frege in his *Begriffsschrift* (1879).

We can't tackle the historical question, but let’s say just a little more about how to understand Fregean quantifiers, still in the friendly context of Loglish, before going fully formal.

Start from the sentence

(1) Romeo loves Juliet.

We can carve up this sentence in various ways. We can, for example, construe it as attributing to Juliet the property of being-loved-by-Romeo – a property we could express using the gappy expression

(2) Romeo loves —.

Suppose we now want to say, not that a particular named person such as Juliet has this property, but just that someone does. Then, in our new notation, and taking the domain of quantification to be people, we can write, e.g.

(3) (∃y) Romeo loves y.

We have filled the gap in (2) with a variable and prefixed an existential quantifier using the same variable. It doesn’t at all matter which variable we use: its sole role is to tie the quantifier to the gap in (2). We can mark this tie with any variable we choose. ‘(∃z) Romeo loves z’, for example, will evidently do just as well.

But now we can iterate this process. So read (3) as attributing to Romeo the complex property of loving someone or other (a property in fact he shares with Juliet and many others). We could express this property using another gappy expression, like

(4) (∃y) . . . loves y.

If we now want to say that everyone has this property, then we can quantify into the gap like this:

(5) (∀x)(∃y) x loves y.
Again, the choice of variable is almost entirely arbitrary – we just mustn’t use ‘\(y\)’ again for the obvious reason that it is already being used to tie another quantifier to one of the slots either side of ‘loves’. (5), then, symbolizes the more natural reading of ‘everyone loves someone’.

Another example. Suppose we start from

(6) If Socrates is a philosopher, then Socrates is wise.

This time we will omit both occurrences of the name to get an expression with two gaps:

(7) If . . . is a philosopher, then . . . is wise.

We have replaced each occurrence of the name with the same dotty gap-marker, to indicate that the gaps are to be filled in the same way. This gappy expression can then be used to attribute the property of being-wise-if-you’re-a-philosopher.

Suppose now that we want to say that everyone has this property (which seems equivalent to saying that every philosopher is wise). Then, replacing ‘if’ with our logical symbol, we can write

(8) \( (\exists x) (\text{x is a philosopher} \rightarrow \text{x is wise}) \).

But again, any other variable will do to tie the ‘\(\forall\)’ to the gaps.

Let’s highlight again the informal syntax here. The idea is that we form a sentence with a quantifier prefix like ‘‘\((\forall x)\)’ or ‘‘\((\exists y)\)’’ by taking a sentence involving one or more occurrences of a particular name; then we (i) remove the name, (ii) replace it with a variable new to the sentence while (iii) prefixing a quantifier using the same variable. And as for interpreting the result, just unpack an initial symbolic quantifier such as ‘‘\((\forall x)\)’’ as saying ‘‘(everything \(x\) [in the relevant domain] is such that)’’, and unpack ‘‘\((\exists y)\)’’ as saying ‘‘(something \(y\) [in the relevant domain] is such that)’’, etc.

(c) It is illuminating to consider briefly an alternative notation. Start again with the simple sentence (1), and omit the name ‘Juliet’ to get (2). But this time, we will tie a ‘someone’ quantifier symbol to the empty place graphically, like so:

(3’’) \( \exists \) Romeo loves —

Now we remove a name from this, to get another gappy expression,

(4’’) \( \exists . . . \) loves —

We can then quantify into the new gap, as follows,

(5’’’) \( \forall \) . . . loves —

to again symbolize the more natural reading of ‘everyone loves someone’. The other, ‘Kylie’, reading can similarly be symbolized by

(5’’’’) \( \exists \forall . . . \) loves —

Again, start from the sentence (6) above and as before omit both occurrences of the name ‘Socrates’ to get the gappy

(7) If . . . is a philosopher, then . . . is wise.

We can now link a quantifier prefix to the gaps in (7) by graphical ties, perhaps like this:
Quantifiers

(8’) \( \forall (\ldots \text{is a philosopher} \rightarrow \ldots \text{is wise}) \)
That, in theory, will do just the same work as (8).

Of course, the gaps-and-braces graphical notation is difficult to typeset, and it gets difficult to read when more than one quantifier prefix gets tied to multiple later gaps. No wonder, then, that we stick to the now conventional quantifier/variable notation. Still, note that the use of a variable to link a quantifier to some place(s) in an expression is really just a variant on our graphical notation. For example,

(5) \( (\forall x)(\exists y) \ x \loves y \),

(5’) \( \exists \ldots \loves \)

are to be explained as ultimately meaning just the same. This again emphasises that the choice of variable-letters is irrelevant, so long as we keep fixed the pattern of ties from quantifiers to places in what follows.

(d) The graphical version of the notation brings out another important point. Obviously, in this second notation, ‘\( \forall \)’ by itself doesn’t yet implement a generalization – it needs to be tied to some gap(s) in an open sentence. And ‘\( \forall \)’ plus half a broken graphical tie obviously doesn’t implement the generalization either! It is ‘\( \forall \)'-plus-the-indication-of-which-gaps(s)-it-is-tied-to which does the work.

It is exactly similar when we use variables. ‘\( \forall \)’ without its variable does not yet implement a generalization, nor does the whole prefix ‘(\( \forall x \))’. It is a prefix plus the later variable(s) – the likes of ‘(\( \forall x \)) \ldots x \ldots x’ or ‘(\( \forall x \)) \ldots x \ldots x \ldots x’ – which does the work.

(Calling the prefix ‘(\( \forall x \))’ by itself a ‘quantifier’, as is conventional, therefore really labels the wrong thing – but it is far too late in the day to complain about that!)

20.7 Arguing with quantifiers: dummy names

(a) Finally in this chapter, we are going to introduce a fourth desideratum for the design of formal apparatus for regimenting arguments involving generality (to add to the requirements that we avoid the redundant multiplicities of ordinary language, prevent scope ambiguities, and fix domains clearly). So start by considering the trite inference

A Everyone loves pizza. Everyone who loves pizza loves ice-cream. So everyone loves ice-cream.

This is quite obviously valid. But suppose we do insist on a supporting proof from the premisses to the conclusion. Then we could proceed informally as follows. We are given

(1) Everyone loves pizza,
(2) Everyone who loves pizza loves ice-cream.

So now choose anyone you like, call them ‘Alex Coe’ (on the model, perhaps, of ‘John Doe’ and ‘Jane Roe’). By (1) we have

(3) Alex loves pizza,

since what applies to everyone applies to Alex in particular. By (2), it follows that

(4) If Alex loves pizza then Alex loves ice-cream
by the same principle that what applies to everyone applies to Alex. Hence, by modus
ponens from (3) and (4), we have

(5) Alex loves ice-cream.

But Alex was arbitrarily chosen, hence what applies to Alex applies to everyone. So

(6) Everyone loves ice-cream.

True, that’s a bit laborious! But the proof-idea, the argumentative strategy illustrated
here, is surely a very familiar one, one we repeatedly use when arguing for general
conclusions. Thus:

Suppose want to establish conclusion (C): everything (or at least, everything in
some class) has a certain property. Then one way of proceeding is to pick an
arbitrary representative (arbitrary in the sense that we rely on no special knowledge
about that individual as contrasted with other members of the class). We then
show that this arbitrary representative has the property in question. We can then
generalize and deduce the target conclusion (C).

Now: what is the expression ‘Alex’ doing in sentences (3) to (5) of our argument?
This plainly isn’t serving as an ordinary fixed proper name. On the other hand, it isn’t a
bound pronoun either. It is a sort of ‘dummy name’, or ‘temporary name’, or ‘ambiguous
name’, or ‘arbitrary name’ – perhaps none of those labels is entirely happy, though all
are in use. Another common, colourless, label is ‘parameter’.

Prefiguring what will happen when we go fully formal, let’s now put all this into
Loglish (as a half-way house). Using ‘is $F$’ for ‘is a pizza-lover’ (or ‘loves pizza’),
and ‘is $G$’ for ‘is an ice-cream-lover’ (or ‘loves ice-cream’), the inference $A$ can be
represented like this:

$$B \quad (\forall x) x \text{ is } F. \quad (\forall x)(x \text{ is } F \rightarrow x \text{ is } G). \quad \text{So: } (\forall x) x \text{ is } G.$$

Our sketched proof then becomes this:

$$B' \quad (1) \quad (\forall x) x \text{ is } F \quad \text{(premiss)}$$

$$\quad (2) \quad (\forall x)(x \text{ is } F \rightarrow x \text{ is } G) \quad \text{(premiss)}$$

Arbitrarily choose some object in the domain, $a$:

$$\quad (3) \quad a \text{ is } F \quad \text{(from 1)}$$

$$\quad (4) \quad a \text{ is } F \rightarrow a \text{ is } G \quad \text{(from 2)}$$

$$\quad (5) \quad a \text{ is } G \quad \text{(from 3, 4)}$$

But $a$ was arbitrarily chosen; so

$$\quad (6) \quad (\forall x) x \text{ is } G \quad \text{(from 5)}$$

We won’t pause, however, to mull over the details of this Loglish version; we will see
a properly formalized version of the proof later, when discussing natural deduction
for quantified arguments. The point we want for the moment is simply this: if we are
going to be able to replicate this sort of proof in a $\mathcal{QL}$ language, we will need symbols
available inside our formal language to play the role played here by ‘$a$’, i.e. to serve as
a ‘dummy/temporary/ambiguous/arbitrary name’ for an arbitrarily selected individual.

A related point. Looking ahead, we will also be discussing truth trees for quantifier
inferences. In Loglish terms, when tree-building, we will find ourselves repeatedly taking
steps which look like this: we have ‘(∃x) x is F’, telling us that there is at least one F; we can pick one of the Fs and arbitrarily dub it ‘a’; then we add ‘a is F’ to the tree. Again, without going into any further details now, we will be using dummy names. So, when we go fully formal, our QL languages will also need symbols for dummy names in order for us to deal with truth trees too.

Therefore, in headline terms,

Our QL languages will need symbols to serve as dummy names for use in formalized arguments.

(b) Suppose, however, that we adopt the principle which goes back to Frege that in a good formal language significant differences of semantic role should get marked syntactically, by differences in symbolic style. Well, as we have just stressed, dummy names are not proper names with a fixed significance, nor are they pronouns bound to quantifiers. So that suggests that a formal language for quantified arguments ought to have (among other apparatus) three distinguished kinds of symbols, respectively for proper names (like ‘n’), for pronouns linked to quantifiers (like ‘x’), and now also for dummy names or parameters (like ‘a’ as used in the Loglish argument above).

This key idea will guide the presentation in the coming chapters.

20.8 Summary

Ordinary English expressions of generality behave in complex ways – consider e.g. the availability of ‘all’, ‘every’, ‘any’, and ‘each’ to express universal generalizations, and compare the behaviour of these quantifiers in complex contexts. Vernacular sentences involving quantifiers are also prone to scope ambiguities. Following the ‘divide and rule’ strategy introduced in §8.6, we therefore want to devise QL languages for expressing generality, formal languages avoid the complexities and scope ambiguities of English.

There is some slightly prolix apparatus already available in English which can be used to express generalizations unambiguously, i.e. the use of prefixed quantifier operators of the kind ‘everyone/everything is such that’, ‘someone/something is such that’, tied to later pronouns. We will model our formal apparatus on these quantifier prefixes.

Semi-formalizing, we can abbreviate these quantifier prefixes by ‘(∀x)’, ‘(∃y)’, etc., tied to variables like ‘x’, ‘y’, etc. which act as corresponding pronouns.

A QL language will have a fixed domain of quantification for its quantifiers to range over. Restricted quantifiers ranging over some subclass of the domain can be expressed using conditionals and conjunctions.

We will also want our languages to have (as well as ordinary proper names and variables) symbols to serve as parameters or dummy names, for use in certain sorts of arguments involving generality.