Philosophy Faculty Reading Lists 2011-2012

PART II PAPER 07:
MATHEMATICAL LOGIC

This paper, with something quite like the current syllabus, is a very long-standing Cambridge tradition, rather distinctively covering both some core technical results in mathematical logic, but also exploring a selection of philosophical issues they give rise to or throw light on. (Other issues, less technical, issues in the philosophy of mathematics belong in Paper 8.]

Anyone who takes this paper will need to get to understand the formal results (and have some sense of how they are proved). But then you can concentrate more on the technical side and really master the "bookwork" proofs of some key results (which are in fact, approached slowly, relatively straight-forward); or you can concentrate on the philosophical issues; or – as most people taking the paper choose to do – you can work on some of both.

Here, then, is the current **syllabus**, which falls into three parts.

> **A) First-order and second-order logic:** completeness, compactness, conservativeness, expressive power and Löwenheim-Skolem theorems. First and second order theories; categoricity, non-standard models of arithmetic.
>
> **B) Recursive functions and computability:** decidability, axiomatizability, Church's thesis, Gödel's incompleteness theorems, Hilbert's programme.
>
> **C) Set theory:** embedding mathematics in set theory, the cumulative iterative hierarchy, elements of cardinal and ordinal arithmetic, the axiom of choice.

(A) We take a more detailed look at core logic -- the quantification theory familiar from earlier years of the tripos -- and then at its second-order extension. Recall: first-order logic allows us to move from e.g. Fa to $\exists xFx$; second order logic allows us to "quantify into predicate position" and move from Fa to $\exists X\, Xa$. Arguably, second-order logic looks to be the "natural" logic for informal mathematics: but how does it work in more detail? Why has it so often been thought problematic?

(B) Logic, of course, provides the deductive apparatus for formalized theories. And the second component of the paper looks at what we can think of as the canonical theory of finite mathematical objects (i.e. suites of objects that can all be finitely characterized), namely arithmetic. For note, all the natural numbers can be finitely characterized, and other kinds of finite objects can be numbered off and talked

about via their numerical codes. We discuss the contrasts between first- and second-order arithmetics, and also consider the limitations revealed by Gödel's Theorems that apply to any rich enough formalized arithmetic. We discuss the idea of "mechanical" formal computations on numbers, leading up to a discussion of Church's Thesis which claims that the mechanically computable functions are just the so-called recursive functions. We also consider Hilbert's programme which gives a special role to finitary mathematics.

(C) The third part of the paper looks at theories of suites of objects that can't all be finitely characterized together. The canonical such theory is set theory, which allows us in the first instance to talk not just about numbers but about arbitrary infinite sets of numbers, and then – by various more or less natural constructions – talk about real numbers, functions from real numbers to real numbers, complex numbers, and onwards through the whole bestiary of mathematics. Because other infinitary objects can be defined in set theoretical terms, there is a sense in which set theory can be thought of as "foundational". But here's a deep and fundamental issue about the supposedly foundational notion of a set because the naïve set theory you might first dream up is very easily seen to be inconsistent (by Russell's Paradox). So how should we think of sets? What sets are there?

You should certainly go to the lectures on each part: but it is possible to then really concentrate on (A) + (B), or (A) + (C).

CORE FORMAL READING

The core technical reading for this paper has been produced "in house". First, Dr. Oliver gives out a full set of handouts for his course on core logic.

Second, Dr Smith has worked up his old handouts for this paper on computation and Gödel's Theorems into a recent book (though, as is the way with these things, its contents now overshoot by quite a way what you really need to know for this paper!):

SMITH, P., *An Introduction to Gödel's Theorems* (Cambridge, Cambridge University Press, 2007) [The 4th reprint (2009) has been heavily corrected.]

And third, Prof. Potter has rewritten his earlier 1990 mathematics book on set theory into a text with a lot more philosophical commentary (though again this rather overshoots as far as the needs of the paper are concerned):

POTTER, M., *Set Theory and Its Philosophy* (Oxford, Oxford University Press, 2004). [Available online at www.myilibrary.com/?id=75496]

Lecturers will, of course, indicate the "must-read" portions of these books, and indicate too which bits of "bookwork" – i.e. which standard proofs – might be directly examinable.

But don't stick to just one book on a formal topic, for it is always good to read more than one presentation of formal results. The Casimir Lewy library shelves a wide range of text books which cover the formal parts of the Mathematical Logic syllabus, at various degrees of depth and assuming various degrees of mathematical sophistication. Most of these books have virtues (or they wouldn't have got published!); many are really quite excellent. We mention a number of the best in this list: but different books will "work" for different readers – if you don't get on with one, try another on the list or neighbours on the library shelves.

PRELIMINARY READING ON THE FORMAL MATERIAL

First, here are a few of the most accessible treatments, some of which could very usefully be looked at as preliminary reading before the course starts (or later used in parallel with the other core readings).

On basic logic:

CHISWELL, I., and W. HODGES, *Mathematical Logic* (Oxford: Oxford University Press, 2007).
LEARY, C. C., *A Friendly Introduction to Mathematical Logic* (London: Prentice Hall, 2000). [Chs 1-3]

Both of these are pretty accessible, Chiswell and Hodges perhaps more so: it goes a bit slower and looks around a bit more widely: highly recommended. Both books get to high point of the completeness and compactness theorems for first-order logic, which are the key theorems of the purely logical component of the course.

(Don't worry if a few of the examples in these or other books require more mathematical background than you have: you can very often just skip. Though it is worth noting that Wikipedia's mathematical entries are usually pretty reliable if you want to discover the meaning of some unfamiliar mathematical term.)

On arithmetic, computability etc.

For an accessible introduction, with some very nice historical excerpts thrown in, see:

EPSTEIN, R. L., and W. CARNIELLI, *Computability: Computable Functions, Logic, and the Foundations of Mathematics*. 2nd ed. (London: Wadsworth, 2000). [You can skip the "optional" chapters.]

On set theory:

There are a number of excellent "entry-level" options here. Here's three:

DEVLIN, K., *The Joy of Sets*. 2nd ed. (New York: Springer-Verlag, 1993). [chs 1, 2]
GOLDREI, D., *Classic Set Theory* (London: Chapman & Hall/CRC, 1996).
HALMOS, P., *Naïve Set Theory* (New York: Springer, 1974).

Devlin is more conceptually/philosophically alert than many mathematicians, and the opening chapters of his book give a very accessible presentation of the motivation for standard Zermelo-Fraenkel set theory: but this is perhaps the most challenging of the three books. Goldrei is at the Open University and his book is subtitled 'For Guided Independent Study': it is very carefully written as a teach-yourself-the-basics book. Halmos's old book (originally from 1960) is the most accessible, and deservedly a classic: it is only a hundred pages long, but will give you a good preliminary sense of what set theory is about and what it might be useful for.

GENERAL FORMAL SURVEYS

Two very useful, more discursive surveys, standing back a bit from the nitty gritty of proofs, but trying to give a sense of how results fit together with an indication of their wider significance are:

ROGERS, R., *Mathematical Logic and Formalized Theories* (Amsterdam: North-Holland, 1971).
WOLF, R. S., *A Tour through Mathematical Logic* (Washington: Mathematical Association of America, 2005).

Rogers's now rather old book is very useful and very accessible though relatively introductory. Wolf's newer book goes further but is a rather bumpier ride because somewhat uneven in level of difficulty (though he gives some useful proof sketches). These books will make very useful companions to formal work over the year, and could be especially helpful whenever you feel in danger of not seeing the wood for the trees.

RELATING THE FORMAL TO THE PHILOSOPHICAL

One recent book stands out as a reliable, accessible, and thought-provoking guide into quite a few of the philosophical issues on the syllabus which arise from the formal work:

GIAQUINTO, M., *The Search for Certainty* (Oxford, Clarendon Press, 2002). [Also available online at http://lib.myilibrary.com/?id=91428]

This is very good on what the project of set theory is, on Hilbert's programme, on Skolem's Paradox, and on the significance of Gödel's incompleteness theorems, all core philosophical topics for this paper. Another very useful and insightful survey essay on the philosophy of mathematics, which touches on some of the technical material in the paper is:

DUMMETT, M., 'The Philosophy of Mathematics', in *Philosophy 2: Further Through the Subject*, ed. by A. Grayling (Oxford: Oxford University Press, 1998).

THE WAY THE REST OF THIS READING LIST IS STRUCTURED

Now to get down to more details! The rest of this reading list is divided into three main parts corresponding to parts A, B and C of the syllabus. To repeat, for formal expositions, we'll list some possible alternatives, because different presentations are to the taste of different readers. **No one expects you to read all of them!**

For philosophical topics, we usually distinguish core readings – in something like a sensible reading order – from a selection of possible further readings. Such divisions are inevitably somewhat arbitrary, and different supervisors will want to take different views about what is basic – needed to make a shot at a supervision essay – and what pushes on the debate rather further. Still, it is better to make some crude divisions than to present undifferentiated and perhaps daunting lists without any commentary.

PART A: LOGIC

FIRST ORDER LOGIC

Formal Expositions

The key things you'll need to understand are the ideas of soundness/ completeness theorems, the compactness theorem, and the Löwenheim-Skolem theorems – and how to prove them.

The books by Chiswell/Hodges and by Leary already mentioned of course cover first-order logic in an accessible way. And almost any standard middle or advanced level text will cover the needed ground. But a stand-out presentation is:

HODGES, W., 'Elementary Predicate Logic', in *Handbook of Philosophical Logic, Vol. 1*, ed. by D. Gabbay and F. Guenthner (Dordrecht: Reidel, 1984-89). Also an expanded version of this appears in the 2nd edition of the *Handbook*.

For a more discursive introduction to some main ideas see:

ROGERS, R., *Mathematical Logic and Formalized Theories* (Amsterdam: North-Holland, 1971). [Chs 2, 3]

What is the relation between the informal idea of a first-order quantifier inference being valid and its being valid according to the standard model-theoretic definition given in the formal texts? Arguably, such an inference is valid in the intuitive sense if and only if it is model-theoretically valid. For a defence of that claim, see the important:

KREISEL, G., 'Informal Rigour and Completeness Proofs' *Problems in the Philosophy of Mathematics*, ed. Imre Lakatos (Amsterdam: North-Holland, 1967), pp. 138–157. Reprinted in *The Philosophy of Mathematics*, ed. J. Hintikka (Oxford: Oxford University Press, 1969).

Those readings should more than suffice alongside Dr. Oliver's notes. But for another overview treatment highlighting the main ideas, though in more detail, you could see:

BOOLOS, G., J. BURGESS and R. C. JEFFREY, *Computability and Logic*. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007). [Chs 9, 10, 12, 13, 14]

And finally let's mention two more standard full-dress textbook treatments for enthusiasts:

ENDERTON, H., *A Mathematical Introduction to Logic.* 2nd ed. (San Diego: Harcourt/Acadmic Press, 2002). [Ch. 2]
MENDELSON, E., *Introduction to Mathematical Logic*. 4th ed. (Pacific Grove, California: Wadsworth, 1997). [§§2.1—2.9.]

Enderton is the more approachable, while Mendelson's book is something of a classic that just might appeal to those with a mathematics background (or who want to see the book their lecturers first learnt logic from!).

Philosophical Issues Arising: Skolem's Paradox

Some issues about first-order logic (the interpretation of the connectives, how to construe quantification) were topics in IA and IB. A key philosophical issue for this paper is the so-called Skolem paradox. So what is the supposed paradox? How should we respond? For basic discussion see:

GIAQUINTO, M., *The Search for Certainty* (Oxford, Clarendon Press, 2002). [pp. 130–136. Also available online at http://lib.myilibrary.com/?id=91428]
BENACERRAF, P., 'Skolem and the Sceptic', *Aristotelian Society Supplementary Volume,* 59 (1985), 85-115. Also in S. Shapiro, ed., *The Limits of Logic* (Aldershot: Dartmouth, 1996).

Also see, for slightly more brisk discussion, the opening third of:

BAYS, T., 'Skolem's Paradox'*.* In E.N. Zalta, ed. *The Stanford Encyclopedia of Philosophy.* 2009. Retrieved 11/09/09 from http://plato.stanford.edu/archives/spr2009/entries/paradox-skolem.

For further reading, try the rest of Bays's article, and some of:

BOOLOS, G., J. BURGESS and R. C. JEFFREY, *Computability and Logic*. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007). [Pp. 251-253.]
GEORGE, A., 'Skolem and the Lowenheim-Skolem Theorem: A Case Study of the Philosophical Significance of Mathematical Results.' *History and Philosophy of Logic* 6 (1985): 75-89. [http://dx.doi.org/10.1080/01445348508837077]
MCINTOSH, C., 'Skolem's Criticisms of Set Theory', *Noûs,* 13 (1979), 313-334.
MELIA, J., 'The Significance of Non-Standard Models', *Analysis,* 55 (1995), 127-134.
MOORE, A. W., *The Infinite* (London: Routledge, 1990). [Ch. 11]
PUTNAM, H., 'Models and Reality', *Journal Symbolic Logic,* 45 (1980), 464-482. Also in his *Realism and Reason*, Philosophical Papers, vol. 3 (Cambridge: Cambridge University Press, 1983).
SHAPIRO, S., *Foundations without Foundationalism* (Oxford: Oxford University Press, 1991). [Ch. 8. Also available from www.oxfordscholarship.com]
SKOLEM, T., 'Some Remarks on Axiomatised Set Theory', in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, ed. by J. van Heijenoort (Cambridge, Massachusetts: Harvard University Press, 1967). [esp. § 3; *locus classicus* of the paradox]
WRIGHT, C., 'Skolem and the Skeptic', *Aristotelian Society Supplementary Vol.,* 59 (1985). Also in S. Shapiro, ed., *The Limits of Logic* (Aldershot: Dartmouth, 1996) [§ IV]

SECOND ORDER LOGIC

Formal Expositions

You need some sense of the difference between first- and second-order logic in terms of axiomatizability, compactness, L-S theorems etc. You'll need to understand why e.g. the second-order Peano arithmetic is categorical and first-

order Peano-arithmetic isn't.

For a useful introductory overview, see:

ROGERS, R., *Mathematical Logic and Formalized Theories: A Survey of Basic Concepts and Results* (Amsterdam: North-Holland, 1971). [Ch. 4, §§ 1–5]

Another introductory formal survey is:

ENDERTON, H., 'Second-Order and Higher-Order Logic'*.* In E.N. Zalta, ed. *The Stanford Encyclopedia of Philosophy.* 2009. Retrieved 11/09/09 from http://plato.stanford.edu/entries/logic-higher-order.

But the classic modern presentation is no doubt:

SHAPIRO, S., *Foundations without Foundationalism* (Oxford: Oxford University Press, 1991). [chs 3-5. Also available online at: www.oxfordscholarship.com]

(Shapiro's book is subtitled "A Case for Second-Order Logic"). That would give you more than enough. If you prefer a much brisker overview, see:

BOOLOS, G., J. BURGESS and R. C. JEFFREY, *Computability and Logic*. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007). [Ch. 22]

Enthusiasts who are looking for other textbook treatments can try:

ROBBIN, J. W., *Mathematical Logic: A First Course* (New York: Benjamin, 1969). [Ch. 6]
VAN DALEN, D, *Logic and Structure* 4th ed. (Berlin: Springer, 2004). [Ch. 4].

Finally, for more discussion of first- vs. second-order arithmetic, you should look at:

SMITH, P., *An Introduction to Gödel's Theorems* (Cambridge, Cambridge University Press, 2007). [Ch. 22]

And for a discussion of the sort of non-standard models that first-order arithmetic can have, see (probably after having looked at some of the background reading for Part B of the syllabus):

BOOLOS, G., J. BURGESS and R. C. JEFFREY, *Computability and Logic*. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007). [Ch. 25]

Philosophical Issues Arising: 1, On The Status Of Second-Order Logic As Logic

Is second-order logic just set-theory in disguise (with the second-order quantifiers running over sets)? That's the view of:

QUINE, W. V., *Philosophy of Logic*. 2nd ed. (Cambridge, Massachusetts: Harvard University Press, 1986). [ch. 5]

For discussion see:

BOOLOS, G., 'On Second-Order Logic', *Journal of Philosophy,* 72 (1975): 502-526. Also in his *Logic, Logic and Logic* (Cambridge, Massachusetts: Harvard University Press, 1998); and in S. Shapiro, ed., *The Limits of Logic* (Aldershot: Dartmouth, 1996).

SHAPIRO, S., *Foundations without Foundationalism* (Oxford: Oxford University Press, 1991). [ch. 2, §§ 3-5. Also available: www.oxfordscholarship.com]

For further reading, see:

BOOLOS, G., 'A Curious Inference', *Journal of Philosophical Logic,* 16 (1987), 1-12. Also in his *Logic, Logic and Logic* (Cambridge, Massachusetts: Harvard University Press, 1998); and in S. Shapiro, ed., *The Limits of Logic* (Aldershot: Dartmouth, 1996).

THARP, L., 'Which Logic Is the Right Logic?' *Synthese,* 31 (1975), 1-21. Also in S. Shapiro, ed., *The Limits of Logic* (Aldershot: Dartmouth, 1996).

VÄÄNÄNEN, J. 'Second-order Logic and Foundations of Mathematics'*, The Bulletin of Symbolic Logic,* 7 (2001) 504–520. Postscript file: www.math.ucla.edu/~asl/bsl/0704/0704-003.ps.

Philosophical Issues Arising: 2, The Connections With Plural Quantification And Natural Language

George Boolos has argued that we can "tame" second-order logic (and see it as genuinely part of logic) by interpreting second-order quantifiers as (akin to) plural quantifiers. For a basic exchange, see:

BOOLOS, G., 'To Be Is to Be a Value of a Variable (or to Be Some Values of Some Variables)', *Journal of Philosophy,* 81 (1984), 430-449. Also in his *Logic, Logic and Logic* (Cambridge, Massachusetts: Harvard University Press, 1998); and in S. Shapiro, ed., *The Limits of Logic* (Aldershot: Dartmouth, 1996).

RESNIK, M., 'Second Order Logic Still Wild', *Journal of Philosophy,* 85 (1988), 75-87. Also in S. Shapiro, ed., *The Limits of Logic* (Aldershot: Dartmouth, 1996).

For further discussion see:

BOOLOS, G., 'Nominalist Platonism', *Philosophical Review,* 94 (1985), 327-344. Also in his *Logic, Logic and Logic* (Cambridge, Massachusetts: Harvard University Press, 1998); and in S. Shapiro, ed., *The Limits of Logic* (Aldershot: Dartmouth, 1996).

HIGGINBOTHAM, J., 'On Higher-Order Logic and Natural Language', in *Philosophical Logic*, ed. by T. J. Smiley (Oxford: Oxford University Press, 1998).

LINNEBO, Ø., 'Plural Quantification'*.* In E.N. Zalta, ed. *Stanford Encyclopedia of Philosophy.* 2008. Retrieved 16/09/2009 from http://plato.stanford.edu/archives/spr2009/entries/plural-quant.

RAYO, R. and S. YABLO, 'Nominalism Through De-Nominalization'", *Noûs* 35 (2001): 74-92.

A HISTORICAL ASIDE

Modern textbooks present a pretty united front! They may differ in their preferred type of formal proof system – e.g. an axiom system in Mendelson, a natural deduction system in van Dalen, a 'semantic tableaux' or tree system (as in Peter Smith's *Introduction to Formal Logic*) in Smullyan's classic *First-Order Logic*. But the proof systems are equivalent; the books agree on giving centre stage to the equivalent versions of "first-order" logic. It wasn't always like this. One rainy afternoon in the library, flick through the pages of:

FREGE, G., *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (Halle: L. Nebert, 1879). [Translated in (i) *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, ed. by J. van Heijenoort (Cambridge, Massachusetts: Harvard University Press, 1967), also as 'Conceptual Notation' in (ii) *Frege, Conceptual Notation and related articles* trans T. W Bynum (Oxford: OUP, 1972).]

RUSSELL, B., and A.N. WHITEHEAD, *Principia Mathematica* (Cambridge: Cambridge University Press, 1910-13; 2nd ed. 1925-27)

These immediately *look* very different from modern logical systems; and they *are.* Both go beyond what we now think of as first-order logic, and don't isolate their first-order subsystems as a core. It perhaps isn't until we get to

HILBERT, D. and W. ACKERMANN *Grundzüge der Theoretishcen Logik* (Berlin: Springer-Verlag, 1928). 2nd ed. translated as *Principles of Mathematical Logic* (New York, Chelsea Publishing Co, 1950).

that a system of first-order logic is extracted and treated as central (it is worth dipping into Hilbert and Ackermann's wonderful book to note how recognizable their discussion is, compared with Frege and Russell/Whitehead). So what was the story that lead to the emergence of first-order logic as the acknowledged privileged core theory? It's well worth following this up, e.g. in:

FERREIROS, J., 'The Road to Modern Logic – an Interpretation", *Bulletin of Symbolic Logic* 7 (2001): 441-84. [Available online at: www.math.ucla.edu/~asl/bsl/0704/0704-001.ps]

which is long, but full of interest.

PART B : ARITHMETIC AND COMPUTABILITY

BACKGROUND

You at least need to know what "Robinson Arithmetic, "First-Order Peano Arithmetic" and "Second-Order Peano Arithmetic" are, and have a sense of their relative strengths. For an introduction to formal theories of arithmetic see:

SMITH, P., *An Introduction to Gödel's Theorems* (Cambridge: Cambridge University Press, 2007, 2009). [Chs 8, 10, 22]

For a more mathematical introduction, see:

MENDELSON, E., *Introduction to Mathematical Logic*. 4th ed. (Pacific Grove, California: Wadsworth, 1997). [§3.1]

GÖDEL'S FIRST INCOMPLETENESS THEOREM

Formal Expositions

For a nice introduction (in a splendidly sane short book, which you should eventually read all of), see:

FRANZÉN, T., *Gödel's Theorem: An Incomplete Guide to Its Use and Abuse* (Wellesley: A. K. Peters, 2005). [Chs 1–3]

And for another introductory survey see:

ROGERS, R., *Mathematical Logic and Formalized Theories* (Amsterdam: North-Holland, 1971). [ch. 8]

There's a bit more detail again in:

GEORGE, A., and D. J. VELLEMAN, *Philosophies of Mathematics* (Oxford: Blackwell, 2001). [ch. 7]

But for a full-dress proof with all the trimmings see:

SMITH, P., *An Introduction to Gödel's Theorems* (Cambridge: Cambridge University Press, 2007, 2009). [Especially chs 16,17 -those chapters more or less follow Gödel's original proof.]

Those recommendations on the formalities should more than suffice for philosophical purposes. But some might like the rather different approach of the enviably elegant:

SMULLYAN, R., *Gödel's Incompleteness Theorems* (Oxford: Oxford University Press, 1992). [Chs 1-5]

which will particularly appeal to the mathematically minded.

Note by the way that Gödel proved his First Theorem in 1931, before the beginnings of the general theory of computability really got underway in 1936: the original version of the Theorem appeals only to the restricted notion of a "primitive recursive" function. Many modern books, however, approach things in a non-historical order, first explaining the general theory of computability, and then moving on to Gödel's Theorem. Two notable books which do things this way around are:

BOOLOS, G., J. BURGESS and R. C. JEFFREY, *Computability and Logic*. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007). [Gets to Gödel's Theorem in Ch. 17.]

EPSTEIN, R. L., and W. CARNIELLI, *Computability: Computable Functions, Logic, and the Foundations of Mathematics*. 2nd ed. (London: Wadsworth, 2000).

Philosophical Issues Arising: 1, Minds And Machines

LUCAS, J. R., 'Minds, Machines and Gödel', *Philosophy,* 36 (1961), 112-127. Also in A. R. Anderson, ed., *Minds and Machines* (Englewood Cliffs, New Jersey: Prentice-Hall, 1964), pp. 43-59.

famously argues that Gödel's theorem shows that minds are not machines. (It is not really essential, but might help if you know what a Turing machine is before you start reading this debate). For a classic riposte, see:

PUTNAM, H., 'Minds and Machines', in his *Mind, Language and Reality, Philosophical Papers, Vol. 2* (Cambridge: Cambridge University Press, 1975). [§ 1]

Others have tried to rescue Lucas's argument, in particular:

PENROSE, R., *Shadows of the Mind* (Oxford: Oxford University Press, 1994). [chs 2 & 3, esp. §§ 2.5-3.10]

For a stern critique of that see:

FEFERMAN, S. 'Penrose's Gödelian Argument', in *PSYCHE* 2 (1996): 21-32, http://math.stanford.edu/~feferman/papers/penrose.pdf

(There's much more on Penrose to be found in the same issue of *PSYCHE:* http://journalpsyche.org/ojs-2.2/index.php/psyche/issue/view/116)

For other related discussion see:

GÖDEL, K., 'Some Basic Theorems in the Foundations of Mathematics and Their Philosophical Implications', in his *Collected Works, Vol. III* (Oxford: Oxford University Press, 1995). [This, the "Gibbs Lecture" from 1951 is not easy but is remarkably rich.]

LEWIS, D., 'Lucas against Mechanism', *Philosophy,* 44 (1969), 231-233. Reprinted in Lewis's *Papers in Philosophical Logic* (Cambridge: Cambridge University Press, 1998).

PENROSE, R., *The Emperor's New Mind* (Oxford: Oxford University Press, 1989). [esp. pp. 129-146 & 538-541 – this is Penrose's first shot at extracting philosophical morals from Gödel, in an earlier book.]

SMITH, P., *An Introduction to Gödel's Theorems* (Cambridge: Cambridge University Press, 2007). [§ 28.6 – relates to the argument in Gödel's paper which springs from the *Second* Incompleteness Theorem.]

Philosophical Issues Arising: 2, Is The Notion Of Natural Number Open-Ended?

DUMMETT, M., 'The Philosophical Significance of Gödel's Theorem', *Ratio,* 7 (1963), 140-155. Also in his *Truth and other Enigmas* (London: Duckworth, 1978).

WRIGHT, C., 'About "The Philosophical Significance of Gödel's Theorem": Some Issues', in his *Realism, Meaning and Truth*. 2nd ed. (Oxford: Blackwell, 1993).

MOORE, A., 'More on "The Philosophical Significance of Gödel's Theorem"', in *New Essays on the Philosophy of Michael Dummett*, eds Johannes L. Brandl and Peter M. Sullivan (Grazer philosophische Studien,1998).

GÖDEL'S SECOND INCOMPLETENESS THEOREM

Formal Expositions

GEORGE, A., and D. J. VELLEMAN, *Philosophies of Mathematics* (Oxford: Blackwell, 2001). [ch. 7]

Could be a useful place to start. And for quite a bit more detail see:

SMITH, P., *An Introduction to Gödel's Theorems* (Cambridge: Cambridge University Press, 2007). [chs 24–26]

That should put you in a position to appreciate Boolos's wonderful jeu d'esprit:

BOOLOS, G., 'Gödel's Second Incompleteness Theorem Explained in Words of One Syllable', *Mind,* 103 (1994), 1-3. Also in his *Logic, Logic and Logic* (Cambridge, Massachusetts: Harvard University Press, 1998).

For useful commentary, see:

MOORE, A., 'What Does Gödel's Second Incompleteness Theorem Show?' *Noûs,* 22 (1988): 573-584.

HILBERT'S PROGRAMME

As we'll see, the main philosophical issue arising from Gödel's Second Theorem (at least as far as this paper is concerned) is it's impact on Hilbert's Programme For a very good introduction to Hilbert, see:

GIAQUINTO, M., *The Search for Certainty* (Oxford: Oxford University Press, 2002). [pt IV, chs. 3 & 4. Also available online at http://lib.myilibrary.com/?id=91428]

But do read the man himself:

HILBERT, D., 'On the Infinite', in *Philosophy of Mathematics: Selected Readings*, ed. by P. Benacerraf and H. Putnam (Oxford: Blackwell, 1964; 2nd ed.., 1983). Also in J. van Heijenoort, ed., *From Frege to Gödel: a Source Book in Mathematical Logic, 1879-1931* (Cambridge, Massachusetts: Harvard University Press, 1967).

For further elaboration see also:

GEORGE, A., and D. J. VELLEMAN, *Philosophies of Mathematics* (Oxford: Blackwell, 2001). [ch. 6]

And for further discussion, see:

KREISEL, G., 'Hilbert', in *Philosophy of Mathematics: Selected Readings,* ed. by P. Benacerraf and H. Putnam (Oxford: Blackwell, 1964). [1st ed. only*]*

PARSONS, C., 'Finitism and Intuitive Knowledge', in *The Philosophy of Mathematics Today*, ed. by M. Schirn (Oxford: Oxford University Press, 1998), 249-270.

POTTER, M. D., *Reason's Nearest Kin* (Oxford: Oxford University Press, 2000). [ch. 9. Also online at: www.oxfordscholarship.com]

TAIT, W. W., 'Finitism', *Journal of Philosophy,* 78 (1981): 524-546.

Philosophical Issue Arising: What Was Hilbert's Programme? Do Gödel's Incompleteness Theorems Undermine It?

A standard answer to the second question is given by:

GIAQUINTO, M., *The Search for Certainty* (Oxford: Oxford University Press, 2002). [pt V, ch. 2. Also available online at http://lib.myilibrary.com/?id=91428]

And similarly by:

SMITH, P., *An Introduction to Gödel's Theorems* (Cambridge: Cambridge University Press, 2007). [§§ 28.1–28.5]

For dissent see:

DETLEFSEN, M., 'On an Alleged Refutation of Hilbert', in *Proof, Logic and Formalization*, ed. by M. Detlefsen (London: Routledge, 1991).

For more discussion see;

GENTZEN, G., 'The Concept of Infinity', in *The Collected Papers of Gerhard Gentzen*, ed. by M. E. Szabo (Amsterdam: North-Holland, 1969).

RAATIKAINEN, P., 'Hilbert's Program Revisited', *Synthese* 137 (2003): 157–17 www.mv.helsinki.fi/home/praatika/Hilbert's%20Program%20Revisited.pdf

SIMPSON, S. 'Partial Realizations of Hilbert's Program', *Journal of Symbolic Logic* 53 (1988): 349–363. www.math.psu.edu/simpson/papers/hilbert.pdf [For enthusiasts who want to know something of the afterlife of Hilbert's Programme.]

ZACH, R., 'Hilbert's Program', In E.N. Zalta, ed. *The Stanford Encyclopedia of Philosophy,*2009. Retrieved 17/09/09: http://plato.stanford.edu/entries/hilbert-program/

ZACH, R. 'Hilbert's Program, Then and Now', in *Philosophy of Logic*, ed. by D. Jacquette (Amsterdam: Elsevier, 2006). [Also available from www.ucalgary.ca/~rzach/static/hptn.pdf]

RECURSIVE FUNCTIONS AND COMPUTABILITY

Expositions

What is a Turing computable function? What is a recursive function? Why are they the same class of functions? For explanations, see:

SMITH, P., *An Introduction to Gödel's Theorems* (Cambridge: Cambridge University Press, 2007). Chs 29, 31, 32]

For other alternatives, see:

BOOLOS, G., J. BURGESS and R. C. JEFFREY, *Computability and Logic*. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007). [Chs 3 - 8] Though many think the treatment in the same chapters of the 3rd ed. – when the authors were just Boolos and Jeffrey – is nicer.

CUTLAND, N. J., *Computability* (Cambridge: Cambridge University Press, 1980). [Chs 1–5. A classic book that will appeal to mathematicians]

HAMILTON, A., *Logic for Mathematicians* (Cambridge: Cambridge University Press, 1978). [ch. 7]

ROGERS, H., *Theory of Recursive Functions and Effective Computability* (Cambridge, Massachusetts: MIT Press, 1987). [Another old classic from 1967 which is well worth reading the first chapter of, esp. §§1.1-1.7]

Philosophical issue arising: What is the status of Church's Thesis?

It is a mathematical theorem that a function is Turing computable if and only if it is recursive (if and only if it register computable, if and only if it is Herbrand-Gödel computable, etc.). Different attempts to regiment the intuitive notion of a computable function all converge. Church's Thesis (a.k.a. the Church-Turing Thesis) claims that indeed the *intuitively* computable functions are just the Turing computable/recursive functions.

For some initial clarifications, see:

SMITH, P., *An Introduction to Gödel's Theorems* (Cambridge: Cambridge University Press, 2007, 2009). [ch. 34]

Then read:

SHAPIRO, S., 'Understanding Church's Thesis', *Journal of Phil. Logic* 10 (1981): 353–366. www.springerlink.com/content/u246822601741106/fulltext.pdf

BLACK, R., 'Proving Church's Thesis. *Philosophia Mathematica* 8 (2000): 244–258.

SHAPIRO, S., 'Computability, Proof and Open-Texture', in *Church's Thesis after 70 Years*, ed. by Olszewski et al. (Heusenstaam: Ontos Vergag 2006).

SMITH, P., *An Introduction to Gödel's Theorems* (Cambridge: Cambridge University Press, 2007, 2009). [ch. 35 takes an opinionated minority line]

See also:

COPELAND, B. J., 'Church's Thesis', In E.N. Zalta, ed., *The Stanford Encyclopedia of Philosophy,* http://plato.stanford.edu/entries/church-turing.

MENDELSON, E., 'Second Thoughts About Church's Thesis and Mathematical Proofs', *Journal of Philosophy,* 87 (1990): 225-233.

SMITH, P., Review of in *Church's Thesis after 70 Years* http://bit.ly/RDLSJ

There's an interesting local sub-debate here:

HOGARTH, M., 'Non-Turing Computers and Non-Turing Computability', *Proceedings of the Biennial Meetings of the Philosophy of Science Association,* 1 (1994): 126-138.

HOGARTH, M., 'Deciding Arithmetic Using SAD Computers', *British Journal for the Philosophy of Science,* 55 (2004): 681-91.

BUTTON, T., 'SAD Computers and Two Versions of the Church-Turing Thesis', *British Journal for the Philosophy of Science (2009) 60 (4): 765-792.* [Criticizes Hogarth].

PART C : SET THEORY

ZFC AND CLOSELY RELATED SET THEORIES

Formal expositions

For an informal outline of some key ideas, see:

GEORGE, A., and D. J. VELLEMAN, *Philosophies of Mathematics* (Oxford: Blackwell, 2001). [ch. 3]

WOLF, R. S., *A Tour through Mathematical Logic* (Washington: Mathematical Association of America, 2005). [Ch.2]

The next is a classic and pleasingly slim volume that also introduces some key ideas in a very accessible way:

HALMOS, P., *Naïve Set Theory*, (New York: Springer, 1974).

The opening chapters of the following classic are still very worth reading:

FRAENKEL, A. A.*, et al.*, *Foundations of Set Theory* (Amsterdam: North-Holland, 1958). [Chs. 1, 2]

But two texts stand out.

POTTER, M., *Set Theory and Its Philosophy* (Oxford: Oxford University Press, 2003) [Also online at: http://www.myilibrary.com/?id=75496].

DEVLIN, K., *The Joy of Sets*. 2nd ed. (New York: Springer-Verlag, 1993). [chs 1, 2, 3]

Potter's version of set theory – coming to be known as "the Scott-Potter theory" – is elegant but somewhat non-standard: Devlin is a beautifully written presentation of the standard theory. Here's another modern text that is written in a relaxed style (there are even jokes), and is often extremely helpful in the way it introduces concepts and theorems:

JUST, W. and M. WEESE, *Discovering Modern Set Theory I*, (American Mathematical Society, 1996).

Finally, it is worth following up Fraenkel et al.'s semi-historical review by looking at:

ZERMELO, E., 'Investigations in the Foundations of Set Theory I', in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, ed. by J. van Heijenoort (Cambridge, Massachusetts: Harvard University Press, 1967).

and perhaps also:

FERREIROS, J., 'The Early History of Set Theory' In E.N. Zalta, ed. *The Stanford Encyclopedia of Philosophy.* 2009. Retrieved 11/09/09 from http://plato.stanford.edu/entries/settheory-early/.

Those who find historical stories fascinating – and they illuminate *why* one particular set theory has ended up as the canonical one – can follow up Ferreirós by dipping into at least the first half of the longer story told by:

KANIMORI, A., 'The Mathematical Development of Set Theory from Cantor to Cohen'. *Bulletin of Symbolic Logic* 2 (1996): 1–71. Postcript file www.math.ucla.edu/~asl/bsl/0201/0201-001.ps.

Philosophical Issues Arising: 1, Set Theory As A Foundation For Mathematics

In what sense can we say that set theory "provides a foundation for" mathematics?

GIAQUINTO, M., *The Search for Certainty* (Oxford: Oxford University Press, 2002). [Pt. I; also pt. V, §1,§2. Also available online at: http://lib.myilibrary.com/?id=91428]

Gives an introductory discussion of what set theory is supposed to do for us. The set theory texts above have things to say as they go along. For further discussion see:

MADDY, P., *Naturalism in Mathematics* (Oxford: Oxford University Press, 1997). [ch. 2. Also available from www.oxfordscholarship.com]

MAYBERRY, J., 'What Is Required of a Foundation for Mathematics?' *Philosophia Mathematica,* 3 (1994), 16-35. Also in Dale Jacquette, ed., *Philosophy of Mathematics: An anthology* (Oxford: Blackwell, 2002).

There are some subversive remarks too in:

OLIVER, A. & T. SMILEY 'What are sets and what are they for?' *Philosophical Perspectives, 20, Metaphysics* (2006): 123–155.

Philosophical Issues Arising: 2, What Conception Of Sets Is Supposed To Be Reflected In Standard Set Theories? Does That Conception Justify The Axioms?

BOOLOS, G., 'The Iterative Concept of Set', *Journal of Philosophy,* 68 (1971), 215-231. Also in his *Logic, Logic and Logic* (Cambridge, Massachusetts: Harvard University Press, 1998); and in P. Benacerraf & H. Putnam, eds, *Philosophy of Mathematics: selected readings*, 2nd ed. only (Cambridge: Cambridge University Press, 1983).

BOOLOS, G., 'Iteration Again', *Philosophical Topics,* 17 (1989), 5-21. Also in his *Logic, Logic and Logic* (Cambridge, Massachusetts: Harvard University Press, 1998).

PARSONS, C., 'What Is the Iterative Conception of Set?' in his *Mathematics in Philosophy: Selected Essays* (Ithaca, New York: Cornell University Press, 1983). Also in P. Benacerraf & H. Putnam, eds, *Philosophy of Mathematics: Selected Readings*, 2nd ed. only (Cambridge University Press, 1983).

See also:

FORSTER, T., 'The Iterative Conception of Set' *Review of Symbolic Logic* 1 (2008): 97–110.

GÖDEL, K., 'What Is Cantor's Continuum Problem?' *American Mathematical Monthly,* 54 (1947), 515-525. Also in his *Collected Works*, Vol. II (Oxford: Oxford University Press, 1990); and in P. Benacerraf & H. Putnam, eds, *Philosophy of Mathematics: selected readings* (Oxford: Blackwell, 1964; 2nd ed., Cambridge: Cambridge University Press, 1975).

PASEAU, A., 'Boolos on the Justification of Set Theory', *Philosophia Mathematica* 15 (2007): 30–53.

POTTER, M., 'Iterative Set Theory', *Philosophical Quarterly,* 43 (1993), 178-193.

WANG, H., 'The Concept of Set', in his *From Mathematics to Philosophy* (London: Routledge & Kegan Paul, 1974) [Ch. 6]. Reprinted in P. Benacerraf & H. Putnam, eds, *Philosophy of Mathematics: Selected Readings* (2nd ed., Cambridge: Cambridge University Press, 1975).