Our next task must be to complete our proof system for PL inferences by adding rules for dealing with disjunction. Initially, we’ll continue in the same rather rough-and-ready style as the last chapter. Then we’ll say something about ‘theorems’, i.e. wffs in our system that can be proved without invoking any premisses. Finally we’ll need to give a more careful, official, presentation of the rules for our proof system.

A2.1 Rules for disjunction—a first shot

The general idea of a proof-system, and in particular the idea of suppositional sub-proofs, is now (let’s hope!) reasonably clear. Extending the system we’ve got in order to cope with disjunction introduces no new issues of principle; but there is some interest in completing the job. So in this section we’ll add a pair of rules for ‘∨’, one introduction and one elimination rule (the resulting system will be complete—i.e. it will capture all the tautologically valid arguments in the three connectives ‘∧’, ‘∨’, and ‘¬’).

The introduction rule for the connective ‘∨’, expressing inclusive disjunction, is immediate and needn’t delay us: given a proposition A you can infer the disjunction of that proposition with any proposition at all—i.e from A we can infer \((A \lor B)\), or equally infer \((B \lor A)\), where B is any wff (see the box for a schematic representation). Call that rule ‘∨-introduction’ or ‘∨I’ for short.

\[
\begin{array}{ccc}
A & A \\
\vdash & \vdash \\
(A \lor B) & \lor I & (B \lor A) & \lor I
\end{array}
\]

The corresponding elimination rule is more complex but equally intuitive. Consider the following example. Sherlock knows that

Moriarty saw the jewels being hidden either by looking through the window or by peeping through the keyhole.

Now, Sherlock argues,

Suppose Moriarty looked through the window. ... He could only have seen through the window by standing on someone’s shoulders ... So Moriarty had an accomplice.
(Imagine the gaps filled in by appeal to premisses we are granting him, so this bit of reasoning is deductive.) He continues,

Suppose, alternatively, Moriarty looked through the keyhole. ... He could only have got into the hall and seen through the keyhole if he was let in to the house ... So Moriarty had an accomplice.

Then, if Sherlock has reasoned soundly so far, he is evidently entitled to conclude that either way, whether Moriarty saw through the window or through the keyhole,

Moriarty had an accomplice.

More generally, suppose you accept that either \( A \) or \( B \); and you also know that, on the supposition that case \( A \) holds, \( C \) follows, and also on the supposition that case \( B \) holds, \( C \) again follows. These three bits of information taken together entitle you to conclude that \( C \).

This type of informal ‘argument by cases’ is mirrored in the following formal rule for PL arguments: from \((A \lor B)\), together with a proof of \( C \) from the supposition \( A \), and a proof of \( C \) from the supposition \( B \), we can infer \( C \). This rule enables us to argue from a disjunction: it is standardly called ‘\( \lor \)-elimination’ or ‘\( \lor \)E’ for short. And an application of \( \lor \)E will be set out as in the schema on the right.

Here's the rationale for the layout: we first temporarily suppose \( A \), indenting the line of argument a step to the right, and then derive conclusion \( C \); but then we drop the supposition \( A \) and start afresh with a new supposition \( B \) and again derive conclusion \( C \). Since the first supposition is discharged after we get to \( C \) the first time, we (as it were) momentarily move left to the previous column, only to immediately make a second supposition and so indent the proof to the right again. That's why the second supposition appears in the same indented column as the first one. To signal what is going on, and indicate that the first sub-proof does indeed finish when \( C \) is established the first time, we draw a horizontal bar under the first sub-proof, to separate it from second sub-proof. Wffs above the bar, since they are derived while the supposition \( A \) is in play, can’t be appealed to later, below the bar, when the supposition \( B \) is in play instead.

To document an application of the rule, the commentary on the right will cite five line numbers for an application of \( \lor \)E—the line numbers of the original disjunction, and the line numbers of the tops and tails of both of the sub-proofs.

All this will become clear, let’s hope, when we consider a few examples of the two rules for disjunction in operation. First, we show that the argument

\[
A \quad ((P \land Q) \lor (R \land Q)) \vdash Q
\]

is provably valid. Intuitively, the argument is a good one, because the desired conclusion follows whichever limb of the disjunction is the correct one, Suppose the left disjunct is true, then plainly \( Q \) follows; suppose alternatively that the right disjunct is true, then again \( Q \) follows. So given the disjunctive premiss, the
desired conclusion follows either way. This intuitive validation of the argument is exactly mirrored by the following simple formal proof.

\[ A' \]

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<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>(P ∧ Q) v (R ∧ Q)</td>
<td>(Prem)</td>
</tr>
<tr>
<td>2</td>
<td>(P ∧ Q)</td>
<td>(Supp)</td>
</tr>
<tr>
<td>3</td>
<td>Q</td>
<td>(2 ∧E)</td>
</tr>
<tr>
<td>4</td>
<td>(R ∧ Q)</td>
<td>(Supp)</td>
</tr>
<tr>
<td>5</td>
<td>Q</td>
<td>(4 ∧E)</td>
</tr>
<tr>
<td>6</td>
<td>Q</td>
<td>(1,2–3,4–5 vE)</td>
</tr>
</tbody>
</table>

Note also the validity of the following special case

\[ A' \]

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<tbody>
<tr>
<td>1</td>
<td>(P ∧ Q) v Q</td>
<td>(Prem)</td>
</tr>
<tr>
<td>2</td>
<td>(P ∧ Q)</td>
<td>(Supp)</td>
</tr>
<tr>
<td>3</td>
<td>Q</td>
<td>(2 ∧E)</td>
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<tr>
<td>4</td>
<td>Q</td>
<td>(Supp)</td>
</tr>
<tr>
<td>5</td>
<td>Q</td>
<td>(1,2–3,4–4 vE)</td>
</tr>
</tbody>
</table>

Here, the top and the tail of the second sub-proof coincide; but that is intuitively legitimate, since plainly given the supposition that ‘Q’ is true, the conclusion ‘Q’ indeed must be true! We will therefore allow ‘vacuous’ sub-proofs where the initial supposition is the final wff (see §A1.5, the discussion of inference L).

Next we establish the validity of the inference

\[ B \]

\[(P v Q) \therefore (Q v P)\]

The proof is immediate:

\[ B' \]

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<tbody>
<tr>
<td>1</td>
<td>P∧Q</td>
<td>(Prem)</td>
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<tr>
<td>2</td>
<td>P</td>
<td>(Supp)</td>
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<tr>
<td>3</td>
<td>Q v P</td>
<td>(2 vI)</td>
</tr>
<tr>
<td>4</td>
<td>Q</td>
<td>(Supp)</td>
</tr>
<tr>
<td>5</td>
<td>Q v P</td>
<td>(4 vI)</td>
</tr>
<tr>
<td>6</td>
<td>Q v P</td>
<td>(1,2–3,4–5 vE)</td>
</tr>
</tbody>
</table>

Note, we could have added a third basic rule for disjunction, explicitly laying it down that disjunction is symmetric and so, from (A v B), you can always infer (B v A). But the availability of the proof-strategy just illustrated shows that the addition of such a rule would be redundant.

‘But hold on: isn’t it odd that v-elimination leads here to a disjunctive result?’

No! The disjunction being ‘eliminated’, being ‘used up’, is the initial disjunction, the one whose disjuncts appear at the head of the two sub-proofs.

Next, we establish some results about the interplay of disjunction and conjunction. Consider the intuitively valid inference

\[ C \]

\[(P ∧ (Q v R)) \therefore ((P ∧ Q) v (P ∧ R))\]

The first steps of the proof are entirely routine:

\[ C' \]

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<tbody>
<tr>
<td>1</td>
<td>P ∧ (Q v R)</td>
<td>(Prem)</td>
</tr>
<tr>
<td>2</td>
<td>P</td>
<td>(1 ∧E)</td>
</tr>
<tr>
<td>3</td>
<td>Q v R</td>
<td>(1 ∧E)</td>
</tr>
</tbody>
</table>
We now need to show that the desired conclusion ‘((P ∧ Q) ∨ (P ∧ R))’ follows whichever of the disjuncts ‘Q’ or ‘R’ obtains:

\[ \begin{align*}
(4) & \quad Q \quad \text{(Supp)} \\
(5) & \quad (P \land Q) \quad (2,4 \land I) \\
(6) & \quad ((P \land Q) ∨ (P \land R)) \quad (5 \lor I) \\
(7) & \quad R \quad \text{(Supp)} \\
(8) & \quad (P \land R) \quad (2,7 \land I) \\
(9) & \quad ((P \land Q) ∨ (P \land R)) \quad (8 \lor I) \\
(10) & \quad ((P \land Q) ∨ (P \land R)) \quad (3,4–6,7–9 \lor E)
\end{align*} \]

Similarly, we can show the converse also obtains, i.e. that the argument

D' \[ ((P \land Q) ∨ (P \land R)) \therefore (P \land (Q ∨ R)) \]

is also provably valid. Here’s the proof without further comment (make sure you understand the strategy):

\[ \begin{align*}
D' & \quad (1) \quad ((P \land Q) ∨ (P \land R)) \quad \text{(Prem)} \\
 & \quad (2) \quad (P \land Q) \quad \text{(Supp)} \\
 & \quad (3) \quad P \quad (2 \land E) \\
 & \quad (4) \quad Q \quad (2 \land E) \\
 & \quad (5) \quad (Q ∨ R) \quad (4 \lor I) \\
 & \quad (6) \quad (P \land (Q ∨ R)) \quad (3,5 \lor I) \\
 & \quad (7) \quad (P ∨ R) \quad \text{(Supp)} \\
 & \quad (8) \quad P \quad (7 \land E) \\
 & \quad (9) \quad R \quad (7 \land E) \\
 & \quad (10) \quad (Q ∨ R) \quad (9 \lor I) \\
 & \quad (11) \quad (P ∨ (Q ∨ R)) \quad (8,10 \land I) \\
 & \quad (12) \quad (P ∨ (Q ∨ R)) \quad (1,2–6,7–11 \lor E)
\end{align*} \]

Now, for an example of a slightly more complex proof, consider the inference

E \[ ((P \lor Q) \land (P \lor R)) \therefore (P \lor (Q \land R)) \]

The first steps of a proof warranting this inference write themselves:

\[ \begin{align*}
E' & \quad (1) \quad ((P \lor Q) \land (P \lor R)) \quad \text{(Prem)} \\
 & \quad (2) \quad (P \lor Q) \quad (1 \lor E) \\
 & \quad (3) \quad (P \lor R) \quad (1 \lor E)
\end{align*} \]

But now what? Well, let’s try to show that the conclusion follows whichever limb of disjunction (2) holds. The first limb is an entirely trivial sub-proof, and so we need to complete a proof of the following shape:

\[ \begin{align*}
(4) & \quad P \quad \text{(Supp)} \\
(5) & \quad (P \lor (Q \land R)) \quad (4 \lor I) \\
(6) & \quad Q \quad \text{(Supp)} \\
 & \quad \vdots \\
(n) & \quad (P \lor (Q \land R)) \\
(n+1) & \quad (P \lor (Q \land R))
\end{align*} \]
But now we can’t proceed further without making some new supposition. Now, 
we haven’t yet made use of the disjunctive premiss at line (3). The obvious thing 
to do next, then, is to assume each of its disjuncts in turn, with a view to using 
‘∨E’ on (3) to get to line (n). Since these will be new suppositions made while the 
supposition that ‘Q’ is in play, the proof indents another column. So the argu-
ment we want to complete will look like this:

\[
\begin{align*}
E' & \quad (1) \quad (P \lor Q) \land (P \lor R) \\
& \quad (2) \quad (P \lor Q) \\
& \quad (3) \quad (P \lor R) \\
& \quad (4) \quad P \\
& \quad (5) \quad (P \lor (Q \land R)) \\
& \quad (6) \quad Q \\
& \quad (7) \quad P \\
& \quad (8) \quad (P \lor (Q \land R)) \\
& \quad (9) \quad R \\
& \quad (n-1) \quad (P \lor (Q \land R)) \\
& \quad (n) \quad (P \lor (Q \land R)) \\
& \quad (n+1) \quad (P \lor (Q \land R))
\end{align*}
\]

But the remaining gap is now easily filled. And filling in the details, we get

\[
\begin{align*}
& \quad (9) \quad R \\
& \quad (10) \quad (Q \land R) \\
& \quad (11) \quad (P \lor (Q \land R)) \\
& \quad (12) \quad (P \lor (Q \land R)) \\
& \quad (13) \quad (P \lor (Q \land R))
\end{align*}
\]

And we are done! Do make a special point of checking through this proof, and 
ensuring that you understand the argumentative strategy: and note again how 
our convention of indenting suppositions facilitates understanding of the struc-
ture of the proof.

A2.2 Extending the rule vE

The two inference rules that we have given for disjunction are the absolutely 
standard ones for proof system like ours. But in some respects, the standard ver-
sion of vE is not the most natural one possible. To see this, consider the follow-
ing inference

\[
F \quad (P \lor Q), \neg P \vdash Q
\]

This form of inference, commonly called ‘disjunctive syllogism’, is tautologically 
valid, as you would expect. One or other of ‘P’ and ‘Q’ holds; but it can’t be ‘P’, 
as that contradicts the second premiss; so we can infer it must be ‘Q’. But how 
would we warrant this inference by our rules? We have to use ‘vE’, and the 
shape of a proof from those premisses to the desired conclusion must be like this:
Proofs, Continued

The second sub-proof is one of those trivial cases where the supposition at the top also counts as the conclusion at the bottom of the sub-proof. So the interest is in the other sub-proof. How do we get from the supposition ‘P’ to the desired conclusion ‘Q’? Well, we have ‘¬P’ as a premiss too, so we can use the fact that from a contradiction, anything you like follows. In other words, we can plug in a proof from ‘P’ and ‘¬P’ to ‘Q’ as in J, §A1.5.

Now, that would indeed complete the proof using our existing rules of inference. But there’s no getting away from it: it is surely rather odd to suggest that our validation of the intuitively acceptable disjunctive syllogism should depend on the controvertible claim that a contradiction entails anything. Put it this way: suppose we are persuaded to deviate from standard classical logic, and insist instead that we should stop a line of argument and apply reductio immediately a contradiction comes into play, then we won’t get a proof from ‘P’ and ‘¬P’ to ‘Q’ (see §A1.5 again); but surely we could still accept disjunctive syllogism.

Consider again the way we informally warranted disjunctive syllogism a moment ago: we said ‘one or other of P and Q holds; but it can’t be P, as that contradicts the second premiss; so we can infer it must be Q’. A reflection of this natural line of argument in our proof system would look like this,

\[
\begin{array}{c}
F' \quad (1) \quad (P \lor Q) \\
(2) \quad \neg P \\
(3) \quad P \\
(4) \quad Q \\
(5) \quad Q \\
(6) \quad Q \\
\end{array}
\]

\[
\begin{array}{c}
(\lor E) \\
(1,3\neg n, n+1\neg n+1 \lor E)
\end{array}
\]

where \(\lor E\) is the intuitively sound new rule: from \((A \lor B)\), together with a proof of \(C\) from one of the disjuncts as supposition, and a proof of absurdity from the other disjunct, we can infer \(C\). See the box for a schematic display. Adopting this new rule does not allow us to prove any more than we can prove without it: but as we’ve just seen, it does considerably shorten a number of proofs, and it does so in a very natural way. So it is well worth buying.

Here’s a slightly more complex illustration of the same rule in operation.
We'll show that the inference

\[ G, (R \lor (\neg P \land \neg Q)), \neg (R \land \neg Q) \therefore \neg \neg Q \]

is valid by producing a proof. We start as usual by listing the premisses

\[
\begin{align*}
G' & \quad (1) \ P \quad \text{(Prem)} \\
& \quad (2) \ (R \lor (\neg P \land \neg Q)) \quad \text{(Prem)} \\
& \quad (3) \ \neg (R \land \neg Q) \quad \text{(Prem)}
\end{align*}
\]

And now let's try to derive either the desired conclusion or a contradiction from each disjunct of (2). So first suppose ‘R’; this combines with the third premiss in a now familiar way:

\[
\begin{align*}
(4) & \quad R \quad \text{(Supp)} \\
(5) & \quad \neg Q \quad \text{(Supp)} \\
(6) & \quad (R \land \neg Q) \quad (4, 5 \land I) \\
(7) & \quad * \quad (6, 3 \text{ Abs}) \\
(8) & \quad \neg Q \quad \Omega \quad (5-7 \text{ RAA})
\end{align*}
\]

which completes the first sub-proof. We continue:

\[
\begin{align*}
(9) & \quad (\neg P \land \neg Q) \quad \text{(Supp)} \\
(10) & \quad \neg P \quad (9 \land E) \\
(11) & \quad * \quad (1, 10 \text{ Abs})
\end{align*}
\]

which completes the second sub-proof we wanted. So we can now apply \( \lor E^* \) to get

\[ \neg \neg Q \quad (2, 4-8, 9-11 \ lor E^*) \]

For a third example, consider the following proof to warrant the inference

\[ H \quad (P \land Q) \therefore \neg (\neg P \lor \neg Q) \]

And this time try to follow what is going on before reading the remarks on the proof’s structure.

\[
\begin{align*}
(1) & \quad (P \land Q) \quad \text{(Prem)} \\
(2) & \quad (\neg P \lor \neg Q) \quad \text{(Supp)} \\
(3) & \quad \neg P \quad \text{(Supp)} \\
(4) & \quad P \quad (1 \land E) \\
(5) & \quad * \quad (4, 3 \text{ Abs}) \\
(6) & \quad \neg Q \quad \text{(Supp)} \\
(7) & \quad \neg Q \quad (2, 3-5, 6-6 \ lor E^*) \\
(8) & \quad Q \quad (1 \land E) \\
(9) & \quad * \quad (8, 7 \text{ Abs}) \\
(10) & \quad \neg (\neg P \lor \neg Q) \quad (2-9 \text{ RAA})
\end{align*}
\]

Since we want to establish a conclusion of the form \( \neg A \), the obvious strategy is to assume \( A \) and aim for a reductio. But that brings into play the disjunctive supposition (2). To make use of this supposition, we’ll need to use \( \lor E \) or \( \lor E^* \). So we assume the two disjuncts in turn and see what happens … And the rest of the proof more or less writes itself.
It is slightly untidy, to have separate rules \(\lor E\) and \(\lor E^*\) (and indeed we've only presented them as separate rules for introductory clarity). We can readily wrap them up into one compendious rule—so here it is (and this will henceforth be our official version of the rule we'll simply continue to call ‘\(\lor E\)’). From \((A \lor B)\), together with a proof of \(C\) from one of the disjuncts as supposition, and a proof of either \(C\) or absurdity from the other disjunct, we can infer \(C\). To repeat, using this more compendious rule rather than the more standard original version does not allow us to prove any more. Whether we adopt the old style or the new style \(\lor E\), the resulting proof system for the three connectives is complete—that is to say, it has the resources to validate every tautologically valid argument using the three connectives ‘\(\land\)’, ‘\(\lor\)’ and ‘\(\neg\)’ which are indeed tautologically valid. But our revised version of \(\lor E\) speeds up a lot of proofs and we'll adopt it.

### A2.3 Theorems

Consider next the following short proof which starts with no given premisses, just an initial assumption made for the sake of argument:

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<tbody>
<tr>
<td>I</td>
<td>(1)</td>
<td>((P \land \neg P))</td>
<td>(Supp)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td>(P)</td>
<td>(1 (\land E))</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td>(\neg P)</td>
<td>(1 (\land E))</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td>(*)</td>
<td>(2,3 Abs)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(5)</td>
<td>((P \land \neg P))</td>
<td>(1–4 RAA)</td>
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</tbody>
</table>

By the end of this proof, the initial assumption made for the sake of argument is discharged; so we get the conclusion out of thin air—it can be proved using no initial premisses at all!

So, having adopted our set of rules for validating inferences from premisses involving the three connectives ‘\(\land\)’, ‘\(\lor\)’ and ‘\(\neg\)’, we now find that we get as a free bonus some further proofs which show that certain wffs are true independently of any premisses. Such wffs, provable by logic alone, are prime candidates for the status of logically necessary truths. In the present case, ‘\(\neg(P \land \neg P)\)’ is indeed a logical truth. And the proof here obviously generalizes: by substituting the wff \(A\) for each occurrence of ‘\(P\)’, we get a proof of \(\neg(A \land \neg A)\) from no premisses.

Here’s another quick example, this time to show that ‘\(\neg(P \lor \neg P)\)’ can be proved from no assumptions. Given we have no initial premisses to play with, the obvious way to start the proof is to assume the opposite of the desired conclusion and aim for a contradiction. So we start

| J | (1) | \(\neg(P \lor \neg P)\) | (Supp) |

But where do we go now? We can’t apply any of our rules to this wff: so to get anywhere we’ll need to make another assumption. The simplest shot is ‘\(P\)’—which is inconsistent with ‘\(\neg(P \lor \neg P)\)’ so reductio gives us ‘\(\neg P\)’…
An Introduction to Formal Logic

J
(1) \(\neg (P \lor \neg P)\) (Supp)
(2) \(P\) (Supp)
(3) \((P \lor \neg P)\) (2 vI)
(4) \(*\) (3,1 Abs)
(5) \(\neg P\) (2-4 RAA)

But now another contradiction hoves into sight …

(6) \((P \lor \neg P)\) (5 vI)
(7) \(*\) (6,1 Abs)
(8) \(\neg (P \lor \neg P)\) (1-7 RAA)
(9) \((P \lor \neg P)\) (8 DN)

and we are done. Again we can generalize. Systematically replace each occurrence of ‘\(P\)’ by some wff \(A\) and we’ll get a proof of \(A \lor \neg A\) from no premises.

Let’s introduce some standard jargon:

Wffs provable from no premisses are called theorems of the proof system.

Since our rules of inference look to be tautologically sound, leading to conclusions that are always true whenever the premisses of a proof are true, we’d expect theorems to always be true, i.e. to be tautologies. And this can be verified. Every theorem of our proof system for PL is indeed a a tautology; and conversely that every tautology is a theorem.

A2.4 The rules again, a little more carefully

By this stage, you should have a good understanding of the way our proof system for PL arguments lays out proofs, and of the basic rules of inference governing ‘\(\land\)’, ‘\(\lor\)’, and ‘\(\neg\)’. But we’ve left loose ends. For example, should we insist that every premiss is listed at the very outset of a proof? Or are we going to allow new premisses to be introduced at any stage, so long as they are clearly signalled as such? It really doesn’t matter which option we choose; either way, the final conclusion will follow from whatever earlier wffs are premisses. But in a full-dress, official, version of the rules, we’d better pick an option.

We’ve already signalled another choice (immediately after §A1.2, B”): we said that we are not going to treat the line numbers to the left and commentary on the right of a line as belonging to the PL proof itself. So the proof itself consists in the sequence of wffs, moving to and fro between columns as suppositions are made and discharged.

What really matters about the columnar structure is how we shift columns to mark the beginning and end of sub-proofs, or use a bar to separate the two sub-proofs which are used by vE. Thinking a bit more abstractly, a proof in our system consists in a sequence of wffs, absurdity markers, and ‘start sub-proof’ and ‘stop sub-proof’ signals. As a stylistic variant, we could have instead used e.g. ‘\(\triangleright\)’ and ‘\(\triangleleft\)’ as start and stop signals, and written a proof such as \(H\) like this:

\[H' \quad (P \land Q) \triangleright (\neg P \lor \neg Q) \triangleright \neg P, P, * \triangleleft \neg Q \triangleleft Q, Q, * \triangleleft (\neg P \lor \neg Q)\]
But our device of writing the proof vertically and using column shifts (or horizontal bars) instead of ‘>’ and ‘<’ is a lot easier to follow.

We stressed in §A1.5 that when a sub-proof is finished we may no longer use its wffs as premisses to derive further wffs. To put it another way, a wff is only available for use as a premiss a certain stage of a proof if it doesn’t belong to a now finished sub-proof. When is a sub-proof finished? At the first point when followed by one more ‘stop’ signal than ‘start’ signal. Or in terms of our layout conventions, a sub-proof stops when, starting from the supposition at its beginning, we next encounter a a shift back to the left, or hit a horizontal bar in the same column. And which wffs are available at a given stage? It is easy to see that our definition is equivalent to the following—track back up the current column to its head (a supposition or the top of the whole argument, whichever applies); if you encounter a supposition at the top of a sub-proof, move back a column left, and track up that column to its head; if you encounter a supposition at the top of a sub-proof, move back a column left … and keep on going until you get to the top of the argument: then all and only those wffs encountered en route are available to be used as premisses. So in the illustrated proof structure, at stage F, the wffs E, B, A are available; at stage L, the wffs K, J, I, H, G, B, A are all available; but at stage O, the wffs N, G, B, A are now available. Finally at Q, only the previous wffs in the ‘home’ column are still available.

With these definitions to hand, we can now offer an official characterization of a PL proof. For tidiness, we’ll insist that premisses are listed at the outset; that (RAA) and (∨E) are applied immediately after the sub-proof(s) they invoke; and that there aren’t any ‘orphaned’ sub-proofs which are not later used by an application of (RAA) or (∨E). Then, we’ll say that a sequence (of wffs and absurdity markers, plus ‘start sub-proof’ and ‘stop sub-proof’ signals) forms a PL proof of C from the premisses A₁, A₂, A₃, ..., Aₙ just if:

• the sequence starts with A₁, A₂, A₃, ..., Aₙ;
• all sub-proofs in the sequence are finished;
• the sequence concludes with C;
• after the premisses, each further step of the proof satisfies one of the following conditions:
  (Supp) a sub-proof is started, and any wff A is added to the sequence;
  (∧I) a wff of the form (A ∧ B) is added to the sequence, where the wffs A and B are each available at that step;
  (∧E) a wff A is added, where either a wff (A ∧ B) or a wff (B ∧ A),
for some $B$, is available;

(DN) a wff $A$ is added, where is $\neg\neg A$ is available;

(vI) either a wff $(A \lor B)$ or a wff $(B \lor A)$ is added, where is $A$ is available, and $B$ is some wff;

(Abs) the absurdity marker ‘$\ddagger$’ is added, where both $A$ and $\neg A$ are available, for some $A$; then the sub-proof is ended (if in the home column, the whole sequence is ended);

(RAA) Immediately after the application of (Abs) to finish a sub-proof, $\neg A$ is added, where $A$ is the initial wff of the sub-proof just ended.

(vE) Immediately after the end of a sub-proof which itself immediately follows another finished sub-proof, $C$ is added, where a wff $(A \lor B)$ is available at the start of the first of those sub-proofs, and where one sub-proof begins with $A$, the other begins with $B$, and where one ends with $C$ and the other with either $C$ or ‘$\ddagger$’.

• each finished sub-proof is used either by an application of (RAA) or (vE).

And we are done: we have redeemed our promise to give a tidy, official definition of a proof in our proof-system for PL arguments—though whether the gain is quite worth the pain is a moot point!

Note finally that the question whether a sequence of symbols forms PL proof according to our rules is a syntactic question. To explain: the question whether a string of symbols forms a PL wff is just the question whether it can constructed according to the rules for wff-building (§8.1). The meaning of the symbols is irrelevant: a computer can be programmed to check whether a string is a wff without teaching it anything about truth or interpretation. The surface shape of the string is all that matters. Similarly, the question whether a sequence of wffs and markers forms a PL proof from premisses $A_i$ to conclusion $C$ is just the question whether the sequence obeys our conditions for what makes a proof. The meaning of the symbols is again irrelevant: a computer can be programmed to check whether a sequence is a wff without teaching it anything about interpretation or inference. The surface shape of the sequence is all that matters.

A2.5 ‘$\vdash$’ and ‘$|$’

Recall the symbolism we introduced in §13.7. If the argument couched in PL from premisses $A_1, A_2, A_3, \ldots A_n$ to the conclusion $C$ is tautologically valid, then we write

(i) $A_1, A_2, A_3, \ldots A_n \vdash C$.

As we remarked before, the double turnstile ‘$\vdash$’ symbol is not part of PL, but (like the metalogical variables $A_i$) is an extension of the logician’s metalan-
guage—i.e. it belongs to extended English. By definition, this entailment relation holds iff no valuation (of the relevant atoms that occur somewhere in the premises or conclusion) makes all the $A_i$ true and $C$ false. Since the relation is defined in terms of valuations, i.e. assignments of truth or falsity, it sways in the same conceptual orbit as notions of meaning and truth, so it is a semantic relation in that sense. ‘$\vdash$’ is thus sometimes read ‘semantically entails’.

Now for the some symbolism: if there exists a PL proof from the premisses $A_1, A_2, A_3, \ldots A_n$ to the conclusion $C$, then we will write

(ii) $A_1, A_2, A_3, \ldots, A_n \vdash C$.

This single turnstile ‘$\vdash$’ also belongs to extended English, and indicates the relation that holds if we can get from the premisses to the conclusion by allowable inference moves in a PL proof. This relation, as we stressed a moment ago, is defined in terms of symbol-juggling without essential reference to meanings—it is a purely syntactic relation. So ‘$\vdash$’ is often read ‘syntactically entails’.

What is the relation between semantic and syntactic entailment? We plainly need it to be the case this holds:

**Soundness**  If $A_1, A_2, A_3, \ldots, A_n \vdash C$ then $A_1, A_2, A_3, \ldots, A_n \vdash C$.

This says that if you can prove the conclusion $C$ from some collection of PL premisses $A_i$, then these premisses actually do tautologically entail the conclusion. The whole point of our proof system was to codify obviously valid simple modes of inference, and then allow us to verify more complex inferences by chaining together these obviously sound steps, so we should indeed require our proof-system to be a sound one, which won’t enable us to derive conclusions that don’t genuinely follow from the given premisses.

But we also would like our system to be such that the following holds:

**Completeness**  If $A_1, A_2, A_3, \ldots A_n \vdash C$ then $A_1, A_2, A_3, \ldots A_n \vdash C$.

Again unpacking the shorthand, this says that if an argument is valid by the truth-table test then we can get from the premisses to the conclusion in our formal proof system. In other words, completeness holds when our PL proof system has the resources to allow us to derive any conclusion that really is tautologically entailed by a given set of premisses. And ideally, we’d like our proof system to completely capture all the tautologically valid PL inferences.

But the properties of soundness and completeness of course don’t come automatically:

- A proof system can be sound but not complete. For example, the system which only has the rules $\land E$, Abs, RAA and DN is sound (the conclusions that this mutilated system can prove are indeed conclusions that genuinely follow by the truth-table test). But the proof system is obviously not complete.

- A proof system can be complete but not sound. For example, the system with the hyper-permissive rule ‘From any given premisses, infer any conclusion you like’ is complete (since we can infer anything at all, we’ll certainly
be able to infer the conclusions that are semantically entailed by the premises. But this system is obviously not sound!

In fact, the happy result is that our PL proof system hits the right balance: it has enough rules but not too many, it is sound and complete. In other words, the system won’t output garbage, and will output all the truth-table valid arguments. As promised, we’ll show that in Chapter A4.

If we have $A_1, A_2, A_3, \ldots A_n \vdash C$ (if an only if) $A_1, A_2, A_3, \ldots A_n \vdash C$, then in particular this applies to the case where the set of premises is empty, i.e. $\vdash C \iff C$. $\vdash C$, recall, just says that $C$ is a tautology ($\S$13.7). $\vdash C$ says that $C$ can be derived in our proof system without appealing to any premises, i.e. it is a theorem. So the result $\vdash C \iff \vdash C$ says that a wff is a theorem if and only if it is a tautology.

A2.6 Summary

- To complete our PL proof-system we have added a couple of rules for dealing with disjunction, $\lor$ and $\lor E$.
- Our version of $\lor E$ is more complex than the more usual one; the benefit is that we can shorten many proofs (as well as warrant e.g. disjunctive syllogism without relying on inferring an arbitrary wff from a contradiction).
- The resulting full proof-system can be shown to be sound and complete.
- We can produce proofs of some wffs without relying on any premises at all. These are the theorems of our system (and all and only tautologies are theorems).
- We have introduced the single turnstile ‘$\vdash$’ to signify the relation between a bunch of premises $A_1, A_2, A_3, \ldots, A_n$ and a conclusion $C$ that holds when there is a PL proof from the premises to the conclusion.

Exercises A2

Show that the following arguments are valid by giving, in each case, a formal proof of the conclusion from the premises:

i $R, \neg (R \land \neg Q) \vdash (Q \lor P)$

ii $(P \land Q), \neg R \vdash ((P \lor R) \land (\neg P \lor \neg R))$

iii $(P \land R), \neg (P \land Q) \vdash (\neg Q \lor (S \lor \neg R))$

iv $((P \land Q) \lor (R \land \neg P)) \vdash P$

v $(P \lor \neg Q), \neg (P \land Q) \vdash \neg Q$

vi $(P \land (Q \lor R)) \vdash ((P \land Q) \lor (P \land R))$

vii $(P \lor \neg Q), \neg (P \land \neg R) \vdash (R \lor Q)$

viii $((P \lor Q) \lor R) \vdash (P \lor (Q \lor R))$

ix $\neg P, (P \lor \neg Q), (Q \lor R) \vdash R$