Pass it on, . . . That’s the game I want you to learn. Pass it on.
Alan Bennett, The History Boys

The latest version of this Guide can always be downloaded from
logicmatters.net/students/tyl/

URLs in blue are live links to external web-pages or PDF documents. Internal
cross-references to sections etc. are also live. But to avoid the text being covered in
ugly red rashes these internal links are not coloured – if you are reading onscreen, your
cursor should change as you hover over such live links: try here.
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Version history

I realize that it does seem a bit pretentious to talk of a ‘version history’! But on the other hand, if you have seen some earlier incarnation(s) of this Guide, you might well want to know what’s changed so you can skim through to the new bits. Here then are some notes about how things are evolving, starting from Version 9.0, when the Guide first took something like its current form. New readers should just go straight to the Introduction.

Version 10.0: mid-March 2014 The increment in the main version number marks that the structure of the Guide has significantly changed again. The key section on basic first-order logic at the beginning of the mathematical logic chapter was getting more and more sprawling: it has now been hived off into a separate chapter, divided into sections, and further expanded. My sense is that quite a few readers are particularly interested in getting advice on this first step after baby logic, so nearly all the effort in this particular revision of the Guide has been concentrated on making improvements here.

Version 9.4: mid-December 2013 There has been another quite significant restructuring of the Guide. The material in the old Chapter 2 on The Basics has been re-ordered, and is now split into two chapters, one on the classical mathematical logic mainstream curriculum, the other on variant logics (which are mostly likely to be of interest to philosophers). Hopefully, this further chunking up of the Guide will make it look a smidgin less daunting! And by more clearly demarcating topics in the standard mathematical logic curriculum from supplementary topics, it should be easier for readers with different interests to find their way around.

Also new in this version, there are some overdue comments on Peter Johnstone’s Notes on Logic and Set Theory (good), Keith Devlin’s The Joy of Sets (good), Theodore Sider’s Logic for Philosophy (not so good), and Richard Hodel’s An Introduction to Mathematical Logic (patchy).

Version 9.3: mid-November 2013 There are additions to §2.8 on Beginning Set Theory. In the Appendix, there is now a note on Alexander Prestell and Charles N. Delzell’s Mathematical Logic and Model Theory: A Brief Introduction (1986), and a longer entry on Thomas Forster’s Logic, Induction and Sets (2003). But the major addition to the Appendix, which also leads to some changes and additions in earlier parts of the Guide, is a belated substantial entry on Peter G. Hinman’s doorstopper of a book, Fundamentals of Mathematical Logic (2005).

Version 9.2: September 2013 External links in this document are now coloured blue (internal cross-references are live links too, but for aesthetic reasons aren’t also coloured). Apart from minor tidying, and a brief new note on Tourlakis’s Theory of Computation (2012), the main changes are entries in the ‘Big Books’ Appendix on van Dalen (1980/2012) and Hedman (2004), and some consequent earlier changes.
Version 9.1: June 2013  There are minor revisions in phrasing scattered throughout. §1.2 has been rearranged and a suggestion added: §1.4 on Velleman’s book is new. Substantial entries have been added to the ‘Big Books’ Appendix on Enderton (1972, 2002), Leary (2000) and Chiswell and Hodges (2007). These new entries have in turn led to a number of changes earlier, in particular in §2.2 where those books are initially mentioned. There are brief new entries too in §§4.3.2, 5.3, 5.8.1.

Version 9.0: April 2013  The Guide has been significantly restructured, hence the jump in Version number. The old Introduction has been split into a shorter Introduction and a new Chapter 1, which now has a section mapping out how the field of logic divides into areas and also has an expanded section on the ‘baby logic’ which the rest of the Guide assumes philosophers will already know.

Chapter 2, on the basics, is largely unchanged. The order of the next two segments of the Guide has been inverted: The ‘Big Books’ chapter now comes last, relabelled as an Appendix (but otherwise also unchanged in this version). But ‘Exploring further’, now Chapter 3, has been greatly expanded by about 15 pages and is in its first reasonably complete(?) form.

Links to external web-pages are now live.
Chapter 0

Introduction

Note to new readers If you are hoping for help with really introductory logic (e.g. as encountered by philosophers in their first-year courses), then I’m afraid this Guide isn’t designed for you. The only section that pertains to ‘baby logic’ is §1.2; all the rest is about rather more advanced – and eventually very much more advanced – material.

0.1 Why this Guide for philosophers?

It is an odd phenomenon, and a rather depressing one too. Serious logic is seemingly taught less and less, at least in UK philosophy departments, even at graduate level. Yet logic itself is, of course, no less an exciting and intrinsically rewarding subject than it ever was, and the amount of good formally-informed work in philosophy is ever greater as time goes on. Moreover, logic is far too important to be left entirely to the mercies of technicians from maths or computer science departments with different agendas (who often reveal an insouciant casualness about conceptual details that will matter to the philosophical reader).

So how is a competence in logic to be passed on if there are not enough courses, or are none at all? It seems that many beginning graduate students in philosophy – if they are not to be quite dismally uneducated in logic and so be cut off from working in some of the most exciting areas of their discipline – will need to teach themselves from books, either solo or (much better) by organizing their own study groups. That’s perhaps no real hardship, as there are some wonderful books out there. But what to read? Logic books can have a very long shelf life, and one shouldn’t at all dismiss older texts: so there’s more than a fifty year span of publications to select from. I have just counted some three hundred formal logic books of one kind or another on my own shelves – and of course these are only a selection.

Philosophy students evidently need a Study Guide if they are to find their way around the available literature old and new: this is my (on-going, still developing) attempt to provide one.

0.2 Why this Guide for mathematicians too?

The situation of logic teaching in mathematics departments can also be pretty dire. Indeed there are full university maths courses in good UK universities with precisely zero courses on logic or set theory (let alone e.g. category theory), and I believe that the
situation is equally patchy in many other places.

So again, if you want to teach yourself some logic, where should you start? What are the topics you might want to cover? What textbooks are likely to prove accessible and tolerably enjoyable and rewarding to work through? Again, this Guide – or at least, the sections on the core mathematical logic curriculum – will give you some pointers.

True enough, this is written by someone who has – apart from a few guest mini-courses – taught in a philosophy department, and who is no research mathematician. Which probably gives a distinctive tone to the Guide and certainly explains why it ranges into areas of logic of special interest to philosophers. Still, mathematics remains my first love, and these days it is mathematicians whom I get to hang out with. A large number of the books I recommend are very definitely paradigm mathematics texts. So I shouldn’t be leading you too astray.

0.3 Choices, choices

So what has guided my choices of what to recommend in this Guide?

Different people find different expository styles congenial. For example, what is agreeably discursive for one reader is irritatingly verbose and slow-moving for another. For myself, I do particularly like books that are good on conceptual details and good at explaining the motivation for the technicalities while avoiding needless complications or misplaced ‘rigour’, though I do like elegance too. Given the choice, I tend to prefer a treatment that doesn’t rush too fast to become too general, too abstract, and thereby obscures underlying ideas.

The following choices of books no doubt reflect these tastes. But overall, I don’t think that I have been downright idiosyncratic. Nearly all the books I particularly recommend will very widely be agreed to have significant virtues (even if some logicians would have different preference-orderings).

0.4 The shape of the Guide

My conception of the character and shape of the Guide has evolved (or at least, it has mutated) as I’ve gone along, and it does show signs of continuing to grow rather alarmingly. The latest version can always be found at www.logicmatters.net/students/tyl.

The Guide is now divided into four main Chapters and an Appendix:

- The short Chapter 1 says something introductory about the lie of the land, indicating the various subfields of logic comprising both the traditional ‘mathematical logic’ curriculum and also noting other areas such as modal logic which are perhaps of special interest to some philosophers. This initial map of the territory explains why Chapters 2, 3 and 4 are structured as they are.

Chapter 1 also suggests some preliminary reading; this is aimed at those with little or no mathematical background who might benefit from a gentle warm-up before tackling the recommendations in later chapters. This reading may indeed already provide philosophers with enough for their purposes. I certainly think that any philosopher should know this little amount, but these modest preliminaries are not really the topic of this Guide. Our concern is with the ‘serious stuff’ that lies beyond, of somewhat more specialized interest.
Chapters 2 and 3 will then be, for many readers, the core chapters of the Guide. Chapter 2 covers basic first-order logic (at a level a step up from what philosophers will have met in baby logic courses). Nearly everything later depends on this, so it is absolutely imperative to really get on top of this material.

Chapter 3 looks at more-or-less-introductory readings on each of the remaining basic subfields of mathematical logic (at, roughly speaking, an advanced undergraduate or beginning graduate student level for philosophers, or equivalently at a middling-to-upper undergraduate level for mathematicians).

Most of Chapter 4 on ‘Variant Logics’ will be primarily of interest to philosophers. Do note that a number of the topics, like modal logic, are in fact quite approachable even if you know fairly little logic. So the fact that the material in this chapter is mentioned after the mathematical logic topics in the previous two chapters does not reflect a jump in difficulty: far from it.

The introductory books mentioned in the Chapters 2, 3 and 4 already contain numerous pointers to further reading, enough to put you in a position to continue exploring solo. But Chapter 5 adds more suggestions for narrower-focus forays deeper into various subfields, ‘Exploring Further’ by looking at more advanced work, or at work on topics outside the basic menu. This is probably mostly for specialist graduate students (among philosophers) and final year undergraduate/beginning graduate students (among mathematicians).

Appendix A is by way of a supplement, considering some of ‘The Big Books on mathematical logic’. These are typically broader-focus books that cover the basic first-order logic reviewed in Chapter 2 together with one or more subfields from the further menu of mathematical logic at around the level reached in the Chapter 3 readings (or going perhaps rather beyond). Sensibly choosing among them, some of these books can provide very useful consolidating/amplifying reading.

This final chapter is slowly growing into what will probably become a rather long list of mini-reviews. Partly it is written for my own satisfaction, as I occasionally revisit some old friends and take a closer look at other books that have sat unregarded on my shelves. But I’ve also been spurred on by readers: initial versions of this Guide prompted quite a few enquiries of the kind ‘Do you recommend X?’, or ‘We are using Y as a course text: I don’t like it that much, what do you think?’. The Appendix is often much more critical in tone than the previous chapters.

The geographical map in §1.1, together with the overall table of contents for the Guide, should give you a good sense of how I am chunking up the broad field of logic into subfields (in a fairly conventional way). Of course, even the horizontal divisions into different areas can in places be a little arbitrary. And the vertical division between the entry-level readings in earlier chapters and the further explorations in Chapter 5 is evidently going to be a lot more arbitrary. I think that everyone will agree (at least in retrospect!) that e.g. the elementary theory of ordinals and cardinals belongs to the basics of set theory, while explorations of ‘large cardinals’ or independence proofs via forcing are decidedly advanced. But in most areas, there are far fewer natural demarcation lines between the basics and more advanced work.

Still, it is surely very much better to have some such structuring than to heap everything together. And I hope the vertical divisions between chapters will help to make the size of this ever-growing Guide seem quite a bit less daunting.

In fact, you should probably stick to Chapters 2 and 3 and maybe (depending on your interests) parts of Chapter 4 at the outset – they cover the sorts of logical topics that
should be readily accessible to the enthusiastic logic-minded philosophy undergraduate or beginning graduate student, and accessible a year or two earlier in their education to mathematicians.

You can then proceed in two ways. As indicated, Chapter 5 is there in case you find yourself captivated by some particular topic(s) in logic, and want to pursue things further (or investigate new topics). But – not an exclusive alternative, of course! – you might well want to review and consolidate some of the earlier material by looking at two or three of the big single (or jointly) authored multi-topic books which aim to cover some or all of the basic standard menu of courses in mathematical logic: Appendix A reviews some of the options (and mentions a few other books besides).

Within sections in Chapters 2–5, I have put the main recommendations into what strikes me as a sensible reading order of increasing difficulty (without of course supposing you will want to read everything!). And some further books are listed in asides or postscripts. Again, the fine details of the orderings, and the decisions about what to put in asides or postscripts, can all no doubt be warmly debated: people really do get quite heated about this kind of thing.

0.5 A general reading strategy

I very strongly recommend tackling an area of logic (or indeed any new area of mathematics) by reading a series of books which overlap in level (with the next one covering some of the same ground and then pushing on from the previous one), rather than trying to proceed by big leaps.

In fact, I probably can’t stress this advice too much (which is why I am highlighting it here in a separate section). For this approach will really help to reinforce and deepen understanding as you re-encounter the same material from different angles, with different emphases. The multiple overlaps in coverage in the lists below are therefore fully intended, and this explains why the lists are longer rather than shorter. You will certainly miss a lot if you concentrate on just one text in a given area, especially at the outset. Cultivate the habit of judicious skipping and skimming so that you can read enough in the end to build up a good overall picture of an area seen from various angles.

0.6 Other preliminary points

(a) Mathematics is not merely a spectator sport: you should try some of the exercises in the books as you read along to check and reinforce comprehension. On the other hand, don’t obsess about doing exercises if you are a philosopher – understanding proof ideas is very much the crucial thing, not the ability to roll-your-own proofs. And even mathematicians shouldn’t get too hung up on routine exercises (unless you have specific exams to prepare for!): concentrate on the exercises that look interesting and/or might deepen understanding.

Do note however that some authors have the irritating(?) habit of burying important results in the exercises, mixed in with routine homework, so it is often worth skimming through the exercises even if you don’t plan to tackle many of them.

(b) Nearly all the books mentioned here should be held by any university library which has been paying reasonable attention to maintaining a core collection in the area of logic (and any book should be borrowable through your interlibrary loans system). Certainly,
if some recommended books from Chapters 2, 3 and 4 are not on the shelves, do make sure your library orders them (in my experience, university librarians – overwhelmed by the number of publications, strapped for cash, and too familiar with the never-borrowed purchase – are only too happy to get informed recommendations of books which are reliably warranted actually to be useful and used).

Since I’m not assuming you will be buying personal copies, I have not made cost or even being currently in print a significant consideration: indeed it has to be said that the list price of some of the books is just ridiculous. (Though second-hand copies of some books at better prices might be available via Amazon sellers or from abebooks.com.) However, I have marked with one star* books that are available at a reasonable price (or at least are unusually good value for the length and/or importance of the book), and marked with two stars** those books for which e-copies are freely (and legally!) available and URLs are provided here.¹ Most articles can also be downloaded, again with URLs supplied.

(c) And yes, the references here are very largely to published books rather than to on-line lecture notes etc. Many such notes are excellent, but they tend to be a bit terse (as befits material intended to support a lecture course) and so perhaps not as helpful as fully-worked-out book-length treatments for students needing to teach themselves. But I’m sure that there is an increasing number of excellent e-resources out there which do amount, more or less, to free stand-alone textbooks. I’d be very happy indeed to get recommendations about the best.

(d) Finally, the earliest versions of this Guide kept largely to positive recommendations: I didn’t pause to explain the reasons for the then absence of some well-known books. This was partly due to considerations of length which have now quite gone by the wayside; but also I wanted to keep the tone enthusiastic, rather than to start criticizing or carping.

However, as I’ve already noted, enough people have asked what I think about the classic X, or asked why the old warhorse Y wasn’t mentioned, to change my mind. So I have occasionally added – especially in Appendix A – some reasons why I don’t particularly recommend certain books. I should perhaps emphasize that these relatively negative assessments are probably more contentious than the positive ones.

¹Note that some print-on-demand titles can be acquired for significantly less than the official publisher’s price – check Amazon sellers. We will have to pass over in silence the issue of illegal file-sharing of PDFs of e.g. out-of-print books: most students will know the possibilities here.
Before we start

1.1 Logical geography

‘Logic’ is a big field. It is of concern to philosophers, mathematicians and computer scientists. Different constituencies will be particularly interested in different areas and give different emphases. For example, modal logic is of considerable interest to some philosophers, but also parts of this sub-discipline are of concern to computer scientists too. Set theory (which falls within the purview of mathematical logic, broadly understood) is an active area of research interest in mathematics, but – because of its (supposed) foundational status – is of interest to philosophers too. Finite model theory is of interest to mathematicians and computer scientists, but perhaps not so much to philosophers. Type theory started off as a device of philosophy-minded logicians looking to avoid the paradoxes: it has become primarily the playground of computer scientists. And so it goes.

In this Guide, I’m going to have to let the computer scientists largely look after themselves! Our focus is going to be the areas of logic of most interest to philosophers and mathematicians, if only because that’s what I know a little about. Here, then, is a very rough-and-ready initial map to the ground surveyed in Chapters 2, 3 and 4. We start with fundamentals at the top. Then the left hand branch highlights some areas that are likely to be of interest to philosophers and (mostly) not so much to mathematicians: the right hand part covers the traditional ‘Mathematical Logic’ curriculum in advanced (graduate level?) philosophy courses and not-so-advanced mathematics courses.

(Assumed background: ‘Baby logic’)

First order logic

- Modal logic
- Second order logic
- More model theory

Classical variations

- Non-classical variations

Computability, incompleteness

Beginning set theory

6
The dashed lines between the three blocks on the left indicate that they are roughly on
the same level and can initially be tackled in any order; likewise for the three blocks on
the right.

In a bit more detail (though don’t worry at all if at this stage you don’t understand
much of the jargon):

• By ‘Baby logic’, which I say just a little more about in the next section of this
chapter, I mean the sort of thing that is in elementary formal logic courses for
philosophers or is perhaps covered in passing in a discrete mathematics course:
truth-functional logic (the truth-table test for validity, usually a first look at some
formal proof system as well), an introduction to the logic of quantifiers (giving
at least a familiarity with translation to and from quantified wffs and an infor-
mal initial understanding of the semantics for quantifiers), and perhaps also an
introduction to set theoretic notation and the ideas of relations, functions, etc.

• ‘First-order logic’ means a reasonably rigorous treatment of quantification theory,
studying both a proof-system (or two) for classical logic, and the standard classical
semantics, getting as far as a soundness and completeness proof for your favourite
proof system, and perhaps taking a first look at e.g. the Löwenheim-Skolem theo-
rems, etc. This is covered in Chapter 2.

• After briefly looking at second-order logic, we go down the right-hand column, to
take in the three substantial elements (after core first-order logic) which go – in
various proportions – to make up the traditional ‘Mathematical Logic’ curriculum
and which form the topics of Chapter 3. First, and pretty continuous with our
initial study of first-order logic, there is just a little more model theory, i.e. a little
more exploration of the fit between theories cast framed in formal languages and
the structures they are supposed to be ‘about’.

• Next there’s an introduction to the ideas of mechanical computability and decid-
ability, and proofs of epochal results such as Gödel’s incompleteness theorems.
(This is perhaps the most readily approachable area of mathematical logic.)

• Then we start work on set theory.

• Now we move on to the left-hand column of logic-mainly-for-philosophers, covered
in Chapter 4. Even before encountering a full-on-treatment of first-order logic,
philosophers are often introduced to modal logic (the logic of necessity and pos-
sibility) and to its ‘possible world semantics’. You can indeed do an amount of
propositional modal logic with no more than baby logic as background.

• Further down that left-hand column we meet some variations/extensions of stan-
dard logic which are of conceptual interest but which are still classical in favour
(e.g. free logic, plural logic).

• Then there are non-classical variations, of which perhaps the most important – and
the one of interest to mathematicians too – is intuitionist logic (which drops the law
of excluded middle, motivated by a non-classical understanding of the significance
of the logical operators). We could also mention e.g. relevant logics which drop the
classical rule that a contradiction entails anything.

To repeat, don’t be alarmed if (some of) those descriptions are at the moment pretty
opaque to you: we will be explaining things a little more as we go through the Guide.
And as I said, you probably shouldn’t take the way our map divides logic into its fields
too seriously. Still, the map should be a helpful initial guide to the structure of the next three chapters of this Guide.

1.2 Assumed background: ‘baby logic’

If you are a mathematician, you should probably just dive straight into the next chapter on the basics of first-order logic without further ado.

If you are a philosopher without a mathematical background, however, you will almost certainly need to have already done some formal logic if you are going to cope. And if you have only taken a course using some really, really, elementary text book like Sam Guttenplan’s The Languages of Logic, Howard Kahane’s Logic and Philosophy, or Patrick Hurley’s Concise Introduction to Logic, then you might struggle (though these things are hard to predict). If you do only have such a very limited base to work from, it might be a good idea to take a look at some more substantial but still introductory book before pursuing things any further with serious logic.

Here then are couple of initial suggestions of books that start from scratch again but do go far enough to provide a good foundation for further work:

1. My Introduction to Formal Logic* (CUP 2003, corrected reprint 2013): for more details see www.logicmatters.net/ifl, where there are also answers to the exercises). This is intended for beginners, and was the first year text in Cambridge for a decade. It was written as an accessible teach-yourself book, covering propositional and predicate logic ‘by trees’. But it gets as far as for a completeness proof for the tree system of predicate logic without identity.

2. Paul Teller’s A Modern Formal Logic Primer** (Prentice Hall 1989) predates my book, is now out of print, but is freely available online at tellerprimer.ucdavis.edu, which makes it unbeatable value! It is in many ways quite excellent, and had I known about it at the time (or listened to Paul’s good advice, when I got to know him, about how long it takes to write an intro book), I’m not sure that I’d have written my own book, full of good things though it is! As well as introducing trees, Teller also covers a version of ‘Fitch-style’ natural deduction (I didn’t have the page allowance to do this, regrettably). Like me, he also gets as far as a completeness proof. Notably user-friendly. Answers to exercises are available at the website.

Of course, those are just two possibilities from very many. I have not latterly kept up with the seemingly never-ending flow of books of entry-level introductory logic books, some of which are no doubt excellent too, though there are also some poor books out there. Mathematicians should be particularly warned that some of the books on ‘discrete mathematics’ cover elementary logic pretty badly. And there are also more books like Guttenplan’s that will probably not get philosophers to the starting line as far as this Guide is concerned. This is not the place, however, to discuss lots more options for elementary texts. I will mention here just two other books:

3. I have been asked a number of times about Jon Barwise and John Etchemendy’s widely used Language, Proof and Logic (CSLI Publications, 1999: 2nd edition 2011 – for more details, see http://ggweb.stanford.edu/lpl/toc). The unique selling point for the book is that it comes with a CD of programs, including a famous one by the authors called ‘Tarski’s World’ in which you build model worlds and can query whether various first-order sentences are true of them. Some students really like it, and but at least equally many don’t find this kind of thing particularly useful.
And I believe that the CD can’t be registered to a second owner, so you have to buy the book new to get the full advantage.

On the positive side, this is another book which is in many respects user-friendly, goes slowly, and does Fitch-style natural deduction. It is a respectable option. But Teller is rather snappier, and no less clear, and wins on price!

4. Nicholas Smith’s recent Logic: The Laws of Truth (Princeton UP 2012) is very clearly written and seems to have many virtues. The first two parts of the book overlap very largely with mine (it too introduces logic by trees). But the third part ranges wider, including natural deduction – indeed the coverage goes as far as Bostock’s book, mentioned below in \S 2.2. It seems a particularly readable addition to the introductory literature. Answers to exercises can be found at http://www-personal.usyd.edu.au/~njjsmith/lawsoftruth/

1.3 Do you really need more logic?

This section is for philosophers! It is perhaps worth pausing to ask such readers if they are sure – especially if they already worked through a book like mine or Paul Teller’s – whether they really do want or need to pursue things much further, at least in a detailed, formal, way. Far be it from me to put people off doing more logic: perish the thought! But for many purposes, you can survive by just reading the likes of

1. David Papineau, Philosophical Devices: Proofs, Probabilities, Possibilities, and Sets (OUP 2012). From the publisher’s description: ‘This book is designed to explain the technical ideas that are taken for granted in much contemporary philosophical writing, … The first section is about sets and numbers, starting with the membership relation and ending with the generalized continuum hypothesis. The second is about analyticity, a prioricity, and necessity. The third is about probability, outlining the difference between objective and subjective probability and exploring aspects of conditionlization and correlation. The fourth deals with metalogic, focusing on the contrast between syntax and semantics, and finishing with a sketch of Gödel’s theorem.’

Or better – since perhaps Papineau gives rather too brisk an overview – you could rely on

2. Eric Steinhart, More Precisely: The Math You Need to Do Philosophy (Broadview 2009) The author writes: ‘The topics presented … include: basic set theory; relations and functions; machines; probability; formal semantics; utilitarianism; and infinity. The chapters on sets, relations, and functions provide you with all you need to know to apply set theory in any branch of philosophy. The chapter of machines includes finite state machines, networks of machines, the game of life, and Turing machines. The chapter on formal semantics includes both extensional semantics, Kripkean possible worlds semantics, and Lewisian counterpart theory. The chapter on probability covers basic probability, conditional probability, Bayes theorem, and various applications of Bayes theorem in philosophy. The chapter on utilitarianism covers act utilitarianism, applications involving utility and probability (expected utility), and applications involving possible worlds and utility. The chapters on infinity cover recursive definitions, limits, countable infinity, Cantor’s diagonal and power set arguments, uncountable infinities, the aleph and beth numbers, and definitions by transfinite recursion. More Precisely is designed both as a
text book and reference book to meet the needs of upper level undergraduates and graduate students. It is also useful as a reference book for any philosopher working today.’

Steinhart’s book is admirable, and will give many philosophers a handle on some technical ideas going beyond baby logic and which they really should know just a little about, without the hard work of doing a ‘real’ logic course. What’s not to like?

1.4 How to prove it

Assuming, however, that you do want to learn more serious logic, before getting down to business in the next chapter, let me mention one other – rather different and often-recommended – book:

Daniel J. Velleman, *How to Prove It: A Structured Approach* (CUP, 2nd edition, 2006). From the Preface: ‘Students of mathematics … often have trouble the first time that they’re asked to work seriously with mathematical proofs, because they don’t know ‘the rules of the game’. What is expected of you if you are asked to prove something? What distinguishes a correct proof from an incorrect one? This book is intended to help students learn the answers to these questions by spelling out the underlying principles involved in the construction of proofs.’

There are chapters on the propositional connectives and quantifiers, and informal proof-strategies for using them, and chapters on relations and functions, a chapter on mathematical induction, and a final chapter on infinite sets (countable vs. uncountable sets). This truly excellent student text could well be of use both to some (many?) philosophers and to some mathematicians reading this Guide. By the way, if you want to check your answers to exercises, you will find a long series of blog posts starting at http://technotes-himanshu.blogspot.co.at/2010/04/how-to-prove-it-intro-exercises.html useful.

True, if you are a mathematics student who has got to the point of embarking on an upper level undergraduate course in some area of mathematical logic, you should certainly have already mastered nearly all the content of Velleman’s splendidly clear book. However, an afternoon or two skimming through this text (except perhaps for the very final section), pausing over anything that doesn’t look very comfortably familiar, could still be time extremely well spent.

What if you are a philosophy student who (as we are assuming) has done some baby logic? Well, experience shows that being able to handle *formal* tree-proofs or natural deduction proofs doesn’t always translate into being able to construct good *informal* proofs. For example, one of the few meta-theoretic results usually met in a baby logic course is the expressive completeness of the set of formal connectives \{∧, ∨, ¬\}. The (informal!) proof of this result is really easy, based on a simple proof-idea. But many students who can ace the part of the end-of-course exam asking for quite complex *formal* proofs find themselves all at sea when asked to replicate this *informal* bookwork proof. Another example: it is only too familiar to find philosophy students introduced to set notation not even being able to make a start on a good informal proof that \{(a), (a, b)\} = \{(a′), (a′, b′)\} if and only if \(a = a'\) and \(b = b'\).

Well, if you are one of those students who jumped through the formal hoops but were unclear about how to set out elementary mathematical proofs (from the ‘meta-theory’ theory of baby logic, or from baby set theory), then again working through Velleman’s book from the beginning could be just what you need to get you prepared for the serious
study of logic. And even if you were one of those comfortable with the informal proofs, you will probably still profit from skipping and skimming through (perhaps paying especial attention to the chapter on mathematical induction).
Chapter 2

First-order logic

And so, at long last, down to work! This chapter begins with

2.1 A very brief outline of what you need to know about the fundamentals of first-order logic (FOL).

We then continue with . . .

2.2 The main recommendations of books for starting first-order logic.

2.3 Some other positive recommendations, with a postscript of some more negative comments on other books.

We will take this slowly, in some detail, as obviously it is very important to get these foundations securely in place before doing moving on to anything more exotic.

2.1 Getting to grips with first-order logic

A reminder for philosophers: We are assuming that you have already done some baby logic (e.g. you have encountered quantifiers before, know how to translate in and out of quantifier-variable notation, maybe have encountered a proof-system). There are now four tasks: (i) To nail down more securely your initial understanding of FOL. (ii) To push on as far as a completeness proof for some specific deductive system. (iii) To develop an initial appreciation of how other different styles of logical proof-systems work. (iv) To get used to the more mathematical mode of presentation that you find in books that go beyond baby logic.

A reminder for mathematicians: You have probably already seen fragments of logical notation (and maybe a bit more) scattered through elementary mathematics courses, and you have met a few logical ideas, but haven’t yet done a logic course. So there are now two tasks: (i’) To work through a first systematic treatment of FOL. (ii’) Looking sideways from your initial main textbook, to develop an initial appreciation of how alternative styles of logical proof-systems work.

Both philosophers and mathematicians will probably already have a rough idea of what these tasks are likely to involve, given their background (however fragmentary) in baby logic. But it may be worth spelling things out just a little – though don’t worry if at present you don’t fully grasp the import of all of these bullet points:

• Starting with syntax, you obviously need to know how first-order languages are constructed. And now you need to be able to prove things that you previously
took for granted, e.g. that ‘bracketing works’, meaning that every wff has a unique parsing (a unique construction tree).

Two variants to look out for: some syntaxes typographically distinguish variables from ‘parameters’; some syntaxes not only have a primitive absurdity sign \( \bot \) but take negation to be defined rather than primitive (by putting \( \neg \varphi \overset{\text{def}}{=} \varphi \rightarrow \bot \)).

Note too that baby logic courses for philosophers often ignore functions; but given that FOL is deployed to regiment everyday mathematical reasoning, and that functions are crucial to mathematics, function expressions now become centrally important.

- On the semantic side, you need to understand the idea of first-order structures and the idea of an interpretation of a language in such a structure. You’ll need, of course, a proper formal semantic story with the bells and whistles required to cope with quantifiers adequately (I say ‘a story’ as again there are variants on the market).\(^1\) Once you know what it takes for sentences to be true on an interpretation, you’ll be able to give a formal account of, and prove results about, semantic entailment: \( \Gamma \models \varphi \) if every interpretation which makes the sentences in \( \Gamma \) true makes \( \varphi \) true. (Another variation to look out for: does your story extend to allowing the semantic entailment relation to apply to non-sentences, wffs with variables dangling free?)

- Back to syntax: you need to get to know one deductive proof-system for FOL reasonably well, and you ought ideally (and at quite an early stage) acquire at least a glancing acquaintance with some alternatives. But where to concentrate your efforts? It is surely natural to give centre stage, at least at the outset, to so-called natural deduction systems.

The key feature of natural deduction systems is that they allow us to make temporary assumptions ‘for the sake of argument’ and then later discharge them (as we do all the time in everyday reasoning, mathematical or otherwise). Different formal natural deduction systems then offer different ways of handling temporary assumptions and of showing where in the argument they get discharged. Suppose, for example, we want to show that from \( \neg(P \land \neg Q) \) we can infer \( P \rightarrow Q \) (where \( \neg \), \( \land \) and \( \rightarrow \) are of course our symbols for, respectively, not, and, and [roughly] implies.) Then one way of laying out a natural deduction proof would be like this (a so-called Fitch-style layout) – where we have added informal commentary on the right:

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \neg(P \land \neg Q) )</td>
<td>premiss</td>
</tr>
<tr>
<td>2</td>
<td>( P )</td>
<td>supposition for the sake of argument</td>
</tr>
<tr>
<td>3</td>
<td>( \neg Q )</td>
<td>additional supposition for the sake of argument</td>
</tr>
<tr>
<td>4</td>
<td>( P \land \neg Q )</td>
<td>from 2, 3 by and-introduction</td>
</tr>
<tr>
<td>5</td>
<td>( \bot )</td>
<td>marking that 1 and 4 give us a contradiction</td>
</tr>
<tr>
<td>6</td>
<td>( \neg \neg Q )</td>
<td>supposition 3 must be false, given 1 and 2, by reductio</td>
</tr>
<tr>
<td>7</td>
<td>( Q )</td>
<td>from 6, eliminating the double negation</td>
</tr>
<tr>
<td>8</td>
<td>( P \rightarrow Q )</td>
<td>if the supposition 2 is true, so is 7.</td>
</tr>
</tbody>
</table>

\(^1\)For cognoscenti: the stories vary along two dimensions. Do we augment an interpretation of the language by assigning (temporary) values to the variables already in the language or to new constants added to the language? Do we assign such values to all the variables/new constants at once, or only one at a time as needed?
Here, the basic layout principle is that, whenever we make a new temporary assumption we indent the line of argument a column to the right (vertical bars marking the indented columns), and whenever we discharge an assumption at the top of an indented column we move back to the left.

An alternative layout (going back to Genzten) would display the same proof like this, where the guiding idea is (roughly speaking) that a wff below an inference line follows from what is immediately above it:

\[
\begin{array}{c}
(P \land \neg Q)^{(2)} \\
\hline
(P \land \neg Q) \\
\hline
\neg \neg Q^{(1)} \\
\hline
Q^{(2)} \\
\hline
(P \rightarrow Q)
\end{array}
\]

Here labels are used to indicate where a supposition gets discharged, and the supposition to be discharged gets bracketed off and also given the corresponding label.

Fitch-style proofs are perhaps easier to use for beginners (indeed, we might say, more natural, by virtue of more closely follow the basically linear style of ordinary reasoning). But Gentzen-style proofs are preferred for more advanced work, and that’s what the natural deduction texts that I’ll be mentioning use.

• Next – and for philosophers this is likely to be the key move beyond their baby logic course, and for mathematicians this is probably the point at which things get interesting – you need to know how to prove a soundness and a completeness theorem for your favourite deductive system for first-order logic. That is to say, you need to be able to show that there’s a deduction in your chosen system of a conclusion from given premisses only if the inference is indeed semantically valid (the system doesn’t give false positives), and whenever an inference is valid there’s a formal deduction of the conclusion from the premisses (the system captures all the semantically valid inferences).

• As an optional bonus at this stage, it could be good to have a first glimpse of e.g. the (downward) Löwenheim-Skolem theorem and the compactness theorem, i.e. to have an initial engagement with some initial results of model theory, which flow quickly from the completeness theorem.

• Finally, you of course don’t want to get confused by different proof-styles, but the basic ideas driving the different styles are simple and easy to grasp. So it is in fact quite safe to make initial acquaintance with at least a couple of other proof-styles even at an early stage:

  ○ First, and historically very importantly, there are old-school Hilbert-style axiomatic linear proof-systems. A standard such system for e.g. the propositional calculus has a single rule of inference (modus ponens) and a bunch of axioms which can be called on at any stage in a proof. A proof is just a linear sequence of wffs, each one of which is a given premiss, a logical axiom, or follows from earlier wffs in the sequence by modus ponens. Such systems in their unadorned form are pretty horrible to work inside (proofs can be long and very unobvious), even though their Bauhaus simplicity makes them easy to theorize about from the outside. It does strike me as potentially off-putting,
even a bit masochistic, to concentrate primarily on axiomatic systems when beginning serious logic; but for various reasons they are often used in more advanced texts, so you certainly need to get to know about them sooner or later.

○ It is also fun and illuminating to meet a tree (tableau) system if you haven’t already done so (many philosophers get taught a tableau system first, as such a system is relatively easy to manipulate – and hence easy to teach a computerized proof system too!). A tableau proof of the same illustrative inference would run as follows:

\[
\begin{array}{c}
\neg (P \land \neg Q) \\
\neg (P \to Q) \\
P \\
\neg Q \\
\neg P \\
\neg \neg Q \\
\star \\
\star
\end{array}
\]

What we are doing here is starting with the given premiss and the negation of the target conclusion, and we then proceed to unpack the implications of these assumptions. In this case, for the second (negated-conditional) premiss to be true, it must have a true antecedent and false consequent: while for the first premiss to be true we have to consider cases (a false conjunction must have at least one false conjunct but we don’t know in advance which to blame) – which is why the tree splits. But as we pursue either branch we immediately hit contradiction (conventionally marked with a star in tableau systems), showing that we can’t consistently have the premiss true and the negation of the conclusion true too, so the inference is indeed valid.

So, with that menu of topics to pursue, where to begin?

\section{The main recommendations}

Let’s start with three stand-out texts, which philosophers and mathematicians should both find useful, taken in one order or the other.

The first book is by a philosopher and written with philosophy students particularly in mind, as a just-past-baby-logic text that can be used as bridge leading you onwards to more mathematical texts:

1. David Bostock, \textit{Intermediate Logic} (OUP 1997). From the preface: ‘The book is confined to . . . what is called first-order predicate logic, but it aims to treat this subject in very much more detail than a standard introductory text. In particular, whereas an introductory text will pursue just one style of semantics, just one method of proof, and so on, this book aims to create a wider and a deeper understanding by showing how several alternative approaches are possible, and by introducing comparisons between them.’ So this book aims to range rather more widely than most baby logic texts. And Bostock does indeed usefully introduce you to tableaux (trees) and an Hilbert-style axiomatic proof system and natural deduction and even a so-called sequent calculus as well (as noted, it is important to understand what is going on in these different kinds of proof-system). He proves
completeness for tableaux in particular, which I always think makes the needed construction seem particularly natural. And he touches on issues of philosophical interest such as free logic which are not often dealt with in other books at this level. Still, the discussions remain at much the same level of conceptual/mathematical difficulty as the later parts of Teller’s book and my own. So Intermediate Logic should be, as intended, particularly accessible to philosophers who haven’t done much formal logic before and should, as I said, help ease the transition to more advanced work.

Non-philosophers, however, or philosophers who waltzed easily through their baby logic course, will probably prefer to start instead with the following text which – while only a small notch up in actual difficulty – has a notably more mathematical ‘look and feel’ (being written by mathematicians) while still remaining particularly friendly and approachable. It is also considerably shorter than Bostock’s book, though lacking the latter’s wide range.

2. Ian Chiswell and Wilfrid Hodges, *Mathematical Logic* (OUP 2007). Despite the title, this is particularly accessible, very nicely written, and should be entirely manageable even at this very early stage. Indeed if you can’t cope with this lovely book, then I do fear that serious logic might not be for you!

The headline news is that authors explore a (Gentzen-style) natural deduction system. But by building things up in stages over three chapters – so after propositional logic, they consider an interesting fragment of first-order logic before turning to the full-strength version – they make proofs of e.g. the completeness theorem for first-order logic quite unusually manageable. For a more detailed description of how they do this, see §A.17.

Very warmly recommended for beginning the serious study of first-order logic. (The book has brisk solutions to some exercises. A demerit mark to OUP for not publishing C&H more cheaply.)

Next, complement C&H by reading the first half of

3. Christopher Leary’s *A Friendly Introduction to Mathematical Logic* (Prentice Hall 2000: currently out of print, but a slightly expanded new edition is being planned). There is a great deal to like about this book. Chs. 1–3 do indeed make a very friendly and helpful introduction to first-order logic, now done in axiomatic style, in under 125 pages. Unusually, Leary dives straight into a full treatment of FOL without spending an introductory chapter or two on propositional logic, which means that you won’t feel that you are labouring through the very beginnings of logic one more time than is really necessary. The book is again written by a mathematician for a mostly mathematical audience so some illustrations of ideas can presuppose a smattering of elementary background mathematical knowledge; but you will miss very little if you occasionally have to skip an example. I like the tone very much indeed, and say more about this admirable book in §A.13.

In sum, then, if your background is in philosophy, read Bostock first then carefully work through C&H, followed by Leary to reinforce and expand your knowledge. If you are more mathematically minded, proceed the other way about: work through C&H and Leary and then skim quite a bit faster through Bostock for consolidation of some key ideas and to get an impression of how some further proof-systems work.

Leary’s book might not be so widely available, however, so let me mention a good alternative – a much used fourth text which should be in any library:
4. Herbert Enderton’s *A Mathematical Introduction to Logic* (Academic Press 1972, 2002), which focuses on a Hilbert-style axiomatic system, is often regarded as a classic of exposition. However, it does strike me as somewhat more difficult than C&H, so I’m not surprised that some report finding it a bit challenging as a student *if used by itself as a first text*. It is a very admirable piece of work, however. And it is another text whose proof-systems are axiomatic, so it should work well as complementary reading following on from C&H. Read up to and including §2.5 or §2.6 at this stage. Later, you can finish the rest of that chapter to take you a bit further into model theory. There is more about this book in the Appendix, §A.6.

Finally in this section, I will mention one more book, and for two reasons. First it is a modern classic which has gone through multiple editions, and hence should be in most libraries, making it a useful natural-deduction based alternative to C&H if the latter isn’t available. Second, later chapters of this book are mentioned below as suggested reading for more advanced work, so even if you have read C&H, it could well be worth making early acquaintance with . . .

5. Dirk van Dalen, *Logic and Structure* (Springer, 1980; 5th edition 2012). This is half-a-notch up from C&H in terms of mathematical style/difficulty. And the book isn’t without its quirks and flaws, so I think it would be occasionally a slightly tough going if taken from a standing start – which is why I have recommended beginning with C&H instead. However, mathematicians should be able to cope readily; and even less mathematically minded philosophers should find van Dalen’s more rigorous mode of presentation quite manageable if they have already worked through C&H. (Indeed the combination of the two books – augmented by taking a quick look at some axiomatic system – would indeed make for an excellent foundation in logic.) Read up to and including §3.2 at this stage. For a more extended review of the whole book, see §A.8.

Note to philosophers: If you have already carefully read and mastered the whole of e.g. Teller’s introductory baby-logic-plus-a-little book, the recommended reading in this section won’t in fact teach you *very* many more basic big ideas or major new results (apart from introducing you to some entry-level model-theoretic notions like compactness and the L-S theorems). However, as promised, you will begin to see this largely familiar material being re-presented in the sort of mathematical *style* and with the sort of rigorous detail that you will necessarily encounter more and more as you progress in logic, and that you very much need to start feeling entirely comfortable with at an early stage.

### 2.3 Some other texts for first-order logic

The material covered in the last section is so very fundamental, and the alternative options so very many, that I really need to say at least something about some other books. The Appendix will tell you about how some other Big Books on mathematical logic handle the basics. But in this section I will list – in order of date of first publication – a small handful of books that have particular virtues of one kind or another, books that will either make for illuminating parallel reading or will usefully extend your knowledge of first-order logic or both. There follows a postscript where I respond to comments on early versions of the Guide which asked why I *didn’t* recommend certain texts.

We start with two ‘golden oldies’:
1. Raymond Smullyan, *First-Order Logic* (Springer 1968, Dover Publications 1995). is a short book, terse but absolutely packed with good things. In some ways, this is the most sophisticated book I’m mentioning at this stage. But enthusiasts can try reading Parts I and II, just a hundred pages, after C&H. Those with a taste for mathematical neatness should be able to cope with these chapters and will appreciate their great elegance. This beautiful little book is the source of so many modern treatments of logic based on tree/tableau systems, such as my own – though it packs in a lot more than I do at three times the length. Not always easy as the book progresses, but wonderful.

2. Geoffrey Hunter, *Metalogic* (Macmillan 1971, University of California Press 1992). This is not groundbreaking in the way Smullyan’s book is, but rather is an exceptionally good early student textbook from a time when there were few to choose from. Read Parts One to Three at this stage. And if you are enjoying it, then do continue to finish the book, where it treats a formal theory of arithmetic and proves the undecidability of first-order logic, topics we revisit in §3.3. Unfortunately, the typography – from pre-L\LaTeX\ days – isn’t at all pretty to look at: this can make the book’s pages appear rather unappealing. But in fact the treatment of an axiomatic system of logic is in fact exceptionally clear and accessible. Still well worth blowing the dust off your library’s copy.

Next we come to a classic presentation of logic via natural deduction:

3. Neil Tennant, *Natural Logic* (Edinburgh UP 1978, 1990). Out of print, but freely available from http://people.cohums.ohio-state.edu/tennant9/Natural_Logic.pdf. All credit to Tennant for writing the first textbook at its level which does Gentzen-style natural deduction, a couple of years before van Dalen. Tennant thinks that this approach to logic is philosophically highly significant, and in various ways this shows through. Although not as mathematical in look-and-feel as van Dalen, this book also isn’t always an easy read, despite its being intended as a first logic text for philosophers. However, it is freely available to sample, and you may well find it highly illuminating parallel reading to C&H.

Now for four more recent books:

4. Derek Goldrei’s *Propositional and Predicate Calculus: A Model of Argument* (Springer, 2005) is another exceptionally clear book written by a mathematician for a mostly mathematical audience, using an axiomatic system. Unlike Leary, Goldrei does spend a gentle hundred pages on propositional logic (but done with some sophistication). And unlike Leary who goes on to deal with Gödel’s incompleteness theorem, Goldrei only strays beyond the basics of FOL to touch on a few model-theoretic ideas. But this means that the book can proceed at a pretty gentle pace, and the resulting treatment as far as it goes strikes me as an unusually accessible treatment. Anyone struggling a little with faster-paced books with an axiomatic flavour, or who just wants a comfortably manageable additional text, should find this particularly useful. (If I have a very small complaint, Goldrei stays too fond of subscripts on predicate letters etc. for too long – but that shouldn’t cause more than a moment’s aesthetic grumbling! It shouldn’t much affect ease of reading.)

5. Don’t be put off by the title of Melvin Fitting’s *First-Order Logic and Automated Theorem Proving* (Springer, 1990, 2nd end. 1996). Yes, at various places in the book there are illustrations of how to implement various algorithms in Prolog. But you can easily pick up the very small amount of background knowledge about Prolog
that’s needed to follow what’s going on (and that’s a fun thing to do anyway) – or
you can just skip.

As anyone who has tried to work inside an axiomatic system knows, proof-
discovery for such systems is hard. Which axiom-schema should we instantiate
with which wffs at any given stage of a proof? Natural deduction systems are
nicer: but since we can make any new temporary assumption at any stage in a
proof, again we need to keep our wits about us, or at least learn to recognize some
common patterns of proof. By contrast, tableau proofs very often write themselves
(which is why I used to introduce formal proofs to students that way – thereby
largely separating the business of getting across the idea of formality from having
to teach heuristics of proof-discovery). And because tableau proofs very often write
themselves, they are also good for automated theorem proving.

Now, an open tableau for a single propositional calculus wff \( A \) at the top of the
tree in effect constructs a disjunctive normal form for \( A \) – just take the conjunction
of atoms or negated atoms on each open branch of a completed tree and disjoin the
results. And a tableau proof that \( C \) is a valid in effect works by seeking to find a
disjunctive normal form for \( \neg C \) and showing it to be empty. When this is pointed
out, you might well think ‘Aha! Then there ought to be a dual proof-method which
works with conjunctive normal forms, rather than disjunctive normal forms!’ And
you of course must be right. This alternative is called the resolution method, and
indeed is the more common approach in the automated proof community.

Fitting explores both the tableau and resolution methods in this exceptionally
clearly written book. The emphasis is, then, notably different from most of the
other recommended books: but the fresh light thrown on first-order logic makes
the detour through this book \textit{vaut le voyage}, as the Michelin guides say. (By the
way, if you don’t want to take the full tour, then there’s a nice introduction to
proofs by resolution in Shawn Hedman, \textit{A First Course in Logic} (OUP 2004):
§1.8, §§3.4–3.5.)

6. Raymond Smullyan’s \textit{Logical Labyrinths} (A. K. Peters, 2009) won’t be to everyone’s
taste. From the blurb: ‘This book features a unique approach to the teaching of
mathematical logic by putting it in the context of the puzzles and paradoxes of
common language and rational thought. It serves as a bridge from the author’s
puzzle books to his technical writing in the fascinating field of mathematical logic.
Using the logic of lying and truth-telling, the author introduces the readers to
informal reasoning preparing them for the formal study of symbolic logic, from
propositional logic to first-order logic, … The book includes a journey through the
amazing labyrinths of infinity, which have stirred the imagination of mankind
as much, if not more, than any other subject.’ Smullyan starts with puzzles of the
kind where you are visiting an island where there are Knights (truth-tellers) and
Knaves (persistent liars) and you have to work out what’s true from what they say
about each other and the world; and he ends with first-order logic (using tableaux),
completeness, compactness and more. Not a substitute for more conventional texts,
of course, but – for those with a taste for being led up to the serious stuff via
sequences of puzzles – an entertaining and illuminating supplement.

7. Jan von Plato’s \textit{Elements of Logical Reasoning*} (CUP, 2014) is based on the au-
thor’s introductory lectures. I rather suspect that without his lectures and class-
room work to round things out, the fairly bare bones presented here in a relatively
short compass would be quite tough as a first introduction, as von Plato talks
about a number of variant natural deduction and sequent calculi. But suppose you
have already met one system of natural deduction, and want to know rather more about ‘proof-theoretic’ aspects of this and related systems. Suppose, for example, that you want to know about variant ways of setting up ND systems, about proof-search, about the relation with so-called sequent calculi, etc. Then this is a very clear, very approachable and interesting book. Experts will see that there are some novel twists, with deductive systems tweaked to have some very nice features: beginners will be put on the road towards understanding some of the initial concerns and issues in proof theory.

Postcript

Obviously, I have touched on only a very small proportion of books that cover first-order logic. The Appendix covers another handful. But I end this chapter responding to some particular Frequently Asked Questions prompted by earlier versions of the Guide.

What about Mendelson? Oddly, perhaps the most frequent question I have been asked is ‘But what about Mendelson, Chs. 1 and 2’? Well, Elliot Mendelson’s Introduction to Mathematical Logic (Chapman and Hall/CRC 5th edn 2009) was first published in 1964 when I was a student and the world was a great deal younger. The book was the first modern textbook at its level (so immense credit to Mendelson for that), and I no doubt owe my career to it – it got me through tripos! And it seems that some who learnt using the book are in their turn still using it to teach from. But it has to be said that the book is often terse to the point of unfriendliness, and the basic look-and-feel of the book hasn’t changed as it has run through successive editions. Mendelson’s presentation of axiomatic systems of logic are quite tough going, and as the book progresses in later chapters through formal number theory and set theory, things if anything get even less reader-friendly. Nearly fifty years on, there are (unsurprisingly) many rather more accessible and more amiable alternatives for beginning serious logic. Mendelson’s book is certainly a monument worth visiting one day: but I can’t recommend starting there – if you really want an old-school introduction, Hunter’s book mentioned above is more approachable. (I say more about this book in the Appendix, §A.2.)

What about Hinman? Similar remarks apply to P. Hinman’s recent 2005 blockbuster Fundamentals of Mathematical Logic which has a similar range to Mendelson’s book. But aiming as it does for a more sophisticated level of treatment, and reflecting that the subject has moved on in the intervening decades, the book is more than twice the size. I can see why it might appeal to those now running a modern year-long graduate-level mathematical logic course, looking for a one-stop shop where their students can find (more than) everything they could need. But, this is surely not the place to begin if you have not met a serious treatment of first-order logic before. For a start, Hinman isn’t really very interested in formal proof systems (a deductive system for FOL doesn’t appear until over two hundred pages into the book), so in fact the opening chapters are really mostly on model theory, much of it at least a step up from introductory material. (I say more about this book in §A.16.)

What about Bell, DeVidi and Solomon? I suggested that if you concentrated at the outset on a one-proof-style book, you would do well to widen your focus at an early stage to look at other logical options. And I recommended Bostock’s book for, inter alia, telling you about different styles of proof-system. An alternative which initially looks promising is the enticingly titled John L. Bell, David DeVidi and Graham Solomon’s Logical Options: An Introduction to Classical and Alternative Logics (Broadview Press 2001). This book covers a lot pretty snappily – for the moment, just Chapters 1 and 2 are relevant – and a few years ago I used it as a text for second-year seminar for undergraduates who had used my own tree-based book for their first year course. But many students found it quite hard going, as the exposition is terse, and I found myself having to write very extensive seminar notes. For example, see http://www.logicmatters.net/resources/pdfs/ProofSystems.pdf, which gives a brisk overview of some different proof-styles (written for those who had first done logic using by tableau-based introductory book).

And what about Sider? Theodore Sider – a very well-known philosopher – has written a text called Logic for Philosophy* (OUP, 2010) aimed at philosophers, which I’ve been asked to com-
ment on. The book in fact falls into two halves. The second half (about 130 pages) is on modal logic, and I will comment on that in §4.1. The first half (almost exactly the same length) is on propositional and first-order logic, together with some variant logics, so is very much on the topic of this chapter. But while the coverage of modal logic is quite good, I can’t at all recommend the first half of this book.

Sider starts with a system for propositional logic of sequent proofs in what is pretty much the style of E. J. Lemmon’s 1965 book *Beginning Logic*. Which, as anyone who spent their youth teaching a Lemmon-based course knows, students do not find user-friendly. Why on earth do things this way? We then get shown a Hilbertian axiomatic system with a bit of reasonably clear explanation about what’s going on. But there are much better presentations for the marginally more mathematical.

Predicate logic gets only an axiomatic deductive system. Again, I can’t think this is the best way to equip philosophers who might have a perhaps shaky grip on formal ideas with a better understanding of how a deductive calculus for first-order logic might work, and how it relates to informal rigorous reasoning. But, as I say, if you are going to start with an axiomatic system, there are better alternatives. The explanation of the semantics of a first-order language is quite good, but not outstanding either.

Some of the decisions about what technical details then to cover in some depth and what to skimp over are pretty inexplicable. For example, there are pages tediously proving the mathematically unexciting deduction theorem for axiomatic propositional logic: yet later just one paragraph is given to the deep compactness theorem for first-order logic, which a philosophy student starting on the philosophy of mathematics might well need to know about and understand some applications of. Why this imbalance? By my lights, then, the first-half of Sider’s book certainly isn’t the obvious go-to treatment for giving philosophers a grounding in core first-order logic.

True, a potentially attractive additional feature of this part of Sider’s book is that it does contain discussions about e.g. some non-classical propositional logics, and about descriptions and free logic. But remember all this is being done in 130 pages, which means that things are whizzing by very fast. For example, the philosophically important issue of second-order logic is dealt with far too quickly to be useful. And at the introductory treatment of intuitionistic logic is also far too fast. So the breadth of Sider’s coverage here goes with too much superficiality. If you want some breadth, Bostock is still to be preferred.
Chapter 3

Basic Mathematical Logic

We now press on from an initial look at first-order logic to consider other core elements of mathematical logic. The menu for this chapter:

3.1 A brief consideration of second-order logic.
3.2 Some elements of model theory.
3.3 Formal arithmetic, theory of computation, Gödel’s incompleteness theorems.
3.4 Elements of set theory.

As explained in §0.5, I do very warmly recommend reading a series of books on a topic which overlap in coverage and difficulty, rather than leaping immediately from an ‘entry level’ text to a really advanced one. You don’t have to follow this advice, of course. But I mention it again here to remind you just why the list of recommendations in most sections here and in the next chapter is quite extensive and increments in coverage/difficulty between successive recommendations are often quite small: this level of logic really isn’t as daunting as the overall length of these two chapters might superficially suggest. Promise!

3.1 Second-order logic

We first look at a familiar extension of first-order classical logic of interest to both mathematicians and philosophers, one which is arguably important enough to warrant its having a box of its own in our map of the logical landscape. This is second-order logic, where we allow generalizations which quantify into predicate position. Consider, for example, the intuitive principle of arithmetical induction. Take any property $X$; if 0 has it, and for any $n$ it is passed down from $n$ to $n + 1$, then all numbers must have $X$. It is tempting to regiment this as follows:

$$\forall X[(X0 \land \forall n(Xn \rightarrow X(n + 1))] \rightarrow \forall n Xn]$$

where the second-order quantifier quantifiers ‘into predicate position’ and supposedly runs over all properties of numbers. But this is illegitimate in standard first-order logic. [Historical aside: note that the earliest presentations of quantificational logic, in Frege and in Principia Mathematica, were of logics that did allow this kind of higher-order quantification: the concentration on first-order logic which has become standard was a later development.]

What to do? One option is to go set-theoretic and write the induction principle as

$$\forall X[(0 \in X \land \forall n(n \in X \rightarrow (n + 1) \in X)] \rightarrow \forall n n \in X]$$

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where the variable ‘$X$’ is now a sorted first-order variable running over sets. But arguably this changes the subject (our ordinary principle of arithmetical induction doesn’t seem to be about sets), and there are other issues too. So why not take things at face value and allow that the ‘natural’ logic of informal mathematical discourse often deploys second-order quantifiers that range over properties (expressed by predicates) as well as first-order quantifiers that range over objects (denoted by names), i.e. why not allow quantification into predicate position as well as into name position?

A number of books already mentioned have sections introducing second-order logic. But probably as good as any place to get a brief overview is the article


You could then try one of

2. Dirk van Dalen, Logic and Structure, Ch. 4,


That will be as much as many readers will need. But having got this far, some will want to dive into the simply terrific


And it would be a pity, while you have Shapiro’s book in your hands, to skip the initial philosophical/methodological discussion in the first two chapters here. This whole book is a modern classic, and remarkably accessible.

### 3.2 From first-order logic to elementary model theory

In continuing to explore the mainstream menu of a first ‘Mathematical Logic’ course, we’ll take things under three headings: first, there is introductory model theory (in this section); next, arithmetic and the theory of computable functions; and then thirdly set theory. Things predictably do get rather more mathematical, but hopefully in not too scary a way at the outset. (Note, you can tackle the topics of these next three sections in any order that takes your fancy – eventually, in much more advanced work, the various streams can mingle again in intricately complicated ways, but for now you can take them as running independently.)

So what should you read if you want to move on just a little from those first intimations of classical model theory in some of the readings we mentioned in §2.2?

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**Aside** There is a short old book, the very first volume in the Oxford Logic Guides series, Jane Bridge’s Beginning Model Theory: The Completeness Theorem and Some Consequences (Clarendon Press, 1977) which neatly takes on the story just a few steps. But very sadly, the book was printed in that brief period when publishers thought it a bright idea to save money by photographically printing work produced on electric typewriters. Accustomed as we now are to mathematical texts beautifully \LaTeX ed, the look of Bridges’s book is really quite horribly off-putting, and the book’s virtues do not outweigh that sad handicap. Let’s set it aside.

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Here then are two good places to start; read both as they complement each other nicely.
1. Wilfrid Hodges’s ‘Elementary Predicate Logic’, in the Handbook of Philosophical Logic, Vol. 1, ed. by D. Gabbay and F. Guenthner, (Kluwer 2nd edition 2001). This is a slightly expanded version of the essay in the first edition of the Handbook (read that earlier version if this one isn’t available), and is written with Hodges’s usual enviable clarity. Over a hundred pages long, this serves both as an insightful and fresh overview course on basic first-order logic (more revision!), and as an illuminating introduction to some ideas from model theory.

2. Dirk van Dalen’s Logic and Structure* (Springer, 1980; 5th edition 2012). In §2.2 of this Guide I did recommended reading this text up to and including van Dalen’s §3.2, for coverage of basic first-order logic. Now try reading the whole of his Chapter 3, for a bit of revision and then some more model theory. There are a number of mathematical illustrations which could pass some readers by at this stage, so feel very free to skip if/when the going temporarily gets tough.

Still, van Dalen does go pretty speedily. For a more expansive treatment (though not increasing the level of difficulty, nor indeed covering everything touched on in Hodges’s essay) here is a still reasonably elementary textbook:

3. Maria Manzano, Model Theory, Oxford Logic Guides 37 (OUP, 1999). I seem to recall, from a reading group where we looked at this book, that the translation can leave something to be desired. However, the coverage as far as it goes is rather good, and the treatment gentle and accessible. It starts off by talking about relationships among structures in general before talking about structures-as-models-of-theories (an approach that philosophers who haven’t done a pure maths course will probably find rather helpful).

And this might already be about as far as many philosophers may want/need to go. Many mathematicians, however, will be eager to take up the story again in §5.4.

**Postscript**  Sadly, Hodges’s essay is in an large multi-volume set that some libraries won’t have (though copies of the essay might float around the web); and Manzano’s book is also ludicrously expensive. So I should probably mention some alternatives more likely to be in any library.

Among recent books, there is quite a bit of model theory in Shawn Hedman’s A First Course in Logic (OUP 2004), Chs 2–4; however, the level quickly becomes rather challenging as the topics go quite a bit beyond the basics: see §A.15.

So Ch. 2 and the first two sections of Ch. 3 (and perhaps §3.4 as well) of Peter Hinman’s Fundamentals of Mathematical Logic (A. K. Peters, 2005) would probably serve a bit better as revision-of-the-semantics-of-first-order-logic and an introduction to model theory for the mathematical: see §A.16.

Many of the Big Books surveying Mathematical Logic have presentations of first-order logic which introduce a least some model theory: you can find out more about some of these in Appendix A.

### 3.3 Computability and Gödelian incompleteness

The standard mathematical logic curriculum, as well as looking at some elementary general results about formalized theories and their models, looks at two particular instances of non-trivial, rigorously formalized, axiomatic systems – arithmetic (a paradigm theory of finite whatnots) and set theory (a paradigm theory of infinite whatnots). We’ll take arithmetic first.
In more detail, there are three inter-related topics here: (a) an introduction to formal theories of arithmetic, (b) the elementary theory of arithmetic computations and of computability more generally, leading up to (c) Gödel’s epoch-making proof of the incompleteness of any nice enough theory that can ‘do’ enough arithmetic (a result of profound interest to philosophers).

Now, Gödel’s 1931 proof of his incompleteness theorem uses facts in particular about so-called primitive recursive functions: these functions are a subclass of the computable numerical functions, i.e. a subclass of the functions which a suitably programmed computer could evaluate (abstracting from practical considerations of time and available memory). A more general treatment of the effectively computable functions (arguably capturing all of them) was developed a few years later, and this in turn throws more light on the incompleteness phenomenon.

So, if you are going to take your first steps into this area, there’s a choice to be made. Do you look at things in roughly the historical order, introducing theories of formal arithmetic and learning how to prove initial versions of Gödel’s incompleteness theorem before moving on to look at the general treatment of computable functions? Or do you do some of the general theory of computation first, turning to the incompleteness theorems later?

Here then are two introductory books which take the two different routes:

1. Peter Smith, *An Introduction to Gödel’s Theorems* (CUP 2007, 2nd edition 2013) does things in the historical order. Mathematicians: don’t be put off by the series title ‘Cambridge Introductions to Philosophy’ – putting it in that series was the price I paid for cheap paperback publication. This is still quite a meaty logic book, with a lot of theorems and a lot of proofs, but I hope rendered very accessibly. The book’s website is at http://godelbook.net, where there are supplementary materials of various kinds, including a cut-down version of a large part of the book, *Gödel Without (Too Many) Tears*.


As you’ll already see from these two books, this really is a delightful area. Elementary computability theory is conceptually very neat and natural, and the early Big Results are proved in quite remarkably straightforward ways (just get the hang of the basic ‘diagonalization’ construction, the idea of Gödel-style coding, and one or two other tricks and off you go . . . ).

Just half a notch up in difficulty, here’s another book that focuses first on the general theory of computation before turning to questions of logic and arithmetic:

3. George Boolos, John Burgess, Richard Jeffrey, *Computability and Logic* (CUP 5th edn. 2007). This is the latest edition of an absolute classic. The first version (just by Boolos and Jeffrey) was published in 1974; and there’s in fact a great deal to be said for regarding their 1990 third edition as being the best. The last two versions have been done by Burgess and have grown considerably and perhaps in the process lost elegance and some of the individuality. But whichever edition you get hold of, this is still great stuff. Taking the divisions in the last two editions, you will want to read the first two parts of the book at an early stage, perhaps being more selective when it comes to the last part, ‘Further Topics’.
We’ll return to say a quite a lot more about these topics in Ch. 3.

**Postscript** There are many other introductory treatments covering computability and/or incompleteness. For something a little different, let me mention

A. Shen and N. K. Vereshchagin, *Computable Functions*, (American Math. Soc., 2003). This is a lovely, elegant, little book in the AMA’s ‘Student Mathematical Library’ – the opening chapters can be recommended for giving a differently-structured quick tour through some of the Big Ideas, and hinting at ideas to come.

Then various of the Big Books on mathematical logic have treatments of incompleteness. For the moment, here are three:

I have already very warmly recommended Christopher Leary’s *A Friendly Introduction to Mathematical Logic* (Prentice Hall, 2000) for its coverage of first-order logic. The final chapter has a nice treatment of incompleteness, and one that doesn’t overtly go via computability. (In headline terms you will only understand in retrospect, instead of showing that certain syntactic properties are (primitive) recursive and showing that all primitive recursive properties can be ‘represented’ in theories like $PA$, Leary relies on more directly showing that syntactic properties can be represented.) See also §A.13.

Herbert Enderton *A Mathematical Introduction to Logic* (Academic Press 1972, 2002), Ch. 3 is good on different strengths of formal theories of arithmetic, and then proves the first incompleteness theorem first for $PA$ and then – after touching on other issues – shows how to use the $\beta$-function trick to extend the theorem to apply to Robinson arithmetic. Well worth reading after e.g. my book for consolidation.

Peter G. Hinman’s *Fundamentals of Mathematical Logic* (A. K. Peters, 2005), Chs. 4 and 5 could be read after my book as rather terse revision, and as sharpening the story in various ways.

Finally, I suppose (if only because I’ve been asked about it a good number of times) I should also mention

Douglas Hofstadter, *Gödel, Escher, Bach* (Penguin, first published 1979). When students enquire about this, I helpfully say that it is the sort of book that you might well like if you like that kind of book, and you won’t if you don’t. It is, to say the least, quirky and distinctive. As I far as I recall, though, the parts of the book which touch on techie logical things are pretty reliable and won’t lead you astray. Which is a great deal more than can be said about many popularizing treatments of Gödel’s theorems.

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### 3.4 Beginning set theory

Let’s say that the *elements of set theory* – the beginnings that any logician really ought to know about – will comprise enough to explain how numbers (natural, rational, real) are constructed in set theory (so enough to give us a glimmer of understanding about why it is said that set theory provides a foundation for mathematics). The elements also include the basic arithmetic of infinite cardinals, the development of ordinal numbers and transfinite induction over ordinals, and something about the role of the axiom(s) of choice. These initial ideas and constructions can (and perhaps should) be presented fairly informally: but something else that also belongs here at the beginning is an account of the development of $ZFC$ as the now standard way of formally encapsulating and regimenting the key principles involved in the informal development of set theory.

Going beyond these elements we have e.g. the exploration of ‘large cardinals’, proofs of the consistency and independence of e.g. the Continuum Hypothesis, and a lot more
besides. But readings on these further delights are for Chapter 3: this present section is, as advertised, about the first steps for beginners in set theory. Even here, however, there are many books to choose from, so an annotated Guide should be particularly welcome.

I can start off, though, pretty confidently recommending a couple of ‘entry level’ treatments:

1. Herbert B. Enderton, *The Elements of Set Theory* (Academic Press, 1977) has exactly the right coverage. But more than that, it is particularly clear in marking off the informal development of the theory of sets, cardinals, ordinals etc. (guided by the conception of sets as constructed in a cumulative hierarchy) and the formal axiomatization of ZFC. It is also particularly good and non-confusing about what is involved in (apparent) talk of classes which are too big to be sets – something that can mystify beginners. It is written with a certain lightness of touch and proofs are often presented in particularly well-signposted stages. The last couple of chapters or so perhaps do get a bit tougher, but overall this really is quite exemplary exposition.

2. Derek Goldrei, *Classic Set Theory* (Chapman & Hall/CRC 1996) is written by a staff tutor at the Open University in the UK and has the subtitle ‘For guided independent study’. It is as you might expect extremely clear, it is quite attractively written (as set theory books go!), and is indeed very well-structured for independent reading. The coverage is very similar to Enderton’s, and either book makes a fine introduction (but for what it is worth, I prefer Enderton).

Still starting from scratch, and initially also only half a notch or so up in sophistication from Enderton and Goldrei, we find two more really nice books:

3. Karel Hrbacek and Thomas Jech, *Introduction to Set Theory* (Marcel Dekker, 3rd edition 1999). This goes a bit further than Enderton or Goldrei (more so in the 3rd edition than earlier ones), and you could – on a first reading – skip some of the later material. Though do look at the final chapter which gives a remarkably accessible glimpse ahead towards large cardinal axioms and independence proofs. Again this is a very nicely put together book, and recommended if you want to consolidate your understanding by reading another presentation of the basics and want then to push on just a bit. (Jech is of course a major author on set theory, and Hrbacek once won a AMA prize for maths writing.)


These four books are in print: make sure that your university library has them – though none is cheap (indeed, Enderton’s is quite absurdly expensive). I’d strongly advise reading one of the first pair and then one of the second pair.

My next recommendation might come as a bit of surprise, as it is something of a ‘blast from the past’. But we shouldn’t ignore old classics – they can have a lot to teach us even if we have read the modern books.

5. Abraham Fraenkel, Yehoshua Bar-Hillel and Azriel Levy, *Foundations of Set-Theory* (North-Holland, 2nd edition 1973). Both philosophers and mathematicians should appreciate the way this puts the development of our canonical ZFC set theory into some context, and also discusses alternative approaches. It really is attractively readable, and should be very largely accessible at this early stage. I’m
not an enthusiast for history for history’s sake: but it is very much worth knowing the stories that unfold here.

One intriguing feature of that last book is that it doesn’t at all emphasize the ‘cumulative hierarchy’ – the picture of the universe of sets as built up in a hierarchy of stages or levels, each level containing all the sets at previous levels plus new ones (so the levels are cumulative). This picture is nowadays familiar to every beginner: you will find it e.g. in the opening pages of Joseph Shoenfield ‘The axioms of set theory’, *Handbook of mathematical logic*, ed. J. Barwise, (North-Holland, 1977) pp. 321–344. The picture is also brought to the foreground again in

6. Michael Potter, *Set Theory and Its Philosophy* (OUP, 2004). For philosophers (and for mathematicians concerned with foundational issues) this surely is – at some stage – a ‘must read’, a unique blend of mathematical exposition (mostly about the level of Enderton, with a few glimpses beyond) and extensive conceptual commentary. Potter is presenting not straight ZFC but a very attractive variant due to Dana Scott whose axioms more directly encapsulate the idea of the cumulative hierarchy of sets. However, it has to be said that there are passages which are pretty hard going, sometimes because of the philosophical ideas involved, but sometimes because of unnecessary expositional compression. In particular, at the key point at p. 41 where a trick is used to avoid treating the notion of a level (i.e. a level in the hierarchy) as a primitive, the definitions are presented too quickly, and I know that relative beginners can get lost. However, if you have already read one or two set theory books from earlier in the list, you should be fairly easily be able to work out what is going on and read on past this stumbling block.

It is a nice question how much more technical knowledge of results in set theory a philosophy student interested in logic and the philosophy of maths needs (if she is not specializing in the technical philosophy of set theory). But getting this far will certainly be a useful start, so let’s pause here.

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**Postscript** Books by Ciesielski and by Hajnal and Hamburger, although in LMS Student Text series and starting from scratch, are not really suitable for this list (they go too far and probably too fast). But the following six(!) still-introductory books, listed in order of publication, each have things to recommend them for beginners: good libraries will have them, so browse through and see which might suit your interests and mathematical level.

D. van Dalen, H.C. Doets and H. de Swart, *Sets: Naive, Axiomatic and Applied* (Pergamon, 1978). The first chapter covers the sort of elementary (semi)-naive set theory that any mathematician needs to know, up to an account of cardinal numbers, and then a first look at the paradox-avoiding ZF axiomatization. This is attractively and illuminatingly done (or at least, the conceptual presentation is attractive – sadly, and a sign of its time of publication, the book seems to have been photo-typeset from original pages produced on electric typewriter, and the result is visually not attractive at all).

The second chapter carries on the presentation axiomatic set theory, with a lot about ordinals, and getting as far as talking about higher infinities, measurable cardinals and the like. The final chapter considers about some applications of various set theoretic notions and principles. Well worth seeking out, if you don’t find the typography off-putting.

Keith Devlin, *The Joy of Sets* (Springer, 1979: 2nd edn. 1993). This is mostly remarkably lucid and attractively written (as you would expect from this author). And it starts pretty gently, with the opening chapter exploring ‘naive’ ideas about sets and some set-theoretic constructions, and the next chapter introducing axioms for ZFC pretty gently (indeed, non-mathematicians could particularly like Chs 1 and 2, omitting §2.6). Things then speed up a bit, and by the end of Ch. 3 – some 100 pages – we are pretty much up to the coverage of
Goldrei’s much longer first six chapters, though Goldrei says more about (re)constructing classical maths in set theory. Some will prefer Devlin’s fast-track version. The rest of the book covers non-introductory topics in set theory.

P. T. Johnstone, Notes on Logic and Set Theory (CUP, 1987). Chapters 5–8 (just over fifty pages) introduce the ZF axioms, consider ordinals and well-orderings, the axiom of choice, and cardinal arithmetic. This is a very brisk treatment (augmented with substantive exercises). Mathematicians, though, might find it excellent for revision/consolidation, as the brevity means the Big Ideas get particularly highlighted.

Winfried Just and Martin Weese, Discovering Modern Set Theory I: The Basics (American Mathematical Society, 1996). This covers overlapping ground to Enderton, patchily but perhaps more zestfully and with a little more discussion of conceptually interesting issues. It is at some places more challenging – the pace can be uneven. But this is evidently written by enthusiastic teachers, and the book is very engaging. (The story continues in a second volume.)

I like the style a lot, and think it works very well. I don’t mean the occasional (sightly laboured?) jokes: I mean the in-the-classroom feel of the way that proofs are explored and motivated, and also the way that teach-yourself exercises are integrated into the text. For instance there are exercises that encourage you to produce proofs that are in fact non-fully-justified, and then the discussion explores what goes wrong and how to plug the gaps.

A. Shen and N. K. Vereshchagin, Basic Set Theory (American Mathematical Society, 2002), just over 100 pages, and mostly about ordinals. But very readable, with 151 ‘Problems’ as you go along to test your understanding. Could be very useful by way of revision/consolidation

Ernest Schimmerling, A Course on Set Theory (CUP, 2011) is slightly mistitled: it is just 160 pages, again introductory but with some rather different emphases. Quite an attractive supplementary read at this level.

Finally, what about the chapters on set theory in those Big Books on Mathematical Logic? I’m not convinced that any are to be particularly recommended compared with the stand-alone treatments.
Chapter 4

Variant Logics (of particular interest to philosophers)

Here’s the menu for the chapter:

4.1 We start with modal logic, for two reasons. First, the basics of modal logic don’t involve anything mathematically more sophisticated than the elementary first-order logic covered in Chiswell and Hodges (indeed to make a start on modal logic you don’t even need as much as that). Second, and more much importantly, philosophers working in many areas surely ought to know a little modal logic.

4.2 Classical logic demands that all terms denote one and one thing – i.e. it doesn’t countenance empty terms which denote nothing, or plural terms which may denote more than one thing. In this section, we look at logics which remain classical in spirit but which do allow empty and/or plural terms.

4.3 We then look at some non-classical logics: first, intuitionistic logic (which indeed is of interest to mathematicians too), and then wilder deviations from the classical paradigm.

4.1 Getting started with modal logic

Basic modal logic is the logic of the operators ‘□’ and ‘◇’ (read ‘it is necessarily true that’ and ‘it is possibly true that’); it adopts principles like □ϕ → ϕ and ϕ → ◇ϕ, and investigates more disputable principles like ◇ϕ → □◇ϕ. The place to start is clear:

1. Rod Girle, *Modal Logics and Philosophy* (Acumen 2000, 2009), Part I. Girle’s logic courses in Auckland, his enthusiasm and abilities as a teacher, are justly famous. Part I of this book provides a particularly lucid introduction, which in 136 pages explains the basics, covering both trees and natural deduction for some propositional modal logics, and extending to the beginnings of quantified modal logic. Philosophers may well want to go on to read Part II of the book, on applications of modal logic.

Also pretty introductory, though perhaps a little brisker than Girle at the outset, is

throughout in a way that can be very illuminating indeed. Although it starts from scratch, however, it would be better to come to the book with a prior familiarity with logic via trees, as in my IFL. We will be mentioning this book again in later sections for its excellent coverage of non-classical themes.

If you do start with Priest’s book, then at some point you will need to supplement it by looking at a treatment of natural deduction proof systems for modal logics. One option is to dip into Tony Roy’s comprehensive ‘Natural Derivations for Priest, An Introduction to Non-Classical Logic’ which presents natural deduction systems corresponding to the propositional logics presented in tree form in the first edition of Priest (so the first half of the new edition). This can be downloaded at http://philosophy.unimelb.edu.au/ajl/2006/.

Another possible way in to ND modal systems would be via the opening chapters of

3. James Garson, Modal Logic for Philosophers* (CUP, 2006). This again is certainly intended as a gentle introductory book: it deals with both ND and semantic tableaux (trees), and covers quantified modal logic. It is reasonably accessible, but not – I think – as attractive as Girle.

We now go a step up in sophistication:

4. Melvin Fitting and Richard L. Mendelsohn, First-Order Modal Logic (Kluwer 1998). This book starts again from scratch, but then does go rather more snappily, with greater mathematical elegance (though it should certainly be accessible to anyone who is modestly on top of non-modal first-order logic, as in the previous section). It still also includes a good amount of philosophically interesting material. Recommended.

And we can for the moment stop here. Getting as far as Fitting and Mendelsohn will give most philosophers a good enough grounding in basic modal logic. But for more (including additional reading relevant to Timothy Williamson’s 2013 book on modal metaphysics), see §5.2.

Postscript     Old hands learnt their modal logic from G. E. Hughes and M. J. Cresswell An Introduction to Modal Logic (Methuen, 1968). This was at the time of original publication a unique book, enormously helpfully bringing together a wealth of early work on modal logic in an approachable way.

Nearly thirty years later, the authors wrote a heavily revised and updated version, A New Introduction to Modal Logic (Routledge, 1996). This newer version like the original one concentrates on axiomatic versions of modal logic, which doesn’t make it always the most attractive introduction from a modern point of view. But it is still an admirable book at an introductory level (and going beyond), that enthusiasts will learn from.

I didn’t recommend the first part of Theodore Sider’s Logic for Philosophy* (OUP, 2010). However, the second part of the book which is entirely devoted to modal logic (including quantified modal logic) and related topics like Kripke semantics for intuitionistic logic is significantly better. Compared with the early chapters with their inconsistent levels of coverage and sophistication, the discussion here develops more systematically and at a reasonably steady level of exposition. There is, however, a lot of (acknowledged) straight borrowing from Hughes and Cresswell, and – like those earlier authors – Sider also gives axiomatic systems. In fact, student readers would probably do best by supplementing Sider with a parallel reading of the approachable earlier text. But if you want a pretty clear explanation of Kripke semantics, and want to learn e.g. how to search systematically for countermodels, Sider’s treatment in his Ch. 6 could well work as a basis. And then the later treatments of quantified modal logic in Chs 9 and 10 (and some of the conceptual issues they raise) are also lucid and approachable.
4.2 Other classical extensions and variants

We next look at what happens if you stay first-order in the sense of keeping your variables running over objects, but allow terms that fail to denote (free logic) or which allow terms that refer to more than one thing (plural logic).

4.2.1 Free Logic

Classical logic assumes that any term denotes an object in the domain of quantification, and in particular assumes that all functions are total, i.e. defined for every argument – so an expression like ‘\(f(c)\)’ always denotes. But mathematics cheerfully countenances partial functions, which may lack a value for some arguments. Should our logic accommodate this, by allowing terms to be free of existential commitment? In which case, what would such a ‘free’ logic look like?

For some background and motivation, see

1. David Bostock, *Intermediate Logic* (OUP 1997), Ch. 8,

and also look at a useful and quite detailed overview article from the Stanford Encyclopedia (what would philosophers do without that?):

   http://plato.stanford.edu/entries/logic-free/

Then for another very accessible brief formal treatment, this time in the framework of logic-by-trees, see


For more details (though going rather beyond the basics), you could make a start on


Postscript  Rolf Schock’s *Logics without Existence Assumptions* (Almqvist & Wiskell, Stockholm 1968) is still well worth looking at on free logic after all this time. And for a much more recent collection of articles around and about the topic of free logic, see Karel Lambert, *Free Logic: Selected Essays* (CUP 2003).

4.2.2 Plural logic

In ordinary mathematical English we cheerfully use plural denoting terms such as ‘2, 4, 6, and 8’, ‘the natural numbers’, ‘the real numbers between 0 and 1’, ‘the complex solutions of \(z^2 + z + 1 = 0\)’, ‘the points where line \(L\) intersects curve \(C\)’, ‘the sets that are not members of themselves’, and the like. Such locutions are entirely familiar, and we use them all the time without any sense of strain or logical impropriety. We also often generalize by using plural quantifiers like ‘any natural numbers’ or ‘some reals’ together with linked plural pronouns such as ‘they’ and ‘them’. For example, here is a version of the Least Number Principle: given any natural numbers, one of them must be the least. By contrast, there are some reals – e.g. those strictly between 0 and 1 – such that no one of *them* is the least.
Plural terms and plural quantifications appear all over the place in mathematical argument. It is surely a project of interest to logicians to regiment and evaluate the informal modes of argument involving such constructions. Hence the business of plural logic, a topic of much recent discussion. For an introduction, see


And do read at least two of the key papers listed in Linnebo’s expansive bibliography:


( Oliver and Smiley give reasons why there is indeed a real subject here: you can’t readily eliminate plural talk in favour e.g. of singular talk about sets. Boolos’s classic will tell you something about the possible relation between plural logic and second-order logic.) Then, for much more about plurals, you could look at

4. Thomas McKay, Plural Predication (OUP 2006),

which is clear and approachable. Real enthusiasts for plural logic will want to dive into the long-awaited (though occasionally rather idiosyncratic)

5. Alex Oliver and Timothy Smiley, Plural Logic (OUP 2013).

4.3 Non-classical variants

In this section we turn to consider more radical departures from the classical paradigm. First both historically and in terms of philosophical importance we look at intuitionistic logic. We then consider relevance logic – which suppresses the classical (and intuitionistic) rule that a contradiction implies anything – and we also note some wilder deviancies!

4.3.1 Intuitionist logic

Could there be domains (mathematics, for example) where truth is in some good sense a matter of provability-in-principle, and falsehood a matter of refutability-in-principle? And if so, must every proposition from such a domain be either provable or refutable? Perhaps we shouldn’t endorse the principle that $\varphi \lor \neg \varphi$ is always true. (Maybe the principle, even when it does hold for some domain, doesn’t hold as a matter of logic but as a matter of metaphysics.)

Thoughts like this give rise to one kind of challenge to classical two-valued logic, which of course does assume excluded middle across the board. For more on the intuitionist challenge, see

1. John L. Bell, David DeVidi and Graham Solomon’s Logical Options: An Introduction to Classical and Alternative Logics (Broadview Press 2001), §§5.2, 5.3. Gives an elementary explanation of the constructivist motivation for intuitionist logic, and then explains a tree-based proof system for both propositional and predicate logic.
2. Graham Priest, *An Introduction to Non-Classical Logic* (CUP, much expanded 2nd edition 2008), Chs. 6, 20. These chapters of course flow on naturally from Priest’s treatment in that book of modal logics, first propositional and then predicate. (There is, in fact, a close relation between intuitionistic logic and a certain modal logic.)

There’s a snappy treatment – but again manageable if you have tackled earlier chapters in van Dalen’s book, so you are familiar with the style – in


Likewise, there is quite a good treatment in


Or you could dive straight in to the more expansive and detailed treatment provided accessibly by

5. Melvin Fitting, *Intuitionistic Logic, Model Theory, and Forcing* (North Holland, 1969). The first part of the book Chs 1–6 is a particularly clear stand-alone introduction to the semantics and a proof-system for intuitionist logic (the second part of the book concerns an application of this to a construction in set theory, but don’t let that put you off the first part!).

We will return briefly in §5.9 to note some further explorations of intuitionism.

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**Aside**  One theme not highlighted in these initial readings is that intuitionistic logic seemingly has a certain naturalness compared with classical logic, from a more proof-theoretic point of view. Suppose we think of the natural deduction introduction rule for a logical operator as fixing the meaning of the operator (rather than a prior semantics fixing what is the appropriate rule). Then the corresponding elimination rules surely ought to be in harmony with the introduction rule, in the sense of just ‘undoing’ its effect, i.e. giving us back from a wff \( \varphi \) with \( O \) as its main operator no more than what an application of the \( O \)-introduction rule to justify \( \varphi \) would have to be based on. For the idea of harmony see e.g. Neil Tennant’s *Natural Logic*** (Edinburgh UP 1978, 1990) [http://people.cohums.ohio-state.edu/tennant9/Natural.Logic.pdf](http://people.cohums.ohio-state.edu/tennant9/Natural.Logic.pdf), §4.12. From this perspective the characteristic classical excluded middle rule is seemingly not ‘harmonious’.

There’s a significant literature on this idea: for some discussion, and pointers to other discussions, you could start with Peter Milne, ‘Classical harmony: rules of inference and the meaning of the logical constants’, *Synthese* vol. 100 (1994), pp. 49–94.

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### 4.3.2 Relevance logics (and wilder logics too)

The inference \( P, \neg P \vdash Q \) is classically valid. But doesn’t it commit a fallacy of relevance?

Here’s another, related, issue. Classically, if \( \varphi \vdash \psi \), then \( \varphi, \chi \vdash \psi \) (irrelevant premises can be added without making a valid inference invalid). And if \( \varphi, \chi \vdash \psi \) then \( \varphi \vdash \chi \rightarrow \psi \) (that’s the Conditional Proof rule in action). Presumably we have \( p \vdash p \). So we have \( p, q \vdash p \), whence \( p \vdash q \rightarrow p \). It seems then that classical logic’s carefree attitude to questions of relevance in deduction and its dubious version of the conditional are tied closely together.

Can we do better? What does a more relevance-aware logic look like? For useful introductory reading, see


I can then be very lazy and just refer the beginner (once again!) to the wonderful

3. Graham Priest, *An Introduction to Non-Classical Logic* (CUP, much expanded 2nd edition 2008). Look at Chs. 7–10 for a treatment of propositional logics of various deviant kinds, including relevance logics. Priest goes on to also treat logics where there are truth-value gaps, and – more wildly – logics where a proposition can be both true and false (there’s a truth-value glut). Then, if this excites you, carry on to look at Chs. 21–24 where the corresponding quantificational logics are presented. This book really is a wonderful resource.

If you then want to read one more work on relevant logic, the obvious place to go is

4. Edwin Mares, *Relevant Logic: A Philosophical Interpretation* (CUP 2004). As the title suggests, this book has very extensive conceptual discussion alongside the more formal parts.

Also relevant here is

5. J. C. Beall and Bas van Fraassen’s *Possibilities and Paradox* (OUP 2003) is ‘an introduction to modal and many-valued logics’, discussing – inter alia – the liar paradox, vagueness (truth-value gaps), and paraconsistent reasoning from inconsistent premisses (touching on truth-value gluts again).

You can then follow up some of the references in those two *SEP* encyclopaedia entries.
Chapter 5

Exploring Further

In this chapter, I make some suggestions for more advanced reading on a selection of the areas of logic introduced in Chapters 3 and 4, and also suggest reading on some additional topics.

Before tackling this often significantly more advanced material, however, it might be very well worth first taking the time to look at one or two of the Big Books on mathematical logic which will help consolidate your grip on the basics at the level of Chapter 3 and/or push things on just a bit. See Appendix A for some guidance on what’s available. NB: In terms of the typical level of what they cover, this chapter and Appendix A could as well have appeared in the reverse order (as indeed they did in some earlier versions of this Guide). On the other hand, the often rather critical discussions of the Big Books have a rather different character to the generally commendatory spirit of the previous chapters and this one, and that’s why they are perhaps better thought of as constituting an Appendix to the main Guide.

5.1 How to use this chapter, and the menu

The recommendations in the previous chapters were all pretty mainstream, were mostly fairly manageable by a keen student even if serious mathematics isn’t really his/her thing, and taken together represent the broad sweep of what a student aiming for a wide-ranging but basic logical competence should get to know about.

The recommendations in this chapter are rather more varied. These are books at assorted levels, which I happen to know more or less well and happen to like, clustered round topics that happen to interest me (others would no doubt produce different lists with different emphases, going further in some places, less far in others, and even with new topics). But the general plan is to point to enough works to take you a few more steps on beyond the basics in key areas, so you can then can happily ‘fly solo’ and explore further on your own should you want to. To stress the point, however, I don’t pretend to any consistency in how far we reach beyond the basics. In some cases, I mention recent texts which will take you quite close to the ‘frontiers’ in a sub-subfield; but mostly we still fall some way short of that.

Few people will be keen to pursue all these different subfields of logic: so you will probably want to be a lot more selective and/or follow up suggestions from other sources too. Ideally, then, probably the best thing to do is to choose the relevant section(s) of this chapter to consult while sitting in a good library where you can look at the recommended books, skimming through them to see whether they appear to be what you are looking for. And don’t be daunted by first appearances! Just tell yourself: if you managed the
tougher readings in Chapter 3, then if you take things slowly and put in some work, you ought to be able to manage at least the initial readings in this chapter on any topic you care to pursue further. No doubt you will find some later suggestions tough work at the outset: but then so will almost everyone else. And don’t get hung up on finishing books (unless you have ambitions to do serious work in an area): often you can get a lot out of a lot less than a full end-to-end reading.

So, with those preliminaries, here’s the menu for this chapter:

5.2 Before getting down to the nitty-gritty of more advanced work on mainstream mathematical logic, we first briefly return to consider modal logic again.

5.3 And then it is back to work on math. logic, but starting with an area we haven’t really looked at so far, namely proof theory.

5.4 Next, we look at more advanced model theory.

5.5–5.7 We now separate out three topics which we took together in §3.3, namely computability, Gödelian incompleteness, and theories of arithmetic.

5.8 Now it is time for a long section on serious set theory.

5.9 Set theory quickly becomes quite wildly infinitary: we now look briefly at what happens if we try to keep mathematics on a tighter rein, and restrict ourselves to constructively acceptable methods.

5.10 Finally we glance over towards category theory.

5.2 Aside: next steps beyond the modal basics

Our main focus in this chapter is going to be on the core mathematical logic curriculum, and on topics which naturally extend it. But in this first section we step aside from the main line of development to return to modal logic, which is particular interest to philosophers. Where could you go next after the readings suggested in §4.1?

This will depend on your own further concerns. One initial option is to widen your focus a bit, and (if you haven’t done so) look at a book already mentioned in §4.3.2:

1. J. C. Beall and Bas van Fraassen’s Possibilities and Paradox (OUP 2003) is ‘an introduction to modal and many-valued logics’, discussing – inter alia – the liar paradox, vagueness, and paraconsistent reasoning from inconsistent premisses. So this is an accessible book which keeps philosophical motivations very much at the front of the explorations.

Another book, much longer and more technical, which also stays at the propositional level is

2. Patrick Blackburn, Maarten de Rieke and Yde Venema’s Modal Logic (CUP, 2001). This is one of the Cambridge Tracts in Theoretical Computer Science. But don’t let that provenance put you off – indeed this book could have appeared earlier on the list. A text on propositional modal logics, it is (relatively) accessibly and agreeably written, with a lot of signposting to the reader of possible routes through the book, and interesting historical notes. I think it works pretty well, and will certainly give you an idea about how computer scientists approach modal logic (though I suppose that some of the further developments explored here do take you away from what is likely to be philosophically interesting territory).
The next option is to extend and deepen your appreciation of the quantified modal logics – including second-order logics – that have recently become central to some philosophical debates. Philosophers will want to pursue those debates and learn some technical details by diving into


If you want to further nail down the technical background here, then you will probably want to read

4. Nino B. Cocchiarella and Max A. Freund, *Modal Logic: An Introduction to its Syntax and Semantics* (OUP, 2008). The blurb announces that ‘a variety of modal logics at the sentential, first-order, and second-order levels are developed with clarity, precision and philosophical insight’. However, when I looked at this book with an eye to using it for a graduate seminar a couple of years back, I confess I didn’t find it very appealing: so I do suspect that many philosophical readers will indeed find the treatments in this book rather relentless. However, the promised wide coverage will make the book of particular interest to philosophers concerned with the issues that Williamson discusses.

Going in a different direction, if you are a philosopher interested in the relation between modal logic and intuitionistic logic, then you might want to look at

5. Alexander Chagrov and Michael Zakharyaschev *Modal Logic* (OUP, 1997). This is a volume in the Oxford Logic Guides series and again concentrates on propositional modal logics. It is probably for the more mathematically minded reader: it tackles things in an unusual order, starting with an extended discussion of intuitionistic logic, and is pretty demanding. But enthusiasts should take a look.

In yet another direction, relating to issues to do the logic of the particular modality ‘is formally provable’ (and hence relating to Gödel’s Second Incompleteness Theorem, which we’ll come back to), there’s the terrific


Postscript Three additional books to mention:

Some would say that Johan van Benthem’s *Modal Logic for Open Minds* (CSLI 2010) belongs much earlier in this Guide. But, though developed from a course intended to give ‘a modern introduction to modal logic’, it is not really routine enough in coverage and approach to serve at an elementary level. It takes up some themes relevant to computer science: worth having a look at to get an idea of how modal logic fares in the wider world.

Sally Popkorn, *First Steps in Modal Logic* (CUP, 1994). The author is, at least in this possible world, identical with the mathematician Harold Simmons. This book, which is also entirely on propositional modal logics, is another one written for computer scientists. The Introduction rather boldly says ‘There are few books on this subject and even fewer books worth looking at. None of these give an acceptable mathematically correct account of the subject. This book is a first attempt to fill that gap.’ This considerably oversells the case: but the result is still illuminating and readable.
Finally, if you want to explore even more, there’s the giant *Handbook of Modal Logic*, van Bentham et al., eds, (Elsevier, 2005). You can get an idea of what’s in the volume by looking at [http://www.csc.liv.ac.uk/~frank/MLHandbook](http://www.csc.liv.ac.uk/~frank/MLHandbook) which links to the opening pages of the contributions.

### 5.3 Proof theory

Proof theory has been (and continues to be) something of a poor relation in the standard Mathematical Logic curriculum: the usual survey textbooks don’t discuss it. Yet this is a fascinating area, of interest to philosophers, mathematicians, and computer scientists who, after all, *ought* to be concerned with the notion of proof! So let’s fill this gap next.

For an initial survey see


You should then read the little hundred-page classic


You might also profitably take an early look at the two chapters of this – regrettably, very partial – draft book, which (with a bit of judicious skipping) will help explain more about the conceptual interest of proof theory:


And if you want to follow up in depth Prawitz’s investigations of the proof theory of various systems of logic, the next place to look is

4. Sara Negri and Jan von Plato, *Structural Proof Theory* (CUP 2001). This is a modern text which goes at a reasonable pace, neither too terse, nor too labouried. (When we read it in a graduate-level reading group, however, we did find we needed to pause sometimes to stand back and think about the motivations for various technical developments. A few more ‘classroom asides’ in the text would have made a rather good text even better.)

The path now forks. Going leftwards, we can explore non-standard logics. Reflection on the structural rules of classical and intuitionistic proof systems rather naturally raises the question of what happens when we tinker with these rules. We noted before the classical inference which takes us from the trivial \( p \vdash p \) by ‘weakening’ to \( p, q \vdash p \) and on, via ‘conditional proof’, to \( p \vdash q \rightarrow p \). If we want a conditional that conforms better to intuitive constraints of relevance, then we need to block that proof: is ‘weakening’ the culprit? The investigation of what happens if we tinker with standard structural rules belongs to substructural logic, outlined in


and explored at length in the admirable

6. Greg Restall, *An Introduction to Substructural Logics* (Routledge, 2000), which will also teach you a lot more about proof theory generally in a very accessible way. Do read at least the first seven chapters.
The rightwards path on from the fork more conservatively cleaves to classical themes. In particular, if you want to look at a Gentzen-style famous proof of the consistency of arithmetic using proof-theoretic ideas, I suppose that one obvious place to go is still

7. Gaisi Takeuti, *Proof Theory* (North-Holland 1975. 2nd edn. 1987: reprinted Dover Publications 2013). This is a classic – if only because for a while it was about the only available text. Later chapters won’t really be accessible to beginners. But you could/should try reading Ch. 1 on logic, §§1–7 (and then perhaps the beginnings of §8, pp. 40–45, which is easier than it looks if you compare how you prove the completeness of a tree system of logic); then on Gentzen’s proof, read Ch. 2, §§9–11 and §12 up to p. 114. This isn’t exactly plain sailing – but if you skip and skim over some of the more tedious proof-details you can pick up a basic sense of what happens in the consistency proof.

A wonderful resource on classical themes is the editor’s own two contributions to


   Ch. 1 is a 78 pp. ‘Introduction to Proof Theory’**, which you can download from [http://www.math.ucsd.edu/~sbuss/ResearchWeb/handbookI/index.html](http://www.math.ucsd.edu/~sbuss/ResearchWeb/handbookI/index.html).


   Later chapters of the Handbook will give you pointers for exploring further.

But these do not treat so-called ordinal analysis in proof theory as initiated by Gentzen. For something about this, you could look at the opening sections of

9. Wolfram Pohlers, *Proof Theory: The First Step into Impredicativity* (Springer 2009) . This book has introductory ambitions, but in fact I would judge that it requires an amount of mathematical sophistication from its reader. From the blurb: ‘As a ‘warm up’ Gentzen’s classical analysis of pure number theory is presented in a more modern terminology, followed by an explanation and proof of the famous result of Feferman and Schütte on the limits of predicativity.’ The first half of the book is probably manageable if you already have done some of the other reading. But then the going indeed gets pretty tough.

**Postscript** Let me mention a few more books, in chronological order of publication:

Jean-Yves Girard, *Proof Theory and Logical Complexity, Vol. I* (Bibliopolis, 1987) is intended as an introduction to proof theory [Vol. II was never published]. With judicious skipping, which I’ll signpost, this is readable and insightful.

So: skip the ‘Foreword’, but do pause to glance over ‘Background and Notations’ as Girard’s symbolic choices need a little explanation. Then the long Ch. 1 is by way of an introduction, proving Gödel’s two incompleteness theorem and explaining ‘The Fall of Hilbert’s Program’: if you’ve read some of the recommendations in §3.3 above, you can probably skim this pretty quickly, just noting Girard’s highlighting of the notion of 1-consistency.

Ch. 2 is on the sequent calculus, proving Gentzen’s *Hauptsatz*, i.e. the crucial cut-elimination theorem, and then deriving some first consequences (you can probably initially omit the forty pages of annexes to this chapter). Then also omit Ch. 3 whose content isn’t relied on later. But Ch. 4 on ‘Applications of the *Hauptsatz*’ is crucial (again, however, at a first pass you can skip almost 60 pages of annexes to the chapter).
5.4 Beyond the model-theoretic basics

Back now for a few sections to the traditional core Mathematical Logic curriculum:

5.4.1 Standard model theory

If you want to explore model theory beyond the introductory material in §§2.2 and 3.2, why not start with a quick warm-up, with some reminders of headlines and pointers ahead:


Then a classic choice still remains

2. C. Chang and H. J. Keisler Model Theory* (originally North Holland 1973: the third edition has been inexpensively republished by Dover Books in 2012). This is a weighty book, over 550 pages long; but it proceeds at an engagingly leisurely pace, making it pretty accessible. It is particularly lucid and is very nicely constructed with different chapters on different methods of model-building.

There’s a very general point to be made here. Of course emphases change over time, new techniques become mainstream, etc. But still, don’t ignore the Old Classics in logic – reading them can still be wonderfully illuminating.

Awaiting us next we have the self-selecting

3. Wilfrid Hodges A Shorter Model Theory (CUP, 1997). Deservedly a modern classic – under half the length of Hodges’s encyclopedic original longer version, but full of good things. It does get fairly tough as the book progresses, but the earlier chapters should certainly be manageable.

4. Another, differently organized, book at a similar sort of level of difficulty as the earlier chapters of Hodges is Philipp Rothmaier’s Introduction to Model Theory (Taylor and Francis 2000).
Then a notch or two up again in difficulty and mathematical sophistication – and with later chapters probably going over the horizon for all but the most enthusiastic readers of this Guide – there is another book which has also quickly become something of a standard text:

5. David Marker, Model Theory: An Introduction (Springer 2002). Rightly very highly regarded. (But it isn’t published in the series ‘Graduate Texts in Mathematics’ for nothing!)

And if you can tackle Marker, then you won’t need any further Guide on model theory from me!

Postscript  It is illuminating to read something about the history of model theory. There’s a good, and characteristically readable, unpublished essay here:

Wilfrid Hodges, ‘Model Theory’, http://www.wilfridhodges.co.uk/history07.pdf

Most of the Big Books on mathematical logic have chapters on model theory. I have already mentioned

Shawn Hedman, A First Course in Logic (OUP 2004): Chs 4–6 go surprisingly far for a book describing itself as a first course in logic.

I could also mention the following:

Chs 2 and 3 of Alexander Prestel and Charles N. Delzell’s Mathematical Logic and Model Theory: A Brief Introduction (Springer 1986, 2011) are brisk but clear, and can be recommended to mathematicians. See §A.9 for a little more on this book.

Chs 1–3 of Peter G. Hinman’s doorstep Fundamentals of Mathematical Logic (A. K. Peters, 2005) present mid-level model theory in a fairly accessible way, if you are a mathematician who has already had a first encounter with first-order logic, completeness and compactness. Ch. 7 takes on the story further, but Hodges and Marker are to be preferred. See §A.16 for a little more on this book.

And then there are other general books about model theory at around Marker’s level, of which Bruno Poziat, A Course in Model Theory (Springer, 2000) is perhaps particularly notable.

But I’ll finish this subsection by mentioning two not-too-difficult books which focus in on particular elements of model theory in an attractive way:

Particularly elegant is J. L. Bell and A. B. Slomson’s Models and Ultraproducts* (North-Holland 1969; Dover reprint 2006). As the title suggests, this focuses particularly on the ultra-product construction.

There’s a short little book by Kees Doets Basic Model Theory (CSLI 1996), which concentrates on Ehrenfeucht games which could also appeal to enthusiasts.

5.4.2 A digression on finite model theory

Finite model theory arises from consideration of problems in theory of computation (where, of course, we are interested in finite structures – e.g. finite databases and finite computations over them). What happens, then, to model theory if we restrict our attention to finite models?

Trakhtenbrot’s theorem, for example, tells that the class of sentences true in any finite model is not recursively enumerable. So there is no deductive theory for capturing such finitely valid sentences. As the Wikipedia entry on the theorem wonders, isn’t it counter-intuitive that the property of being valid over all structures is ‘easier’ (can be nicely captured in a sound and complete deductive system) compared with the restricted finite case? It turns out, then, that the study of finite models is surprisingly rich and interesting (at least for enthusiasts!). So why not dip into one or other of


Either is a good standard text to explore the area with, though I prefer Libkin’s.

### 5.5 Computability

In §3.3 we took a first look at the related topics of computability, Gödelian incompleteness, and theories of arithmetic. In this and the next two main sections, we return to these topics, taking them separately (though this division is necessarily slightly artificial).

#### 5.5.1 Computable functions

A recent and very admirable introductory text is

1. Herbert E. Enderton, *Computability Theory: An Introduction to Recursion Theory* (Associated Press, 2011). This short book completes Enderton’s trilogy covering the basics of the mathematical logic curriculum – we’ve already mentioned his *A Mathematical Introduction to Logic* (1972) on first-order logic and a little model theory, and his *The Elements of Set Theory* (1972) (see §§2.2 and 3.4 above, and §A.6). And this present instalment was a candidate for being mentioned in our introductory readings in Chapter 3 instead of Epstein and Carnielli: but Enderton’s book is just a step more challenging/sophisticated/abstract so probably belongs here.

   Enderton writes here with an attractive zip and lightness of touch. The first chapter is on the informal Computability Concept. There are then chapters on general recursive functions and on register machines (showing that the register-computable functions are exactly the recursive ones), and a chapter on recursive enumerability. Chapter 5 makes Connections to Logic (including proving Tarski’s theorem on the undefinability of arithmetical truth and a semantic incompleteness theorem). The final two chapters push on to say something about Degrees of Unsolvability and Polynomial-time Computability. This is all very nicely done, and in under 150 pages too.

Enderton’s book could appeal to, and be very manageable by, philosophers. However, if you are more mathematically minded and/or you have already coped well with the basic reading on computability mentioned in §3.3 you might well prefer to jump up a level, straight to the stand-out text:

2. Nigel Cutland, *Computability: An Introduction to Recursive Function Theory* (CUP 1980). This is a rightly much-reprinted classic and is beautifully put together. This *does* have a bit more of the look-and-feel of a traditional maths text book of its time (so with fewer of Enderton’s classroom asides). However, if you go through most of Boolos and Jeffrey without too much difficulty, you ought certainly to be able to tackle this: it is not conceptually significantly more difficult. Very warmly recommended.

And of more recent general books covering similar ground, I particularly like
3. S. Barry Cooper, *Computability Theory* (Chapman & Hall/CRC 2004, 2nd edn. forthcoming 2013). This is a particularly nicely done modern textbook. Read at least Part I of the book (about the same level of sophistication as Cutland, but with some extra topics), and then you can press on as far as your curiosity takes you, and get to excitements like the Friedberg-Muchnik theorem.

Postscript I’ll mention, if only to set aside, another recent text which I found disappointing: George Tourlakis, *Theory of Computation* (Wiley 2012). Although the author has previously written a quite well-regarded double-decker logic text, and indeed can write with an engaging lightness of touch, this does not strike me as successful. The long opening Chapter 1 here on ‘Mathematical Foundations’ (90 pages) will irritate philosophically minded logicians e.g. with its seeming casualness about use and mention, and is less than ideally clear. Some of the exposition of these introductory basics – e.g. about mathematical induction – is not brilliantly handled. Things do get better when the book, at last, gets down to business in Ch.2 on ‘Algorithms, computable functions and computations’, the core of the book. But of recent books Cooper is definitely to be preferred.

Now to be positive. The inherited literature on computability is huge. But, being very selective, let me mention three classics from different generations:

Rózsa Péter, *Recursive Functions* (originally published 1950: English translation Academic Press 1967). This is by one of those logicians who was ‘there at the beginning’. It has that old-school slow-and-steady un-flashy lucidity that makes it still a considerable pleasure to read. It remains very worth looking at.

Hartley Rogers, Jr., *Theory of Recursive Functions and Effective Computability* (McGraw-Hill 1967) is a heavy-weight state-of-the-art-then classic, written at the end of the glory days of the initial development of the logical theory of computation. It quite speedily gets advanced. But the opening chapters are still excellent reading and are action-packed. At least take it out of the library, read the opening chapter or two, and admire!

Piergiorgio Odifreddi, *Classical Recursion Theory*, Vol. 1 (North Holland, 1989) is well-written and discursive, with numerous interesting historical and conceptual asides. It’s over 650 pages long, so it goes further and deeper than other books on the main list above (and then there is Vol. 2). But it certainly starts off quite gently paced and very accessibly.

A number of books we’ve already mentioned say something about the fascinating historical development of the idea of computability: Richard Epstein offers a very helpful 28 page timeline on ‘Computability and Undecidability’ at the end of the 2nd edn. of Epstein/Carnielli (which we mentioned in §3.3. Cooper’s short first chapter on ‘Hilbert and the Origins of Computability Theory’ also gives some of the headlines. Odifreddi too has many historical details. But here are two more good essays on the history:


5.5.2 Computational complexity

Computer scientists are – surprise, surprise! – interested in the theory of feasible computation, and it is certainly interesting to know at least a little about the topic of computational complexity.
1. Shawn Hedman, *A First Course in Logic* (OUP 2004), Ch. 7 on ‘Computability and complexity’ has a nice review of basic computability theory before some lucid sections introducing notions of computational complexity.

2. Michael Sipser, *Introduction to the Theory of Computation* (Thomson, 2nd edn. 2006) is a standard and very well regarded text on computation aimed at computer scientists. It aims to be very accessible and to take its time giving clear explanations of key concepts and proof ideas. I think this is very successful as a general introduction and I could well have mentioned the book before. But I’m highlighting the book in this subsection because its last third is on computational complexity.


5. You could also look at the opening chapters of the pretty encyclopaedic Sanjeev Arora and Boaz Barak, *Computational Complexity: A Modern Approach* (CUP, 2009). The authors say ‘Requiring essentially no background apart from mathematical maturity, the book can be used as a reference for self-study for anyone interested in complexity, including physicists, mathematicians, and other scientists, as well as a textbook for a variety of courses and seminars.’ And it at least starts very readably. The authors have made a late draft of the book available at http://www.cs.princeton.edu/theory/index.php/Compbook/Draft.

### 5.6 Gödelian incompleteness again

If you have looked at my book and/or Boolos and Jeffrey you should now be in a position to appreciate the terse elegance of

1. Raymond Smullyan, *Gödel’s Incompleteness Theorems*, Oxford Logic Guides 19 (Clarendon Press 1992) is delightfully short – under 140 pages – proving some beautiful, slightly abstract, versions of the incompleteness theorems. This is a modern classic which anyone with a taste for mathematical elegance will find rewarding.

2. Equally short and equally elegant is Melvin Fitting’s, *Incompleteness in the Land of Sets* (College Publications, 2007). This approaches things from a slightly different angle, relying on the fact that there is a simple correspondence between natural numbers and ‘hereditarily finite sets’ (i.e. sets which have a finite number of members which in turn have a finite number of members which in turn . . . where all downward membership chains bottom out with the empty set). [I think that, as with other books from College Publications, it has to be got via print-on-demand from Amazon.]

In terms of difficulty, these two lovely brief books could easily have appeared among our introductory readings in Chapter 3. I have put them here because (as I see it) the simpler, more abstract, stories they tell can probably only be fully appreciated if you’ve first met the basics of computability theory and the incompleteness theorems in a more conventional treatment.
You ought also at some stage read an even briefer, and still officially introductory, treatment of the incompleteness theorems,


After these, where should you go if you want to know more about matters more or less directly to do with the incompleteness theorems?

4. Raymond Smullyan, *Diagonalization and Self-Reference*, Oxford Logic Guides 27 (Clarendon Press 1994) is an investigation-in-depth around and about the idea of diagonalization that figures so prominently in proofs of limitative results like the unsolvability of the halting problem, the arithmetical undefinability of arithmetical truth, and the incompleteness of arithmetic. Read at least Part I.

5. Torkel Franzén, *Inexaustibility: A Non-exhaustive Treatment* (Association for Symbolic Logic/A. K. Peters, 2004). The first two-thirds of the book gives another take on logic, arithmetic, computability and incompleteness. The last third notes that Gödel’s incompleteness results have a positive consequence: ‘any system of axioms for mathematics that we recognize as correct can be properly extended by adding as a new axiom a formal statement expressing that the original system is consistent. This suggests that our mathematical knowledge is inexhaustible, an essentially philosophical topic to which this book is devoted.’ Not always easy (you might want to know something about ordinals before you read this), but very illuminating.

6. Per Lindström, *Aspects of Incompleteness* (Association for Symbolic Logic/As K. Peters, 2nd edn., 2003). This is for enthusiasts. Another terse book, not always reader-friendly in its choices of notation and brevity of argument, but the more mathematical reader will find that it again repays the effort.

Going in a slightly different direction, you will recall from my IGT2 or other reading on the second incompleteness theorem that we introduced the so-called derivability conditions on $\square \varphi$ (where this is an abbreviation for $\varphi$ or at any rate, is closely tied to $\neg \text{Prov}(\overline{\varphi})$, which expresses the claim that the wff $\varphi$, whose Gödel number is $\overline{\varphi}$, is provable in some given theory). The ‘$\square$’ here functions rather like a modal operator: so what is its modal logic? This is investigated in a wonderful modern classic


### 5.7 Theories of arithmetic

The readings in §3.3 will have introduced you to the canonical first-order theory of arithmetic, first-order Peano Arithmetic, as well as to some subsystems of PA (in particular, Robinson Arithmetic) and second-order extensions. And you will already know that first-order PA has non-standard models (in fact, it even has uncountably many models which can be built out of natural numbers!).

So what to read next on arithmetic? There actually seems to be something of a gap in the literature here, and a need for a mid-level book. We have to jump a couple of levels of difficulty to tackle
1. Richard Kaye’s *Models of Peano Arithmetic* (Oxford Logic Guides, OUP, 1991) which tells us a great deal about non-standard models of PA. This will reveal more about what PA can and can’t prove, and will introduce you to some non-Gödelian examples of incompleteness. But this does get pretty challenging in places (it’s probably best if you’ve already worked through some model theory at a more-than-very-basic level), though it is a terrific book.

Kaye’s book is a modern classic, as is the next suggestion:

2. Petr Hájek and Pavel Pudlák, *Metamathematics of First-Order Arithmetic*** (Springer 1993). Now freely available from projecteuclid.org. This is pretty encyclopaedic, but the long first three chapters, say, do remain remarkably accessible for such a work. This is, eventually, a must-read if you have a serious interest in theories of arithmetic and incompleteness.

And what about going beyond first-order PA? We know that full second-order PA (where the second-order quantifiers are constrained to run over all possible sets of numbers) is unaxiomatizable, because the underlying second-order logic is unaxiomatiable. But there are axiomatizable subsystems of second order arithmetic. These are wonderfully investigated in another encyclopaedic modern classic:

3. Stephen Simpson, *Subsystems of Second-Order Logic* (Springer 1999; 2nd edn CUP 2009). The focus of this book is the project of ‘reverse mathematics’ (as it has become known): that is to say, the project of identifying the weakest theories of numbers-and-sets-of-numbers that are required for proving various characteristic theorems of classical mathematics.

We know that we can reconstruct classical analysis in pure set theory, and rather more neatly in set theory with natural numbers as unanalysed ‘urelemente’. But just how much set theory is needed to do the job? The answer is: stunningly little. The project is introduced very clearly and accessibly in the first chapter, which is a must-read for anyone interested in the foundations of mathematics. This introduction can be freely downloaded from http://www.math.psu.edu/simpson/sosoa/.

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*By way of an afterthought* Many philosophers who write about the philosophy of mathematics seem to know surprisingly little ‘real’ mathematics (other than what they recall about set theory). Well, if you are interested in formal theories of arithmetic, ordinary ‘informal’ number theory is fun too, and at least the beginners’ slopes can be managed by the novice relatively easily. So why not take a look at e.g.

John Stillwell, *Elements of Number Theory* (Springer 2002). This is by a masterly expositor, and is particularly approachable.


Alan Baker, *A Comprehensive Course in Number Theory* (CUP 2012) is a nice recent textbook (shorter than its title would suggest, too).

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### 5.8 Serious set theory

§3.4 gave some suggestions for readings on the elements of set theory. These will have introduced you to the standard set theory ZFC, and the iterative hierarchy it seeks
to describe. They also explained how we can construct the real number system in set theoretic terms (so giving you a sense of what might be involved in saying that set theory can be used as a ‘foundation’ for another mathematical theory). You will have in addition learnt something about the role of the axiom of choice, and about arithmetic of infinite cardinal and ordinal numbers. If you looked at the books by Fraenkel/Bar-Hillel/Levy or by Potter, you will also have noted that standard ZFC is not the only set theory on the market.

We now press on to . . .

5.8.1 ZFC, with all the bells and whistles

One option is immediately to go for broke and dive in to the modern bible,

1. Thomas Jech, *Set Theory*, The Third Millennium Edition, Revised and Expanded (Springer, 2003). The book is in three parts: the first, Jech says, every student should know, the second every budding set-theorist should master, and the third consists of various results reflecting ‘the state of the art of set theory at the turn of the new millennium’. Start at page 1 and keep going to page 705 (or until you feel glutted with set theory, whichever comes first).

This is a masterly achievement by a great expositor. And if you’ve happily read e.g. the introductory books by Enderton and then Moschovakis mentioned earlier in this Guide, then you should be able to cope well with Part I of the book while it pushes on the story a little with some material on small large cardinals and other topics. Part II of the book starts by telling you about independence proofs. The Axiom of Choice is consistent with ZF and the Continuum Hypothesis is consistent with ZFC, as proved by Gödel using the idea of ‘constructible’ sets. And the Axiom of Choice is independent of ZF, and the Continuum Hypothesis is independent with ZFC, as proved by Cohen using the much more tricky idea of ‘forcing’. The rest of Part II tells you more about large cardinals, and about descriptive set theory. Part III is indeed for enthusiasts.

Now, Jech’s book is wonderful, but the sheer size makes it a trifle daunting: it goes quite a bit further than many will need, and to get there it does speed along a bit faster than some will feel comfortable with. So what other options are there?

You could start with some preliminary historical orientation. If you looked at the old book by Fraenkel/Bar-Hillel/Levy which was recommended in §3.4, then you will already know something of the early days: or there is a nice short overview


And then you could look through the much longer


You will probably need to skip chunks of this at a first pass: but even a partial grasp will give you a good sense of the lie of the land.

Then to start filling in details, an approachable and admired older book is

4. Frank R. Drake, *Set Theory: An Introduction to Large Cardinals* (North-Holland, 1974), which – at a gentler pace? – overlaps with Part I of Jech’s bible, but also will tell you about Gödel’s Constructible Universe and some more about large cardinals.
But the tough expositional challenge is presenting Cohen’s idea of forcing. Indeed, in the excellent

Chow writes

All mathematicians are familiar with the concept of an open research problem. I propose the less familiar concept of an open exposition problem. Solving an open exposition problem means explaining a mathematical subject in a way that renders it totally perspicuous. Every step should be motivated and clear; ideally, students should feel that they could have arrived at the results themselves. The proofs should be ‘natural’ . . . [i.e., lack] any ad hoc constructions or brilliancies. . . . I believe that it is an open exposition problem to explain forcing.

In short: if you find the idea of forcing tough to get your head around, join the club.

Here though is a very widely used and much reprinted textbook, which nicely complements Drake’s book and which has (inter alia) a pretty good presentation of forcing:

6. Kenneth Kunen, Set Theory: An Introduction to Independence Proofs (North-Holland, 1980). Again, if you have read some of the books in §3.4, you should find this pretty readily accessible, at least until you get to the penultimate chapter on forcing. Kunen has lately published another, totally rewritten, version of this book as Set Theory* (College Publications, 2011). This later book is quite significantly longer, covering an amount of more difficult material that has come to prominence since 1980. Not just because of the additional material, my sense is that the earlier book is a slightly more approachable read. But you’ll probably want to tackle the later version.

Kunen (1980) gives a ‘straight down the middle’ textbook, starting with what is basically Cohen’s original treatment of forcing, though he does relate this to some other approaches. Here are two variant approaches:

7. Raymond Smullyan and Melvin Fitting, Set Theory and the Continuum Problem* (OUP 1996, Dover Publications 2010). This medium-sized book is divided into three parts. Part I is a nice introduction to axiomatic set theory. The shorter Part II concerns matters round and about Gödel’s consistency proofs via the idea of constructible sets. Part III gives a different take on forcing (a variant of the approach taken in Fitting’s earlier Intuitionistic Logic, Model Theory, and Forcing, North Holland, 1969). This is beautifully done, as you might expect from two writers with an enviable knack for wonderfully clear explanations and an eye for elegance.

8. Keith Devlin, The Joy of Sets (Springer 1979, 2nd edn. 1993) Ch. 6 introduces the idea of Boolean-Valued Models and their use in independence proofs. The basic idea is fairly easily grasped, but details get hairy. For more on this theme, see John L. Bell’s classic Set Theory: Boolean-Valued Models and Independence Proofs (Oxford Logic Guides, OUP, 3rd edn. 2005). The relation between this approach and other approaches to forcing is discussed e.g. in Chow’s paper and the last chapter of Smullyan and Fitting.

And after those? Back to Jech’s bible, and then – oh heavens! – there is another blockbuster awaiting you:

**Historical postscript**  We’ve already noted some discussions of the history of set theory. Depending on your interests you might well find one or more of the following to be very helpful/rewarding.

- Michael Hallett, *Cantorian Set Theory and Limitation of Size* (OUP, 1984) is long and discursive but deservedly a modern classic.
- Ivor Grattan-Guinness, *The Search for Mathematical Roots 1870–1940: Logics, Set Theory and the Foundations of Mathematics from Cantor through Russell to Gödel* (Princeton 2000). As the title suggests, this even longer book ranges more widely. Grattan-Guinness has delved deep into the archives and brings to the book an astonishing amount of learning about the period, but this doesn’t always make for a fun read, and on a closer look some of the mathematical exposition isn’t that well done.
- José Ferreirós, *Labyrinth of Thought: A History of Set Theory and its Role in Modern Mathematics* (Birkhäuser 1999). (The most fun of the three?)

### 5.8.2 The Axiom of Choice

But is ZFC the ‘right’ set theory? Let’s start by thinking about the Axiom of Choice in particular. It is comforting to know from Gödel that AC is consistent with ZF (so adding it doesn’t lead to contradiction). But we also know from Cohen’s forcing argument that AC is independent with ZF (so accepting ZF doesn’t commit you to accepting AC too). So why buy AC? Is it an optional extra?

‘Well, AC is just obvious isn’t it?’ Are you really sure? Why so? (See for example Thomas Forster, ‘The Axiom of Choice and Inference to the Best Explanation’, [https://www.dpmms.cam.ac.uk/~tf/cupbook3AC.pdf](https://www.dpmms.cam.ac.uk/~tf/cupbook3AC.pdf), for a very short critical jab at some arguments that AC is ‘obvious’.)

Some of the readings already mentioned will have touched on the question of AC’s status and role. But for an overview/revision of some basics, see


For a very short book also explaining some of the consequences of AC (and some of the results that you need AC to prove), see


That already probably tells you more than enough about the impact of AC: but there’s also a famous book by H. Rubin and J.E. Rubin, *Equivalents of the Axiom of Choice* (North-Holland 1963; 2nd edn. 1985) which gives over two hundred equivalents of AC!

Next there is the nice short classic

- Thomas Jech, *The Axiom of Choice* (North-Holland 1973, Dover Publications 2008). This proves the Gödel and Cohen consistency and independence results about AC (without bringing into play everything needed to prove the parallel results about the Continuum Hypothesis). In particular, there is a nice presentation of the Fraenkel-Mostowski method of using ‘permutation models’. Then later parts of the book tell us something about what mathematics without choice, and about alternative axioms that are inconsistent with choice.

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And for a more recent short book, taking you into new territories (e.g. making links with
category theory), enthusiasts might enjoy


Finally, for more – much more! – there is a page of links to be found at


*Another historical postscript*  Again the history here is fascinating and illuminating. The classic treatment is


### 5.8.3 Alternative set theories

From earlier reading you should have picked up the idea that, although ZFC is the
canonical modern set theory, there are other theories on the market. I mention just a
selection here:

**NBG** You will have come across mention of this already (e.g. even in the early pages
of Enderton’s set theory book). And in fact – in many of the respects that matter – it
isn’t really an ‘alternative’ set theory. So let’s get it out of the way first.

We know that the universe of sets in ZFC is not itself a set. But we might think that
this universe is a *sort* of big collection. Should we explicitly recognize, then, two sorts of
collection, sets and (as they are called in the trade) proper classes which are too big to be
sets? NBG (named for von Neumann, Bernays, Gödel: some say VBG) is one such theory
of collections. So NBG in some sense recognizes proper classes, objects having members
but that cannot be members of other entities. NBG’s principle of class comprehension
is predicative; i.e. quantified variables in the defining formula can’t range over proper
classes but range only over sets, and we get a conservative extension of ZFC (nothing in
the language of sets can be proved in NBG which can’t already be proved in ZFC). For
more on this and on other theories with classes as well as sets, see (briefly) Appendix C
of the book by Potter which we mention again in a moment. Also, for a more extended
textbook presentation of NBG, see

   Ch.4.

**SP** This again is by way of reminder. Recall, earlier in the Guide, we very warmly
recommended


This presents a version of an axiomatization of set theory due to Dana Scott (hence
‘Scott-Potter set theory’). This axiomatization is consciously guided by the conception
of the set theoretic universe as built up in levels (the conception that, supposedly, also
warrants the axioms of ZF). What Potter’s book aims to reveal is that we can get a rich
hierarchy of sets, more than enough for mathematical purposes, without committing our-
selves to *all* of ZFC (whose extreme richness comes from the full Axiom of Replacement).
If you haven’t read Potter’s book before, now is the time to look at it.
ZFA (i.e. ZF − AF + AFA) Here again is the now-familiar hierarchical conception of the set universe. We start with some non-sets (maybe zero of them in the case of pure set theory). We collect them into sets (as many different ways as we can). Now we collect what we’ve already formed into sets (as many as we can). Keep on going, as far as we can. On this ‘bottom-up’ picture, the Axiom of Foundation is compelling (any downward chain linked by set-membership will bottom out, and won’t go round in a circle).

Here’s another alternative conception of the set universe. Think of a set as a gadget that points you at some some things, its members. And those members, if sets, point to their members. And so on and so forth. On this ‘top-down’ picture, the Axiom of Foundation is not so compelling. As we follow the pointers, can’t we for example come back to where we started?

It is well known that in much of the usual development of ZFC the Axiom of Foundation AF does little work. So what about considering a theory of sets which drops AF and instead has an Anti-Foundation Axiom (AFA), which allows self-membered sets?\(^1\)


NF Now for a much more radical departure from ZF. Standard set theory lacks a universal set because, together with other standard assumptions, the idea that there is a set of all sets leads to contradiction. But by tinkering with those other assumptions, there are coherent theories with universal sets. For very readable presentations concentrating on Quine’s NF (‘New Foundations’), and explaining motivations as well as technical details, see


IST Leibniz and Newton invented infinitesimal calculus in the 1660s: a century and a half later we learnt how to rigorize the calculus without invoking infinitely small quantities. Still, the idea of infinitesimals retains a certain intuitive appeal, and in the 1960s, Abraham Robinson created a theory of hyperreal numbers; this yields a rigorous formal treatment of infinitesimal calculus (you will have seen this mentioned in e.g. Enderton’s Mathematical Introduction to Logic, §2.8, or van Dalen’s Logic and Structure,

\(^1\)ZFA’ is sometimes used to label a theory of this kind: but careful, at least as often the label means ZF with atoms, i.e. with urelemente, i.e. it means ZFU.
Later, a simpler and arguably more natural approach, based on so-called Internal Set Theory, was invented by Edward Nelson. As Wikipedia put it, ‘IST is an extension of Zermelo-Fraenkel set theory in that alongside the basic binary membership relation, it introduces a new unary predicate ‘standard’ which can be applied to elements of the mathematical universe together with some axioms for reasoning with this new predicate.’ Starting in this way we can recover features of Robinson’s theory in a simpler framework.


10. Nader Vakin, Real Analysis through Modern Infinitesimals (CUP, 2011). A monograph developing Nelson’s ideas whose early chapters are quite approachable and may well appeal to some.

Aside If we are going to talk about non-standard analysis, we really must also somewhere mention


   John L. Bell, A Primer of Infinitesimal Analysis (CUP 2nd edn. 2008), a lovely, and very short, book, presenting a theory based on quite different framework from IST, this time one connected to topos theory: see below.

ETCS Famously, Zermelo constructed his theory of sets by gathering together some principles of set-theoretic reasoning that seemed actually to be used by working mathematicians (engaged in e.g. the rigorization of analysis or the development of point set topology), hoping to get a theory strong enough for mathematical use while weak enough to avoid paradox. But does he overshoot? We’ve already noted that SP is a weaker theory which may suffice. For a more radical approach, see


12. F. William Lawvere and Robert Rosebrugh, Sets for Mathematicians (CUP 2003) gives a very accessible presentation which in principle doesn’t require that you have already done any category theory.

But perhaps to fully appreciate what’s going on, you will have to go on to dabble in category theory (see §5.10 of this Guide!).

IZF, CZF ZF/ZFC has a classical logic: what if we change the logic to intuitionistic logic? what if we have more general constructivist scruples? The place to start exploring is


Then for one interesting possibility, look at the version of constructive ZF in

Yet more? Well yes, we can keep on going. Take a look, for example, at SEAR (http://ncatlab.org/nlab/show/SEAR). But we must call a halt! For a brisk overview, putting many of these various set theories into some sort of order, and mentioning yet further alternatives, see


If that’s a bit too brisk, then (if you can get access to it) there’s what can be thought of as a bigger, better, version here:


5.9 Constructivism

At the end of the last section on set theory, we briefly mentioned intuitionist and constructivist set theories. Let’s now think about intuitionism and constructivism more generally. Again, the *Stanford Encyclopedia* is there to help us to make a start. Neither of the following two pieces is perhaps ideally accessible, but they are worth skimming to get a first sense of some issues:


And one thing will immediately become clear: there is quite a variety of approaches to constructive mathematics, broadly understood. For more on these, see


On intuitionistic mathematics in particular (picking up from the reading on intuitionist logic mentioned in §4.3.1), see

4. Michael Dummett, *Elements of Intuitionism*, Oxford Logic Guides 39 (OUP 2nd edn. 2000). Another classic – but (it has to be said) quite tough. The final chapter, ‘Concluding philosophical remarks’, is very well worth looking at, even if you bale out from reading all the formal work that precedes it.

On Bishop’s form of constructivism, read the man himself:

5. Errett Bishop and Douglas Bridges, *Constructive Analysis* (Springer 1985). Do read the first two or three lucid chapters which will give you the flavour of the enterprise.

*Postscript* Another line of development of constructivist thought, of particular interest to computer science, is constructivist type theory. Per Martin-Löf’s 1984 lectures on *Intuitionistic Type
5.10 Category theory

Last, but certainly not least, we come to category theory. Philosophers and mathematicians alike will have probably come across claims that category theory – or that division of it which is topos theory – provides a new foundation (or a different sort of foundation) to mathematics, in some sense rivalling set theory in its sweep and generality. So philosophers and mathematicians with foundational interests may well want to know what the fuss is about.

Now, there are various routes in towards category theory, and the best route to take will depend on your background. There is a canonical book by Saunders Mac Lane, *Categories for the Working Mathematician* (Springer 2nd edn 1997): but the ‘working mathematician’ here is assumed to be an already seriously high-powered player in the maths game (not the sort of person likely to be needing to read this Guide!). So set that book aside as a far ambition: the initial suggestions here are directed to a very different audience, philosophers and mathematicians without a high level background in topology etc.

There is, in fact, a brisk introductory encyclopaedia article notionally addressed to philosophers,

1. Jean-Pierre Marquis, ‘Category Theory’, *The Stanford Encyclopedia of Philosophy*: http://plato.stanford.edu/entries/category-theory/, but I suspect most beginners won’t be very much enlightened, as this is already pretty abstract. So let’s start further back and note some particularly introductory and expansive books that I found helpful in trying to teach myself just a bit of category theory a few years ago. I’ll put them into what strikes me – in retrospect – in something like order of difficulty:

2. F. William Lawvere and Stephen H. Schanuel, *Conceptual Mathematics: A First Introduction to Categories* (CUP 2nd edn. 2009). A gentle introduction that slowly but surely introduces you to new categorial ways of thinking about some familiar things. The second edition is notably better than the first because it adds chapters that ease the transitions to more advanced topics.

3. We’ve already mentioned F. William Lawvere and Robert Rosebrugh, *Sets for Mathematicians* (CUP 2003) in §5.8.3 on alternative sets theories. It gives a very accessible presentation which introduces a different way of thinking about sets.

4. Robert Goldblatt, *Topoi: The Categorial Analysis of Logic* (North-Holland 1979, Dover 2009). Again pretty gentle and extremely lucid, and covers a good deal of ground. (Goldblatt’s history of the origins of category theory is arguably unreliable, but the mathematical exposition is very nicely done.)

5. Harold Simmons, *An Introduction to Category Theory* (CUP, 2011) is also introductory, and written for self-study with oodles of exercises (and the solutions available online). Has something of the conversational tone of the lecture room, and you could well find it engaging and helpful.
Certainly read (2) and (3) before going up a notch to one of the next two two books which are again intended for beginners, but somewhat tougher affairs, being brisker and more compressed. Indeed, I wouldn’t recommend diving into either from a cold start: the shock to the system will mean that you probably won’t get very far. However, if you have warmed up with some of those really introductory texts, then both these are excellent:


Read them both (in the order that works for you), and you will be well launched and can explore further under your own steam.

At a similar sort of level you will find e.g. the volumes of Francis Borceux’s *Handbook* and the strange provocations of Paul Taylor’s *Practical Foundations*. And then down the road you can now just about catch sight Saunders Mac Lane and Ieke Moerdijk’s *Sheaves in Geometry and Logic* and the looming presence of Peter Johnstone’s monumental *Elephant*. But enough already . . .
Appendix A

The Big Books – and not so big books – on Mathematical Logic

The traditional menu for a first serious Mathematical Logic course is basic first order logic with some model theory, the basic theory of computability and related matters (like Gödel’s incompleteness theorem), and introductory set theory. The plan in this Appendix is to look at some of the Big (or sometimes not so big) Books that aim to cover an amount of this core menu, to give an indication of what they cover and – more importantly – how they cover it, while commenting on style, accessibility, etc.

These wider-ranging books don’t always provide the best introductions now available to this or that particular area: but they can still be very useful aids to widening and deepening your understanding and can reveal how topics from different areas fit together. I don’t promise to discuss [eventually] every worthwhile Big Book, or to give similarly detailed coverage to those I do consider. But I’m working on the principle that even a patchy guide, very much work in progress, is much better than none.

Sections start with a general indication of the coverage of the book, and some general remarks. Then there is usually, in small print, a more detailed description of contents, and more specific comments. Finally, there’s a summary verdict. The books are listed in chronological order of first publication. Covered so far are:

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A.1 Kleene, 1952

First published sixty years ago, Stephen Cole Kleene's *Introduction to Metamathematics* (North-Holland, 1962; reprinted Ishi Press 2009: pp. 550) for a while held the field as a survey treatment of first-order logic (without going much past the completeness theorem), and a more in-depth treatment of the theory of computable functions, and Gödel's incompleteness theorems.

In a 1991 note about writing the book, Kleene notes that up to 1985, about 17,500 copies of the English version of his text were sold, as were thousands of various translations (including a sold-out first print run of 8000 of the Russian translation). So this is a book with a quite pivotal influence on the education of later logicians, and on their understanding of the fundamentals of recursive function theory and the incompleteness theorems in particular.

But it isn’t just nostalgia that makes old hands continue to recommend it. Kleene’s book remains particularly lucid and accessible: it is often discursive, pausing to explain the motivation behind formal ideas. It is still a pleasure to read (or at least, it ought to be a pleasure for anyone interested in logic enough to be reading this far into the Guide!). And, modulo relatively superficial presentational matters, you’ll probably be struck by a sense of familiarity when reading through, as aspects of his discussions evidently shape many later textbooks (not least my own Gödel book). *The Introduction to Metamathematics* remains a really impressive achievement: and not one to be admired only from afar, either.

Some details  Chs. 1–3 are introductory. There’s a little about enumerability and countability (Cantor’s Theorem); then a chapter on natural numbers, induction, and the axiomatic method; then a little tour of the paradoxes, and possible responses.

Chs. 4–7 are a gentle introduction to the propositional and predicate calculus and a formal system which is in fact first-order Peano Arithmetic (you need to be aware that the identity rules are treated as part of the arithmetic, not the logic). Although Kleene’s official system is Hilbert-style, he shows that ‘natural deduction’ introduction and elimination rules can be thought of as derived rules in his system, so it all quickly becomes quite user-friendly. (He doesn’t at this point prove the completeness theorem for his predicate logic: as I said, things go quite gently at the outset!)

Ch. 8 starts work on ‘Formal number theory’, showing that his formal arithmetic has nice properties, and then defines what it is for a formal predicate to capture (‘numeralwise represent’) a numerical relation. Kleene then proves Gödel’s incompleteness theorem, assuming a Lemma – eventually to be proved in his Chapter 10 – about the capturability of the relation ‘m numbers a proof [in Kleene’s system] of the sentence with number n’.

Ch. 9 gives an extended treatment of primitive recursive functions, and then Ch. 10 deals with the arithmetization of syntax, yielding the Lemma needed for the incompleteness theorem.

Chs. 11-13 then give a nice treatment of general (total) recursive functions, partial recursive functions, and Turing computability. This is all very attractively done.

The last two chapters, forming the last quarter of the book, go under the heading ‘Additional Topics’. In Ch. 14, after proving the completeness theorem for the predicate calculus without and then with identity, Kleene discusses the decision problem. And the final Ch. 15 discusses Gentzen systems, the normal form theorem, intuitionistic systems and Gentzen’s consistency proof for arithmetic.
Summary verdict  Can still be warmly recommended as an enjoyable and illuminating presentation of this fundamental material, written by someone who was himself so closely engaged in the early developments back in the glory days. It should be entirely accessible if you have managed e.g. Chiswell and Hodges and then my Gödel book, and will enrich and broaden your understanding.

A.2  Mendelson, 1964, 2009

Elliot Mendelson’s Introduction to Mathematical Logic (Van Nostrand, 1964: pp. 300) was first published in the distinguished and influential company of The University Series in Undergraduate Mathematics. It has been much used in graduate courses for philosophers since: a 4th edition was published by Chapman Hall in 1997 (pp. 440), with a slightly expanded 5th edition being published in 2009. I will here compare the first and fourth editions, as these are the ones I know.

Even in the later editions this isn’t, in fact, a very big Big Book (many of the added pages of the later editions are due to there now being answers to exercises): the length is kept under control in part by not covering a great deal, and in part by a certain brisk terseness. As the Series title suggests, the intended level of the book is upper undergraduate mathematics, and the book does broadly keep to that aim. Mendelson is indeed pretty clear; however, his style is of the times, and will strike many modern readers as dry and rather old-fashioned. (Some of the choices of typography are not wonderfully pretty either, and this can make some pages look as if they will be harder going than they really turn out to be.)

Some details  After a brief introduction, Ch. 1 is on the propositional calculus. It covers semantics first (truth-tables, tautologies, adequate sets of connectives), then an axiomatic proof system. The treatments don’t change much between editions, and will probably only be of interest if you’ve never encountered a Hilbert-style axiomatic system before. The fine print of how Mendelson regards his symbolic apparatus is interesting: if you read him carefully, you’ll see that the expressions in his formal systems are not sentences, not expressions of the kind that – on interpretation – can be true or false – but are schemata, what he calls statement forms. But this relatively idiosyncratic line about how the formalism is to be read, which for a while (due to Quine’s influence) was oddly popular among philosophers, doesn’t much affect the development.

Ch. 2 is on quantification theory, again in an axiomatic style. The fourth edition adds to the end of the chapter more sections on model theory: there is a longish section on ultra-powers and non-standard analysis, then there’s (too brief) a nod to semantic trees, and finally a new discussion of quantification allowing empty domains. The extra sections in the fourth edition are a definite bonus: without them, there is nothing special to recommend this chapter, if you have worked through the suggestions in §2.2, and in particular the chapters in van Dalen’s book.

Ch. 3 is titled ‘Formal number theory’. It presents a formal version of first-order Peano Arithmetic, and shows you can prove some expected arithmetic theorems within it. Then Mendelson defines the primitive recursive and the (total) recursive functions, shows that these are representable (capturable) in PA. It then considers the arithmetization of syntax, and proves Gödel’s first incompleteness theorem and Rosser’s improvement. The chapter then proves Church’s Theorem about the decidability of arithmetic. One difference between editions is that the later proof of Gödel’s theorem goes via the Diagonalization Lemma; another is that there is added a brief treatment of Löb’s Theorem. At the time of publication of the original addition, this Chapter was a quite exceptionally useful guide thorough the material. But now – at least if you’ve read my Gödel book or the equivalent – then there is nothing to divert you here, except that Mendelson does go through every single stage of laboriously showing that the relation $m$-numbers-a-PA-proof-of-the-sentence-numbered-$n$ is primitive recursive.
Ch. 4 is on set theory, and – unusually for a textbook – the system presented is NBG (von Neumann/Bernays/Gödel) rather than ZF(C). In the first edition, this chapter is under fifty pages, and evidently the coverage can’t be very extensive and it also probably goes too rapidly for many readers. The revised edition doesn’t change the basic treatment (much) but adds sections comparing NBG to a number of other set theories. So while this chapter certainly can’t replace the introductions to set theory recommended in §3.4, it could be worth skimming briskly through the chapter in later editions to learn about NBG and other deviations from ZF.

The original Ch. 5 on effective computability starts with a discussion of Markov algorithms (again, unusual for a textbook), then treats Turing algorithms, then Herbrand-Gödel computability and proves the equivalence of the three approaches. There are discussions of recursive enumerability and of the Kleene-Mostowski hierarchy. And the chapter concludes with a short discussion of undecidable problems. In the later edition, the material is significantly rearranged, with Turing taking pride of place and other treatments of computability relegated to near the end of the chapter; also more is added on decision problems. Since the texts mentioned in §3.3 don’t talk about Markov or Herbrand-Gödel computability, you might want to dip into the chapter briefly to round out your education!

I should mention the appendices. The first edition has a very interesting though brisk appendix giving a version of Schütte’s variation on a Gentzen-style consistency proof for PA. Rather sadly, perhaps, this is missing from later editions. The fourth edition has instead an appendix on second-order logic.

Summary verdict Moderately accessible and very important in its time, but there is now not so much reason to plough through this book end-to-end. It doesn’t have the charm and readability of Kleene 1952, and there are now better separate introductions to each of the main topics. You could skim the early chapters if you’ve never seen axiomatic systems of logic being used in earnest: it’s good for the soul. The appendix that appears only in the first edition is interesting for enthusiasts. Look at the section on non-standard analysis in the revised editions. If set theory is your thing, you should dip in to get the headline news about NBG. And some might want to expand their knowledge of definitions of computation by looking at Ch. 5.

A.3 Shoenfield, 1967

Joseph R. Shoenfield’s Mathematical Logic (Addison-Wesley, 1967: pp. 334) is officially intended as ‘a text for a first-year [maths] graduate course’. It has, over the years, been much recommended and much used (a lot of older logicians first learnt their serious logic from it).

This book, however, is hard going – a significant step or two up in level from Mendelson – though the added difficulty in mode of presentation seems to me not always to be necessary. I recall it as being daunting when I first encountered it as a student. Looking back at the book after a very long time, and with the benefit of greater knowledge, I have to say I am not any more enticed: it is still a tough read.

So this book can, I think, only be recommended to hard-core mathmos who already know a fair amount and can cherry-pick their way through the book. It does have heaps of hard exercises, and some interesting technical results are in fact buried there. But whatever the virtues of the book, they don’t include approachability or elegance or particular student-friendliness.

Some details Chs. 1–4 cover first order logic, including the completeness theorem. It has to be said that the logical system chosen is rebarbative. The primitives are ¬, ∨, 3, and =. Leaving aside the identity axioms, the axioms are the instances of excluded middle, instances of ϕ(τ) → 3ξϕ(ξ), and then there are five rules of inference. So this neither has the cleanness of a Hilbert system
nor the naturalness of a natural deduction system. Nothing is said to motivate this seemingly horrible choice as against others.

Ch. 5 is a brisk introduction to some model theory getting as far as the Ryll-Nardjewski theorem. I believe that the algebraic criteria for a first-order theory to admit elimination of quantifiers given here are original to Shoenfield. But this is surely all done very rapidly (unless you are using it as a terse revision course from quite an advanced base, going beyond what you will have picked up from the reading suggested in our §3.2 above).

Chs. 6–8 cover the theory of recursive functions and formal arithmetic. The take-it-or-leave-it style of presentation continues. Shoenfield defines the recursive functions as those got from an initial class by composition and regular minimization: again, no real motivation for the choice of definition is given (and e.g. the definition of the primitive recursive functions is relegated to the exercises). Unusually for a treatment at this sort of level, the discussion of recursion theory in Ch. 8 goes far enough to cover a Gödelian ‘Dialectica’-style proof of the consistency of arithmetic, though the presentation once more wins no prizes for accessibility.

Ch. 9 on set theory is perhaps the book’s real original raison d’être; in fact, it is a quarter of the whole text. The discussion starts by briskly motivating the ZF axioms by appeal to the conception of the set universe as built in stages (an approach that has become very common but at the time of publication was I think much less usually articulated); but this isn’t the place to look for an in depth development of that idea. For a start, there is Shoenfield’s own article ‘The axioms of set theory’, Handbook of mathematical logic, ed. J. Barwise, (North-Holland, 1977) pp. 321–344.

We get a brusque development of the elements of set theory inside ZF (and then ZFC), and something about the constructible universe. Then there is the first extended textbook presentation of Cohen’s 1963 independence results via forcing, published just four years previous to the publication of this book: set theory enthusiasts might want to look at this to help round out their understanding of the forcing idea. The discussion also touches on large cardinals.

This last chapter was in some respects a highly admirable achievement in its time: but it is equally surely not now the best place to start with set theory in general or forcing in particular, given the availability of later presentations.

Summary verdict This is pretty tough going. Now surely only for very selective dipping into by already-well-informed enthusiasts.

A.4 Kleene, 1967

In the preface to his Mathematical Logic* (John Wiley 1967, Dover reprint 2002: pp. 398), Stephen Cole Kleene writes

After the appearance in 1952 of my Introduction to Metamathematics, written for students at the first-year graduate level, I had no expectation of writing another text. But various occasions arose which required me to think about how to present parts of the same material more briefly, to a more general audience, or to students at an earlier educational level. These newer expositions were received well enough that I was persuaded to prepare the present book for undergraduate students in the Junior year.

You’d expect, therefore, that this later book would be more accessible, a friendlier read, than Kleene’s remarkable IM. But in fact, this doesn’t actually strike me as the case. I’d still recommend reading the older book, augmented by one chapter of this later ‘Little Kleene’. To explain:

Some details The book divides into two parts. The first part, ‘Elementary Mathematical Logic’ has three chapters. Ch. 1 is on the propositional calculus (including a Kalmár-style completeness proof). This presents a Hilbert-style proof system with an overlay of derived rules which look
rather natural-deduction-like (but aren’t the real deal) There is a lot of fussing over details in rather heavy-handed ways. I couldn’t recommend anyone nowadays starting here, while if you’ve already read a decent treatment of the propositional calculus (and e.g. looked at Mendelson to see how things work in a Hilbert-style framework) you won’t get much more out of this.

Much the same goes for the next two chapters. Ch. 2 gives an axiomatic version of the predicate calculus without identity, and Ch. 3 adds identity. (Note, a completeness proof doesn’t come to the final chapter of the book). Again, these chapters are not done with a sufficiently light touch to make them a particularly attractive read now.

The second part of the book is titled ‘Mathematical Logic and the Foundations of Mathematics’. Ch.4 is basically an abridged version of the opening three chapters of IM, covering the paradoxes, the idea of an axiomatic system, introducing formal number theory. You might like to read in particular §§36–37 on Hilbert vs. Brouwer and ‘metamathematics’.

Ch. 5 is a sixty page chapter on ‘Computability and Decidability’. Kleene is now on his home ground, and he presents the material (some original to him) in an attractive and illuminating way, criss-crossing over some of the same paths trodden in later chapters of IM. In particular, he uses arguments for incompleteness and undecidability turning on use of the Kleene T-predicate (compare §33.7 of IGT1 or §43.8 of IGT2). This chapter is certainly worth exploring.

Finally, the long Ch. 6 proves the completeness theorem for predicate logic by Beth/Hintikka rather than by Henkin (as we would now think of it, he in effect shows the completeness of a tree system for logic in the natural way). But nicer versions of this approach are available. The last few sections cover some supplementary material (on Gentzen systems, Herbrand’s Theorem, etc.) but again I think all of it is available more accessibly elsewhere.

**Summary verdict**
Do read Chapter 5 on computability, incompleteness, decidability and closely related topics. This is nicely done, complements Kleene’s earlier treatment of the same material, and takes an approach which is interestingly different from what you will mostly see elsewhere.

### A.5 Robbin, 1969

Joel W. Robbin’s *Mathematical Logic: A First Course* (W. A Benjamin, 1969, Dover reprint 2006: pp. 212) is not exactly a ‘Big Book’. The main text is just 170 pages long. But it does range over both formal logic (first-order and second-order), and formal arithmetic, primitive recursive functions, and Gődelian incompleteness. Robbin, as you might guess, has to be quite brisk (in part he achieves brevity by leaving a lot of significant results to be proved as more or less challenging exercises). However, the book remains approachable and has some nice and unusual features for which it can be recommended.

**Some details**
Ch. 1 is on the propositional calculus. Robbin presents an axiomatic system whose primitives are → and ⊥ – or rather, in his notation, ⊨ and ⊢. The system, including the ‘dotty’ syntax which gives us wffs like \( p_1 \vdash p_2 \), \( p_1 \vdash f \), is a version of Alzono Church’s system in his *Introduction to Mathematical Logic*, Vol. 1 (1944/1956), except that where Church lays down three specific wffs as axioms and has a substitution rule for deriving variant wffs, Robbin lays down three axiom schemas. [Perhaps I should say something about Church’s classic book in this Guide: but that’s for another day.]

As in later chapters, Robbin buries some interesting results in the extensive exercises. Here’s one, pointed out to me by David Auerbach. Robbin defines negation in the obvious way from his two logical primitives, so that \( \sim \varphi =_{\text{def}} (\varphi \rightarrow f) \). And then his three axiom schemas can all be stated in terms of ⊨ and \( \sim \), and his one rule is modus ponens. This system is complete. However, if we take the alternative language with ⊨ and \( \sim \) *primitive*, then the same deductive system (with the same axioms and rules) is not complete. That’s a nice little surprise, and it is worth trying to work out just why it is true.
Ch. 2 briefly covers first-order logic, including the completeness theorem. Then Ch. 3 introduces what Robbin calls ‘First-order (Primitive) Recursive Arithmetic’ (RA). Robbin defines the primitive recursive functions, and then defines a language which has a function expression for each p.r. function \( f \) (the idea is to have a complex function expression built up to reflect a full definition of \( f \) by primitive recursion and/or composition ultimately in terms of the initial functions). RA has axioms for the logic plus axioms governing the expressions for the initial functions, and then there are axioms for dealing with complex functional expressions in terms of their constituents. RA also has all instances of the induction schema for open wffs of the language (so – for cognoscenti – this is a stronger theory than what is usually called Primitive Recursive Arithmetic these days, which normally has induction only for quantifier-free wffs).

Ch. 4 explores the arithmetization of syntax of RA. Since RA has every p.r. function built in, we don’t then have to go through the palaver of showing that we are dealing with a theory which can represent all p.r. functions (in the way we have to if we take standard PA as our base theory of interest). So in Ch. 5 Robbin can prove Gödel’s incompleteness theorem for RA in a more pain-free way.

Ch. 6 then turns to second-order logic, introduces a version of second-order PA\(_2\) with just the successor relation as primitive non-logical vocabulary. Robbin shows that all the p.r. functions can be explicitly defined in PA\(_2\), so the incompleteness theorem carries over.

**Summary verdict** Robbin’s book offers a different route through a rather different selection of material than is usual, accessibly written and still worth reading (you will be able to go through quite a bit of it pretty rapidly if you are up to speed with the relevant basics from this Guide’s §2.2 and §3.3). Look especially at Robbin’s Ch. 3 for the unusually detailed story about how to build a language with a function expression for every p.r. function, and the last chapter for how in effect to do the same in PA\(_2\).

### A.6 Enderton, 1972, 2002

The first edition of Herbert B. Enderton’s *A Mathematical Introduction to Logic* (Academic Press, 1972; pp. 295) rapidly established itself as much-used textbook among the mathematicians it was aimed towards. But it has also been used to in math. logic courses offered to philosophers. A second edition was published in 2002, and a glance at the section headings indicates much the same overall structure: but there are many local changes and improvements, and I’ll comment here on this later version of the book (which by now should be equally widely available in libraries). The author died in 2010, but his webpages live on, including one with his own comments on his second edition: [http://www.math.ucla.edu/~hbe/](http://www.math.ucla.edu/~hbe/).

Enderton’s text deals with first order-logic and a smidgin of model theory, followed by a look at formal arithmetic, recursive functions and incompleteness. A final chapter covers second-order logic and some other matters.

*A Mathematical Introduction to Logic* eventually became part of a logical trilogy, with the publication of the wonderfully lucid *Elements of Set Theory* (1977) and *Computability Theory* (2010). The later two volumes strike me as masterpieces of exposition, providing splendid ‘entry level’ treatments of their material. The first volume, by contrast, is *not* the most approachable first pass through its material. It is good (often *very* good), but I’d say at a notch up in difficulty from what you might be looking for in an *introduction* to the serious study of first-order logic and/or incompleteness.

**Some details** After a brisk Ch. 0 (‘Some useful facts about sets’, for future reference), Enderton starts with a 55 page Ch. 1, ‘Sentential Logic’. Some might think this chapter to be slightly odd. For the usual motivation for separating off propositional logic and giving it an extended treatment
at the beginning of a book at this level is that this enables us to introduce and contrast the key ideas of semantic entailment and of provability in a formal deductive system, and then explain strategies for soundness and completeness proofs, all in a helpfully simple and uncluttered initial framework. But (except for some indications in final exercises) there is no formal proof system mentioned in Enderton’s chapter.

So what does happen in this chapter? Well, we do get a proof of the expressive completeness of \( \{\wedge, \vee, \neg\} \), etc. We also get an exploration (which can be postponed) of the idea of proofs by induction and the Recursion Theorem, and based on these we get proper proofs of unique readability and the uniqueness of the extension of a valuation of atoms to a valuation of a set of sentences containing them (perhaps not the most inviting things for a beginner to be pausing long over). We get a direct proof of compactness. And we get a first look at the ideas of effectiveness and computability.

The core Ch. 2, ‘First-Order Logic’, is over a hundred pages long, and covers a good deal. It starts with an account of first-order languages, and then there is a lengthy treatment of the idea of truth in a structure. This is pretty clearly done and mathematicians should be able to cope quite well (but does Enderton forget his officially intended audience on p. 83 where he throws in an unexplained commutative diagram?!). Still, readers might sometimes appreciate rather more explanation (for example, surely it would be worth saying a bit more than that ‘In order to define \( \sigma \) is true in \( A \)’ for sentences \( \sigma \) and structures \( A \), we will find it desirable [sic] first to define a more general concept involving wffs’, i.e. satisfaction by sequences).

Enderton then at last introduces a deductive proof system (110 pages into the book). He chooses a Hilbert-style presentation, and if you are not already used to such a system, you won’t get much of a feel for how they work, as there are very few examples before the discussion turns to metatheory (even Mendelson’s presentation of a similar Hilbert system is here more helpful). Then, as you’d expect, we get the soundness and completeness theorems. The proof of the latter by Henkin’s method is nicely chunked up into clearly marked stages, and again a serious mathematics student should cope well: but this is still not, I think, a ‘best buy’ among initial presentations.

The chapter ends with a little model theory – compactness, the LS theorems, interpretations between theorems – all rather briskly done, and there is an application to the construction of infinitesimals in non-standard analysis which is surely going to be too compressed for a first encounter with the ideas.

Ch. 3, ‘Undecidability’, is also a hundred pages long and again covers a great deal. After a preview introducing three somewhat different routes to (versions of) Gödel’s incompleteness theorem, we initially meet:

1. A theory of natural numbers with just the successor function built in (which is shown to be complete and decidable, and a decision procedure by elimination of quantifiers is given).
2. A theory with successor and the order relation (also shown to admit elimination of quantifiers and to be complete).
3. Presburger arithmetic (shown to be decidable by a quantifier elimination procedure, and shown not to define multiplication)
4. Robinson Arithmetic with exponentiation.

The discussion then turns to the notions of definability and representability. We are taken through a long catalogue of functions and relations representable in Robinson-Arithmetic-with-exponentiation, including functions for encoding and decoding sequences. Next up, we get the arithmetization of syntax done at length, leading as you’d expect to the incompleteness and undecidability results.

But we aren’t done with this chapter yet. We get (sub)sections on recursive enumerability, the arithmetic hierarchy, partial recursive functions, register machines, the second incompleteness theorem for Peano Arithmetic, applications to set theory, and finally we learn how to use the \( \beta \)-function trick so we can get take our results to apply to any nicely axiomatized theory containing plain Robinson Arithmetic.

As is revealed by that quick description there really is a lot in Ch. 3. To be sure, the material here is not mathematically difficult in itself (indeed it is one of the delights of this area that the initial Big Results come so quickly). However, I do doubt that such an action-packed
presentation is the best way to first meet this material. It would, however, make for splendid revision-consolidation-extension reading after tackling e.g. my Gödel book.

The final Ch. 4 is much shorter, on ‘Second-Order Logic’. This goes very briskly at the outset. It again wouldn’t be my recommended introduction for this material, though it could make useful supplementary reading for those wanting to get clear about the relation between second-order logic, Henkin semantics, and many-sorted first-order logic.

Summary verdict To repeat, A Mathematical Introduction to Logic is good in many ways, but is – in my view – often a step or two more difficult in mode of presentation than will suit many readers wanting an introduction to the material it covers. However, if you have already read an entry-level presentation of first order logic (e.g. Chiswell/Hodges) then you could read Chs 1 and 2 as revision/consolidation. And if you have already read an entry-level presentation on incompleteness (e.g. my book) then it could be well worth reading Ch. 3 as bringing the material together in a somewhat different way.

A.7 Ebbinghaus, Flum and Thomas, 1978, 1994

We now turn to consider H.-D. Ebbinghaus, J. Flum and W. Thomas, Mathematical Logic (Springer, 2nd end. 1994: pp. 289). This is the English translation of a book first published in German in 1978, and appears in a series ‘Undergraduate Texts in Mathematics’, which indicates the intended level: parts of the book, however, do seem more than a little ambitious for most undergraduates.

EFT’s book is often warmly praised and is (I believe) quite widely used. But revisiting it, I can’t find myself wanting to recommend it as a good place to start, certainly not for philosophers but I wouldn’t recommend it for mathematicians either. The presentation of the core material on the syntax and semantics of first-order logic in the first half of the book is done more accessibly and more elegantly elsewhere. In the second half of the book, the chapters do range widely across interesting material. But again most of the discussions will go too quickly if you haven’t encountered the topics before, and – if you want revision/amplification of what you already know – you will mostly do better elsewhere.

Some details The book is divided into two parts. EFT start Part A with a gentle opening chapter talking about a couple of informal mathematical theories (group theory, the theory of equivalence relations), giving a couple of simple informal proofs in those theories. They then stand back to think about what goes on in the proofs, and introduce the project of formalization. So far, so good.

Ch. 2 describes the syntax of first-order languages, and Ch. 3 does the semantics. The presentation goes at a fairly gentle pace, with some useful asides (e.g. on handling the many-sorted languages of informal mathematics using a many-sorted calculus vs. use restricted quantifiers in a single-sorted calculus). EFT though do make quite heavy work of some points of detail.

Ch. 4 is called ‘A Sequent Calculus’. The version chosen is really, really, not very nice. For a start (albeit a minor point that only affects readability), instead of writing a sequent as ‘Γ ⊢ ϕ’, or ‘Γ ⇒ ϕ’, or even ‘Γ: ϕ’, EFT just write an unpunctuated ‘Γ ϕ’. Much more seriously, they adopt a system of rules which many would say mixes up structural rules and classical logical rules for the connectives in an unprincipled way (thereby losing just the insights that a sequent system can be used to highlight). [To be more specific, they introduce a classical ‘Proof by Cases’ rule that takes us from the sequents (in their notation) Γ ψ ϕ and Γ ¬ψ ϕ to Γ ϕ, and then appeal to Proof by Cases to get Cut as a derived rule. This really muddies the waters in various ways!]

Ch. 5 gives a Henkin completeness proof for first-order logic. For my money, there’s too much symbol-bashing and not enough motivating chat here. (I don’t think it is good exegetical policy
to complicate matters as EFT do by going straight for a proof for the predicate calculus with identity: but they are not alone in this.)

Ch. 6 is briskly about The Löwenheim-Skolem Theorem, compactness, and elementarily equivalent structures (but probably fine if you’ve met this stuff before).

Ch. 7, ‘The Scope of First-Order Logic’ is really rather odd. It briskly argues that first-order logic is the logic for mathematics (readers of Shapiro’s book on second-order logic won’t be so quickly convinced). The reason given is that we can reconstruct (nearly?) all mathematics in first-order ZF set theory – which the authors then proceed to give the axioms for. These few pages surely wouldn’t help if you have never seen the axioms before and don’t already know about the project of doing-maths-inside-set-theory.

Finally in Part A, there’s rather ill-written chapter on normal forms, on extending theories by definitions, and (badly explained) on what the authors call ‘syntactic interpretations’.

Part B of the book discusses a number of rather scattered topics. It kicks off with a nice little chapter on extensions of first-order logic, more specifically on second-order logic, on $\mathcal{L}_{\omega_1\omega}$ [which allows infinitely long conjunctions and disjunctions], and $\mathcal{L}_Q$ [logic with quantifier $Qx$, ‘there are uncountably many $x$ such that . . . ’].

Then Ch. 10 is on ‘Limitations of the Formal Method’, and in under forty pages aims to talk about register machines, the halting problem for such machines, the undecidability of first-order logic, Trahtenbrot’s theorem and the incompleteness of second-order logic, Gödel’s incompleteness theorems, and more. This would just be far too rushed if you’d not seen this material before, and if you have then there are plenty of better sources for revising/consolidating/extending your knowledge.

Ch. 11 is by some way the longest in the book, on ‘Free Models and Logic Programming’. This is material we haven’t covered in this Guide. But again it doesn’t strike me as a particular attractive introduction (we will perhaps mention some better alternatives in a future section to be added to Chapter 5).

Ch. 12 is back to core model theory, Fraïssé’s Theorem and Ehrenfeucht Games: but (I’m sorry – this is getting repetitious!) you’ll again find better treatments elsewhere, this time in books dedicated to model theory.

Finally, there is an interesting (though quite tough) concluding chapter on Lindström’s Theorems which show that there is a sense in which standard first-order logic occupies a unique place among logical theories.

Summary verdict  The core material in Part A of the book is covered better (more accessibly, more elegantly) elsewhere.

Of the supplementary chapters in Part B, the two chapters that stand out as worth looking at are perhaps Ch. 9 on extensions of first-order logic, and Ch. 13 (though not easy) on Lindström’s Theorems.


Dirk van Dalen’s popular *Logic and Structure* (Springer: 263 pp. in the most recent edition) was first published in 1980, and has now gone through a number of editions. It is widely used and has a lot to recommend it. A very substantial chapter on incompleteness was added in the fourth edition in 2004. A fifth edition published in 2012 adds a further new section on ultraproducst. Comments here apply to these last two editions.

Some details  Ch. 1 on ‘Propositional Logic’ gives a presentation of the usual truth-functional semantics, and then a natural deduction system (initially with primitive connectives $\rightarrow$ and $\bot$). This is overall pretty clearly done – though really rather oddly, although van Dalen uses in his illustrative examples of deductions the usual practice of labelling a discharged premiss with numbers and using a matching label to mark the inference move at which that premiss is discharged, he doesn’t pause to explain the practice in the way you would expect. Van Dalen then gives a
standard Henkin proof of completeness for this cut-down system, before re-introducing the other
connectives into his natural deduction system in the last section of the chapter. Compared with
Chiswell and Hodges, this has a somewhat less friendly, more conventional, mathematical look-
and-feel: but this is still an accessible treatment, and will certainly be very readily manageable
if you’ve read C&H first. (It should be noted that van Dalen can be surprisingly slapdash. For
example, a tautology is defined is defined to be an (object-language) proposition which is always
true. But then the meta-linguistic schema $\varphi \land \psi \iff \psi \land \varphi$ is said to be a tautology. He means
the instances are tautologies; compare Mendelson who really does think tautologies are schemata.
Again, van Dalen presents his natural deduction system, says he is going to give some ‘concrete
cases’, but then presents not arguments in the object language, but schematic templates for
arguments, written out using ‘$\varphi$’s and ‘$\psi$’s again.)

Ch. 2 describes the syntax of a first-order language, gives a semantic story, and then presents
a natural deduction system, first adding quantifier rules, then adding identity rules. Overall, this
is pretty clearly done (though van Dalen reuses variables as parameters, which isn’t the nicest
way of setting things up). The approach to the semantics is to consider an extension of a first
order language $L$ with domain $A$ to an augmented language $L^A$ which has a constant $\overline{a}$ for every
element $a \in A$; and then we can say $\forall x. \varphi(x)$ is true if all $\varphi(\overline{x})$ are true. Fine: though it would
have been good if van Dalen had paused to say a little more about the pros and cons of doing
things this way rather than the more common Tarskian way that students will encounter. (Let’s
complain some more about van Dalen’s slapdash ways. For example, he talks about ‘the language
of a similarity type’ in §2.3, but gives examples of different languages of the same similarity type
in §2.7. He fusses unclearly about different uses of the identity sign in §2.3, before going on to
make use of the symbolism ‘:=’ in a way that isn’t explained, and is different from the use made
of it in the previous chapter. This sort of thing could upset the more picky reader.)

Ch. 3, ‘Completeness and Applications’, gives a pretty clear presentation of a Henkin-style
completeness proof, and then the compactness and L-S theorems. The substantial third section
on model theory goes rather more speedily, and you’ll need some mathematical background to
follow some of the illustrative examples. The final section newly added in the fifth edition on the
ultraproduct construction speeds up again and is probably too quick to be useful to many.

Ch. 4 is quite short, on second-order logic. If you have already seen a presentation of the
basic ideas, this quick presentation of the formalities could be helpful.

Ch. 5 is on intuitionism – this is a particular interest of van Dalen’s, and his account of
the BHK interpretation as motivating intuitionistic deduction rules, his initial exploration of the
resulting logic, and his discussion of the Kripke semantics are quite nicely done (though again,
this chapter will probably work better if you have already seen the main ideas in a more informal
presentation before).

Ch. 6 is on proof theory, and in particular on the idea that natural deduction proofs (both
classical and intuitionistic) can be normalized. Most readers will find a more expansive and
leisurely treatment much to be preferred.

The final 50 page Ch. 7 is more leisurely. It starts by introducing the ideas of primitive
recursive and partial recursive functions, and the idea of recursively enumerable sets, leading up
to a proof that there exist effectively inseparable r.e. sets. We then turn to formal arithmetic,
and prove that recursive functions are representable (because his version of PA does have the
exponential built in, van Dalen doesn’t need to tangle with the $\beta$-function trick.). Next we get
the arithmetization of syntax and proofs that the numerical counterparts of some key syntactic
properties and relations are primitive recursive. Then, as you would expect, we get the diagno-
M. lemma, and that is used to prove Gödel’s first incompleteness theorem. We then get
another proof relying on the earlier result that there are effectively inseparable r.e. sets, and
going via the undecidability of arithmetic. The chapter finishes by announcing that there is a
finitey axiomatized arithmetic strong enough to represent all recursive properties/relations, so
the undecidability of arithmetic implies the undecidability of first-order logic. There’s nothing,
however, about the second theorem: so most students who get this far will want that bit more
more. However, this chapter is all done pretty clearly, could probably be managed by good stu-
dents as a first introduction to its topics, and would be very good revision/consolidatory reading
for those who’ve already encountered this material.

Summary verdict  Revisiting this book, I find it a rather patchily uneven read. Although
intended for beginners in mathematical logic, the level of difficulty of the discussions rather varies, and the amount of more relaxed motivational commentary also varies. As noted there are occasional lapses where van Dalen’s exposition isn’t as tight as it could be. So this is probably best treated as a book to be read after you’ve had a first exposure to the material in the various chapters: but then it should prove pretty helpful for consolidating/expanding your initial understanding and then pressing on a few steps.

A.9 Prestel and Delzell, 1986, 2011

Alexander Prestell and Charles N. Delzell’s *Mathematical Logic and Model Theory: A Brief Introduction* (Springer, 1986; English version 2011: pp. 193) is advertised as offering a ‘streamlined yet easy-to-read introduction to mathematical logic and basic model theory’. Easy-to-read, perhaps, for those with a fair amount of mathematical background in algebra, for – as the Preface makes clear – the aim of the book is make available to interested mathematicians ‘the best known model theoretic results in algebra’. The last part of the book develops a complete proof of Ax and Kochen’s work on Artin’s conjecture about Diophantine properties of $p$-adic number fields.

So this book is not really aimed at a likely reader of this Guide. Still, Ch. 1 is a crisp and clean 60 page introduction to first-order logic, that could be used as brisk and helpful revision material. And Ch. 2, ‘Model constructions’ gives a nice if pacy introduction to some basic model theoretic notions: again – at least for readers of this Guide – it could serve well to consolidate and somewhat extend ideas if you have already encountered at least some of this material before, based on the reading in our §3.2.

The remaining two chapters, ‘Properties of Model Classes’ and ‘Model Theory of Several Algebraic Theories’ are tougher going, and belong with the more advanced reading in §5.4. But, for those who want to work through this material, it does strike me as well presented.

A.10 Johnstone, 1987

Peter T. Johnstone’s *Notes on Logic and Set Theory* (CUP, 1987: pp. 111) is very short in page length, but very big in ambition. There is an introductory chapter on universal algebra, followed by chapters on propositional and first-order logic. Then there is a chapter on recursive functions (showing that a function is register computable if and only if computable, and that such functions are representable in PA). That is followed by four chapters on set theory (introducing the axioms of ZF, ordinals, AC, and cardinal arithmetic). And there is a final chapter ‘Consistency and independence’ on Gödelian incompleteness and independence results in set theory.

This is a quite remarkably action-packed menu for such a short book. True, the story is filled out a bit by the substantive exercises, but still surely this isn’t the book to use for a first encounter with these ideas (even though it started life as notes for undergraduate lectures for the maths tripos). However, I would warmly recommend the book for revision/consolidation: its very brevity means that the Big Ideas get highlighted in a particularly uncluttered way, and particularly snappy proofs are given.
A.11  Hodel, 1995

Richard E. Hodel’s *An Introduction to Mathematical Logic* (PWS Publishing, 1995, reprinted Dover Publications, 2013: pp. 491) was originally launched into the world by a relatively obscure publisher, but has now been taken up and cheaply republished by Dover. I hadn’t heard of the book before a few people recommended it, commenting on the lack of any mention in earlier versions of this Guide.

The book covers first-order logic and some recursion theory, with a less usual – though not unique – feature being a full textbook treatment of Hilbert’s Tenth Problem (the one about whether there is an algorithm which tells us when a polynomial equation $p(x) = 0$ has a solution in the integers).

So how does Hodel fare against his competitors? Overall, the book is pretty clearly written, though it does have a somewhat old-fashioned feel to it (you wouldn’t guess, for example, that it was written some twenty years after George Boolos and Richard Jeffrey’s *Computability and Logic*). And Hodel does give impressively generous sets of exercises throughout the book – including some getting the student to prove significant results by guided stages. However, I can’t really recommend his treatment of logic in the first half of the book.

**Some details**  Ch. 1 ‘Background’ is an unusually wide-ranging introduction to ideas the budding logician should get her head round early. We get a first informal pass at the notions of a formal system and of an axiomatized system in particular, the idea of proof by induction, a few notions about sets, functions and relations, the idea of countability, the ideas of an algorithmically computable function and of effective decidability. We even get a first pass at the idea of a recursive function (and a look at Church’s Thesis about how the informal idea of being computable relates to the idea of recursiveness). This is very lucidly done, and can be recommended.

Ch. 2 is on ‘The language and semantics of propositional logic’ and is again pretty clearly done.

Ch. 3 turns to formal deductive systems for propositional logic. Unfortunately, Hodel chooses to work primarily with Shoenfield’s system (the primitive connectives are $\neg$ and $\vee$, every instance of $\neg A \vee A$ is an axiom, and there are four rules of inference). I really can’t see the attraction of this system among all the competitors, or why it should be thought as especially appropriate as a starting point for beginners. It neither has the naturalness of a natural deduction system, nor the austere Bauhaus lines of one of the more usual Frege-Hilbert axiomatic systems. The chapter does also consider other systems of propositional logic in the concluding two sections, but goes too quickly to be very helpful. So this key chapter is not a success, it seems to me.

Ch. 4 on ‘First-order languages’, including Tarskian semantics, is again pretty clear (and could be helpful to a beginner who is first encountering the ideas and is looking for reading to augment another textbook). But as we’d expect given what’s gone before, when we turn to Ch. 5 on ‘First-order logic’ we can a continuance of the discussion of a Shoenfield-style system (except that Hodel takes $\lor$ rather than $\exists$ as primitive). Which is, by my lights, much to be regretted. The ensuing discussion of completeness, while perhaps a little laboured, is carefully structured with a good amount of signposting. But there are better presentations of first-order logic overall.

Ch. 6 is called ‘Mathematics and logic’ and touches on an assortment of topics about first-order theories and their limitations (and a probably rather-too-hasty look at set theory as an example).

The next two chapters form quite a nice unit. Ch. 7 discusses ‘Incompleteness, undecidability and indefinability’. Recursive functions are defined Shoenfield-style as those arising from a certain class of initial functions by composition and regular minimization, which eases the proof that all recursive functions are representable (though doesn’t do much to make recursiveness seem a natural idea to beginners). Then, by making the informal assumption that certain intuitively decidable relations are recursive, Hodel proves Gödel’s incompleteness theorem, Church’s Theorem and Tarski’s Theorem. The next chapter fills in enough detail about recursive functions and relations to show how to lift that informal assumption. This seems all pretty clearly done, even if not a first-choice for real beginners.
Then Ch. 9 extends the treatment of computability by showing that the functions computable by an unlimited register machine are just the recursive ones: but, again this sort of thing is done at least as well in other books.

Finally, Ch. 10 deals with Hilbert’s tenth problem (so the first five sections of Ch. 8 and then this chapter form a nice, stand-alone treatment of the negative solution of Hilbert’s problem).

Summary verdict  Beginners could all usefully read Hodel’s opening chapter, which is better than usual in setting the scene, and supplying the student with a useful toolkit of preliminary ideas. The presentation of first-order logic in Chs. 2-6 is based around an unattractive formal system, and while the discussion of the usual meta-theoretic results is pretty clear, it doesn’t stand out from the good alternatives: so overall, I wouldn’t recommend this as your first encounter with serious logic.

Students, however, might find Chs. 7 and 8 provide a nice complement to other discussions of Gödelian incompleteness and Church and Tarski’s Theorems. While more advanced students could revise their grip on basic definitions and results by (re)reading §§8.1–8.5 and then enjoy tackling Ch. 10 on Hilbert’s Tenth Problem.

A.12 Goldstern and Judah, 1995

Half of Martin Goldstern and Haim Judah’s The Incompleteness Phenomenon: A New Course in Mathematical Logic (A.K. Peters, 1995: pp. 247) is a treatment of first-order logic. The rest of the book is two long chapters (as it happens, of just the same length), one on model theory, one mostly on incompleteness and with a little on recursive functions. So the emphasis on incompleteness in the title is somewhat misleading: the book is at least equally an introduction to some model theory. I have had this book recommended to me more than once, but I seem to be immune to its supposed charms (I too often don’t particularly like the way that it handles the technicalities): your mileage may vary.

Some details  Ch. 1 starts by talking about inductive proofs in general, then gives a semantic account of sentential and then first-order logic, then offers a Hilbert-style axiomatic proof system.

Very early on, the authors introduce the notion of $\mathcal{M}$-terms and $\mathcal{M}$-formulae. An $\mathcal{M}$-term (where $\mathcal{M}$ is model for a given first-order language $\mathcal{L}$) is built up from $\mathcal{L}$-constants, $\mathcal{L}$-variables and/or elements of the domain of $\mathcal{M}$, using $\mathcal{L}$-function-expressions; an $\mathcal{M}$-formula is built up from $\mathcal{M}$-terms in the predictable way. Any half-awake student is initially going to balk at this. Re-reading the set-theoretic definitions of expressions as tuples, she will then realize that the apparently unholy mix of bits of language and bits of some mathematical domain in an $\mathcal{M}$-term is not actually incoherent. But she will right wonder what on earth is going on and why: our authors don’t pause to explain why we might want to do things like this at the very outset. (A good student who knows other presentations of the basics of first-order semantics should be able to work out after the event what is going on in the apparent trickery of Goldstern and Judah’s sort of story: but I really can’t recommend starting like this, without a good and expansive explanation of the point of the procedure.)

Ch. 2 gives a Henkin completeness proof for the first-order deductive system given in Ch. 1. This has nothing special to recommend it, as far as I can see: there are many more helpful expositions available. The final section of the chapter is on non-standard models of arithmetic: Boolos and Jeffrey (Ch. 17 in their third edition) do this more approachably.

Ch.3 is on model theory. There are three main sections, ‘Elementary substructures and chains’, ‘ultra products and compactness’, and ‘Types and countable models’. So this chapter – less than sixty pages – aims quite high to be talking e.g. about ultraproducts and about omitting types. You could indeed usefully read it after working through e.g. Manzano’s book: but I certainly don’t think this chapter makes for an accessible and illuminating first introduction.
to serious model theory.

Ch. 4 is on incompleteness, and the approach here is significantly more gentle than the previous chapter. Goldstern and Judah make things rather easier for themselves by adopting a version of Peano Arithmetic which has exponentiation built in (so they don’t need to tangle with Gödel’s β function). And they only prove a semantic version of Gödel’s first incompleteness theorem, assuming the soundness of PA. The proof here goes as by showing directly that – via Gödel coding – various syntactic properties and relations concerning PA are expressible in the language of arithmetic with exponentiation (in other words, they don’t argue that those properties and relations are primitive recursive and then show that PA can express all such properties/relations).

How well, how accessibly, is this done? The authors hack through eleven pages (pp. 207–217) of the arithmetization of syntax, but the motivational commentary is brisk and yet the proofs aren’t completely done (the authors still leave to the reader the task of e.g. coming up with a predicate satisfied by Gödel numbers for induction axioms). So this strikes the present reader as really being neither one thing nor another – neither a treatment with all the details nailed down, nor a helpfully discursive treatment with a lot of explanatory arm-waving. And in the end, the diagonalization trick seems to be just pulled like a rabbit out of the hat. After proving incompleteness, they prove Tarski’s theorem and the unaxiomatizability of the set of arithmetic truths.

To repeat, the authors assume PA’s soundness. They don’t say anything about why we might want to prove the syntactic version of the first theorem, and don’t even mention the second theorem which we prove by formalizing the syntactic version. So this could well leave students a bit mystified when they come across other treatments.

The book ends by noting that the relevant predicates in the arithmetization of syntax are Σ₁, and then defines a set as being recursively enumerable if it is expressible by a Σ₁ predicates (so now talk of recursiveness etc. does get into the picture). But really, if you want to go down this route, this is surely all much better handled in Leary’s book.

Summary verdict The first two chapters of this book can’t really be recommended either for making a serious start on first-order logic or for revision. The third chapter could be used for a brisk revision of some model theory if you have already done some reading in this area. The final chapter about incompleteness (which the title of the book might lead you to think will be a high point) isn’t the most helpful introduction in this style – go for Leary (2000) instead – and on the other hand doesn’t go far enough for revision/consolidatory purposes.

A.13 Leary, 2000

So how friendly is Christopher C. Leary’s A Friendly Introduction to Mathematical Logic (Prentice Hall, 2000: pp. 218)? – meaning, of course, ‘friendly’ by the standard of logic books!

I like the tone a great deal (without being the least patronizing, it is indeed relaxed and inviting), and the level of exposition seems to me to be mostly well-judged for an introductory course (one semester’s worth, in American terms). The book is officially aimed mostly at mathematics undergraduates without assuming any particular background knowledge. But as Leary says in the Preface, it should also be accessible to logic-minded philosophers who are happy to work at following rather abstract arguments (and, I would add, who are also happy to skip over just a few inessential elementary mathematical illustrations).

What does the book cover? Just basic first-order logic (up to the L-S theorems) and the incompleteness theorems. But by being so tightly focused, this short book rarely seems to rush at what it does cover: the pace is pretty even.
If I have a general reservation about the book, it is that Leary opts for a Hilbertian axiomatic system of logic, with fairly brisk explanations. If you’ve never seen before a serious formal system for first-order logic this initially makes for quite a dense read: if on the other hand you have been introduced to logic by trees or seen a natural deduction presentation, you would welcome some paragraphs explaining the advantages for present purposes of the choice of an axiomatic approach here.

Some details  Ch. 1, ‘Structures and languages’, starts by talking of first-order languages (Leary makes the good choice of not starting over again with propositional logic, but assumes students know their truth-tables!), and then moves on to explaining the idea of first order structures, and truth-in-a-structure. There is a good amount of motivational chat as Leary goes through, and the exercises – as elsewhere in the book – seem particularly well-designed to aid understanding.

Ch. 2, ‘Deductions’, introduces an essentially Hilbertian logical system and proves its soundness: it also considers systems with additional non-logical axioms. The logical primitives are ‘¬’, ‘→’, ‘∀’ and ‘=’. Logical axioms are just the identity axioms, an axiom-version of ∀-elimination (and its dual, ∃-introduction): the inference rules ∀-introduction (and its dual) and a rule which allows us to infer φ from a finite set of premisses Γ if it is an instance of a tautological entailment.

I don’t think this is the friendliest ever logical system (and no doubt for reasons of brevity, Leary doesn’t pause to consider alternative options); but it is not unusually horrible either. If you take it slowly, the exposition here should be quite manageable.

Ch. 3, ‘Completeness and compactness’, gives a nice version of a Henkin-style completeness theorem for the described deductive system, then proves compactness and the upward and downward Löwenheim-Skolem theorems (the latter in the version ‘if L is a countable language and B is an L-structure, then B has a countable elementary substructure’ [the proof might be found a bit trickier though]). So there is a little model theory here as well as the completeness proof: and you could well read this chapter without reading the previous ones if you are already reasonably up to speed on structures, languages, and deductive systems.

In Ch. 4, ‘Incompleteness—Groundwork’, Leary (re)introduces the theory he calls N, a version of Robinson Arithmetic with exponentiation built in. He then shows that (given a scheme of Gödel coding) that the usual numerical properties and relations involved in the arithmetization of syntax – such as, ultimately, Prf(m, n), i.e. m codes for an N-proof of the formula numbered n – can be represented in N.

He does this by the direct method. That is to say, instead of [like my IGT] showing that those properties/relations are (primitive) recursive, and that N can represent all (primitive) recursive relations, Leary directly writes down ∆₀ wffs which express and hence represent them. This is inevitably gets more than a bit messy: but he has a very good stab at motivating every step as he works up to showing that N can express Prf(m, n) by a ∆₀ wff. If you want a full-dress demonstration of this result, then this is one of the most user-friendly available.

Ch. 5, ‘The Incompleteness Theorems’, is the shortest in the book: but Leary has done all the groundwork to enable him to now give a brisk but pretty clear presentation, at least after he has proved the Diagonalization Lemma. A little unfortunately, the proof of that is rather too rabbit-out-of-a-hat for my liking. But once the Lemma is in place, the rest of the chapter goes very nicely and acessibly (we get the first incompleteness theorem in its semantic version, the undecidability of arithmetic, Tarksi’s theorem, the syntactic version of incompleteness and then Rosser’s improvement. Then there is nice section giving Boolos’s proof of incompleteness echoing the Berry paradox. Finally, the second theorem is proved by assuming (though not proving) the derivability conditions.

Summary verdict  If you have already briefly met an axiomatically presented deductive systems for first-order semantics, and an informal account of its semantics, then you’ll find the opening two chapters of Leary’s book very manageable (if you haven’t they’ll be a bit more work). The treatment of completeness etc. in Ch. 3 would make for a nice stand-alone treatment even if you don’t read the first two chapters. Or you could just start the book by reading §2.8 (where N is first mentioned), and then read the excellent Chs 4 and 5 on incompleteness with a lot of profit.
A Friendly Introduction is indeed in many ways an unusually likeable introduction to the material it covers, and has a lot to recommend it. (The book is currently out of print but a revised edition is in preparation).

A.14 Forster, 2003

Thomas Forster’s Logic, Induction and Sets (CUP, 2003: pp. x + 234) is rather quirky, and some readers will enjoy it for just that reason. It is based on a wide-ranging lecture course given to mathematicians who – such being the oddities of the Cambridge tripos syllabus – at the beginning of the course already knew a good deal of maths but very little logic. The book is very bumpily uneven in level, and often goes skips forward very fast, so I certainly wouldn’t recommend it as an ‘entry level’ text on mathematical logic for someone wanting a conventionally systematic approach. But it is often intriguing.

Some details  Ch. 1 is called ‘Definitions and notations’ but is rather more than that, and includes some non-trivial exercises: but if you are dipping into later parts of the book, you can probably just consult this opening chapter on a need-to-know basis.

Ch. 2 discusses ‘Recursive datatypes’, defined by specifying a starter-pack of ‘founders’ and some constructors, and then saying the datatype is what you can get from the founders by applying and replying the constructors (and nothing else). The chapter considers a range of examples, induction over recursive datatypes, well-foundedness, well-ordering and related matters (with some interesting remarks about Horn clauses too).

Ch. 3 is on partially ordered sets, and we get a lightning tour through some topics of logical relevance (such as the ideas of a filter and an ultrafilter).

Chs. 4 and 5 deal slightly idiosyncratically with propositional and predicate logic, and could provide useful revision material (there’s a slip about theories on p. 70, giving two non-equivalent definitions).

Ch. 6 is on ‘Computable functions’ and is another lightning tour, touching on quite a lot in just over twenty pages (getting as far as Rice’s theorem). Again, could well be useful to read as revision, especially if you want to highlight again the Big Ideas and their interrelations.

Ch. 7 is on ‘Ordinals’. Note that Forster gives us the elements of the theory of transfinite ordinal numbers before turning to set theory in the next chapter. It’s a modern doctrine that ordinals just are sets, and that the basic theory of ordinals is part of set theory; and in organizing his book as he does, Forster comes nearer than most to getting the correct conceptual order into clear focus (though even he wobbles sometimes, e.g. at p. 182). However, the chapter could have been done more clearly.

Ch. 8 is called ‘Set Theory’ and is perhaps the quirkiest of them all – though not because Forster is here banging the drum for non-standard set theories (surprisingly given his interests, he doesn’t). But the chapter is oddly structured, so for example we get a quick discussion of models of set theory and the absoluteness of $\Delta_0$ properties before we actually encounter the ZFC axioms. The chapter is probably only for those, then, who already know the basics.

Ch. 9 comprises answers to some of the earlier exercises – exercises are indeed scattered through the book, and some of them are rather interesting.

Summary verdict  Different from the usual run of textbooks, not a good choice for beginners. However, if you already have encountered some of the material in one way or the other, Forster’s book could very well be worth looking through for revision and/or to get some new perspectives.
A.15  Hedman, 2004

Shawn Hedman’s A First Course in Logic (OUP, 2004: pp. xx + 214) is subtitled ‘An Introduction to Model Theory, Proof Theory, Computability and Complexity’. So there’s no lack of ambition in the coverage! And I do like the general tone and approach at the outset. So I wish I could be more enthusiastic about the book in general. But, as we will see, it is decidedly patchy both in terms of the level of the treatment of various topics, and in terms of the quality of the exposition.

Some details  After twenty pages of mostly rather nicely done ‘Preliminaries’ – including an admirably clear couple of pages the $P = NP$ problem, Ch. 1 is on ‘Propositional Logic’. On the negative side, we could certainly quibble that Hedman is a bit murky about object-language vs meta-language niceties. The treatment of induction half way through the chapter isn’t as clear as it could be. Much more importantly, the chapter offers a particularly ugly formal deductive system. It is in fact a (single conclusion) sequent calculus, but with proofs constrained to be a simple linear column of wffs. So – heavens above! – we are basically back to Lemmon’s Beginning Logic (1965). Except that the rules are not as nice as Lemmon’s (thus Hedman’s $\land$-elimination rule only allows us to extract a left conjunct; so we need an additional $\land$-symmetry rule to get from $P \land Q$ to $Q$). I can’t begin to think what recommended this system to the author out of all the possibilities on the market. On the positive side, there’s quite a nice treatment of a resolution calculus for wffs in CNF form, and a proof that this is sound and complete. This gives Hedman a completeness proof for derivations in his original calculus with a finite number of premisses, and he gives a compactness proof to beef this up to a proof of strong completeness.

Ch. 2, ‘Structures and first-order logic’ should really be called ‘Structures and first-order languages’, and deals with relations between structures (like embedding) and relations between structures and languages (like being a model for a sentence). I’m not sure I quite like its way of conceiving of a structure as always some $V$-structure, i.e. as having an associated first-order vocabulary $V$ which it is the interpretation of – so structures for Hedman are what some would call labelled structures. But otherwise, this chapter is clearly done.

Ch. 3 is about deductive proof systems for first-order logic. The first deductive system offered is an extension of the bastardized sequent calculus for propositional logic, and hence is equally horrible. Somehow I sense that Hedman just isn’t much interested in standard proof-systems for logic. His heart is in the rest of the chapter, which moves towards topics of interest to computer scientists, about Skolem normal form, the Herbrand method, unification and resolution, so-called ‘SLD-resolution’, and Prolog – interesting topics, but not on my menu of basics to be introduced at this very early stage in a first serious logic course. The discussions seem quite well done, and will be accessible to an enthusiast with an introductory background (e.g. from Chiswell and Hodges) and who has read the section of resolution in the first chapter.

Ch. 4 is on ‘Properties of first-order logic’. The first section is a nice presentation of a Henkin completeness proof (for countable languages). There is then a long aside on notions of infinite cardinals and ordinals (Hedman has a policy of introducing background topics, like the idea of an inductive proof, and now these set theoretic notions, only when needed: but it can break the flow). §4.3 can use the assumed new knowledge about non-countable infinities to beef up the completeness proof, give upwards and downwards LS theorems, etc., again done pretty well. §§4.4–4.6 does some model theory under the rubrics ‘Amalgamation of structures’. ‘Preservation of formulas’ and ‘Amalgamation of vocabularies’: this already gets pretty abstract and uninviting, with not enough motivating examples. §4.7 is better on ‘The expressive power of first-order logic’.

The next two chapters, ‘First order theories’ and ‘Models of countable theories’, give a surprisingly (I’d say, unrealistically) high level treatment of some model theory, going well beyond e.g. Manzano’s book, eventually talking about saturated models, and even ending with ‘A touch of stability’. This hardly chimes with the book’s prospectus as being a first course in logic. The chapters, however, could be useful for someone who wants to push onwards, after a first encounter with some model theory.

Ch. 6 comes sharply back to earth: an excellent chapter on ‘Computability and complexity’ back at a sensibly introductory level. It begins with a well done review of the standard material on primitive recursive functions, recursive functions, computing machines, semi-decidable decision
problems, undecidable decision problems. Which is followed by a particularly clear introduction to ideas about computational complexity, leading up to the notion of $\text{NP}$-completeness. An excellent chapter.

Sadly, the following Ch. 8 on the incompleteness theorems again isn’t very satisfactory as a first pass through this material. In fact, I doubt whether a beginning student would take away from this chapter a really clear sense of what the key big ideas are, or of how to distinguish the general results from the hack-work needed to show that they apply to this or that particular theory. And things probably aren’t helped by proving the first theorem initially by Boolos’s method rather than Gödel’s. Still, just because it gives an account of Boolos’s proof, this chapter can be recommended as supplementary reading for those who have already seen some standard treatments of incompleteness.

The last two chapters ratchet up the difficulty again. Ch. 9 goes ‘Beyond first-order logic’ by speeding through second-order logic, infinitary logics (particularly $\mathcal{L}_{\omega_1\omega}$), fixed-point logics, and Lindström’s theorem, all in twenty pages. This will probably go too fast for those who haven’t encountered these ideas before. It should be noted that the particularly brisk account of second-order logic gives a non-standard syntax and says nothing about Henkin vs full semantics. The treatment of fixed-point logics (logics that are ‘closed under inductive definitions’) is short on motivation and examples. But enthusiasts might appreciate the treatment of Lindström’s theorem.

Finally, Ch. 10 is on finite model theory and descriptive complexity. Beginners doing a first course in logic will again find this quite tough going.

**Summary verdict**  A very uneven book in level, with sections that work well at an introductory level and other sections which will only be happily managed by considerably more advanced students. An uneven book in coverage too. By my lights, this couldn’t be used end-to-end as a course text: but in the body of the Guide, I’ve recommended parts of the book on particular topics.

### A.16 Hinman, 2005


The author says the book was written over a period of twenty years, as he tried out various approaches ‘to enable students with varying levels of interest and ability to come to a deep understanding of this beautiful subject’. But I suspect that you will need to be mathematically quite strong to really cope with this book: whatever Hinman’s intentions for a wider readership, this is not for the fainthearted.

The book’s daunting size is due to its very wide coverage rather than a slow pace – so after a long introduction to first-order logic (or more accurately, to its model theory) and a discussion of the theory of recursive functions and incompleteness and related results, there follows a very substantial survey of set theory, and then lengthy essays on more advanced model theory and on recursion theory. As too often, proof theory is the poor relation here – indeed Hinman is very little interested in deductive systems for logic, which don’t make an appearance until over two hundred pages into the book.

Let me mention at the outset what strikes me as a pretty unfortunate global notational convention, which might puzzle casual browsers or readers who want to start some way through the book. Given the two-way borrowing of notation between informal mathematics and the formal languages in which logicians regiment that mathematics, it is good to have some way of visually distinguishing the formal from the informal (so we don’t just rely on context). One common method is font selection. Thus, even in an informal context, we may snappily say that addition commutes by writing e.g.
∀x∀y x + y = y + x; the counterpart wff for expressing this in a fully formalized language may then be, e.g., ∀x∀y x + y = y + x. But instead of using sans serif or boldface for formal wffs, or another font selection, Hinman prefers using an ‘(informal) mathematical sign with a dot over it to represent a formal symbol in a formal language which denotes the informal object’, so he’d write ∀x∀y x + y = y + x for the formal wff. As you can imagine, this convention eventually leads to really nasty rashes of dots – for example, to take a relatively tame example from p. 459, we get

\[ \bigcup x = \{ z : \exists v [ v \in x \land z \in v] \} \]

(note how even opening braces in formal set-former notation get dotted). This dottiness quite surely isn’t a happy choice!

Some details  
Hinman himself in his Preface gives some useful pointers to routes through the book, depending on your interests.

The Introduction gives a useful and approachable overview of some key notions tied up with the mathematical logician’s project of formalization (and talks about a version of Hilbert’s program as setting the scene for some early investigations).

Ch. 1 is on ‘Propositional Logic and other fundamentals’. §§1.1, 1.3 and 1.4 are devoted to the language of propositional logic, and give the usual semantics, define the notion tautological entailment and explore its properties, giving a proof of the compactness theorem. But note, there is no discussion at all here – or in the other sections of this chapter – of a proof-system for propositional logic.

§1.2 is a rather general treatment of proofs by induction and the definition of functions by recursion (signposted as skippable at this early stage – and indeed the generality doesn’t make for a particularly easy read for a section so early in the book). §§1.6 and 1.7 also cover more advanced material, mainly introducing ideas for later use: the first briskly deals e.g. with ultrafilters and ultraproducts (we get another take on compactness), and the second relates compactness to topological ideas and also introduces the idea of a Boolean algebra.

Ch. 2, ‘First-order logic’, presents the syntax and semantics of first-order languages, and then talks about first-order structures (isomorphisms, embeddings, extensions, etc), and proves the downward L-S theorem. We then get a general discussion of theories (thought of as sets of sentences closed under semantic consequence), and an extended treatment of some examples (the theory of equality, the theory of dense linear orders, and various strengths of arithmetic). There’s some quite sophisticated stuff here, including discussion of quantifier elimination. But there is still no discussion yet of a proof-system for first-order logic, so the chapter could as well, if not better, have been called ‘Elements of model theory’.

Ch. 3, ‘Completeness and compactness’, starts with a compactness proof for countable languages. Then we at last have a very brisk presentation of an old-school axiomatic system for first-order logic (I told you that Hinman is not interested in proof-systems!), and a proof of completeness using the Henkin construction that has already been used in the compactness proof.

We next get – inter alia – an algebraic proof of compactness for first-order consequence via ultraproducts, and a return to Boolean algebras and e.g. the Rasiowa-Sikorski theorem (§3.3); an extension of the compactness and completeness results to uncountable languages (§3.4); and some heavy-duty applications of compactness (§3.5).

Finally in this action-packed chapter, we have some rather unfriendly treatments of higher-order logic (§3.6) and infinitary logic (§3.7).

Let’s pause for breath. We are now a bit over 300 pages into the book. Things have already got pretty tough. The book is not quite a relentless march along a chain of definitions/theorems/corollaries; there are just enough pauses for illustrations and helpful remarks en route to make it a bearable. But Hinman does have a taste for going straight for abstractly general formulations (and his notational choices can sometimes be unhappy too). So as indicated in my preamble, the book will probably only appeal to mathematicians already used to this sort of fairly hardcore approach. In sum, therefore, I’d only recommend the first part of the book to the mathematically minded who already know their first-order logic and a bit of model theory; but such readers might then find it quite helpful as a beginning/mid-level model theory resource.
On we go. Next we have two chapters (almost 150 pages between them) on recursive functions, Gödelian incompleteness, and related matters. Perhaps it is because these topics are conceptually easier, more ‘concrete’, than what’s gone before, or perhaps it is because the topics are closer to Hinman’s heart, but these chapters seem to me to work better as an introduction to their topics. In particular, while not my favourite treatment, Ch. 4 is clear, very sensibly structured, and should be accessible to anyone with some background in logic and who isn’t put off by a certain amount of mathematical abstraction. The chapter opens with informal proofs of the undecidability of consistent extensions of $\mathcal{Q}$, the first incompleteness theorem and Tarski’s theorem on the undefinability of truth (as well as taking a first look at the second incompleteness theorem). These informal proofs depend on the hypothesis that effectively calculable functions are expressible or the hypothesis that such functions are representable (we don’t yet have a formal story about these functions). Unsurprisingly, given I do something in the same ball-park in my Gödel book, I too think this is a good way to start and to motivate the ensuing development. There follows, as you’d expect, the necessary account of the effectively calculable in terms of recursiveness, and then we get proofs that recursive functions can be expressed/represented in arithmetic, leading on to formal versions of the theorems about undecidable and incompleteness. This presentation takes a different-enough path through the usual ideas to be worth reading even if you’ve already encountered the material a couple of times before.

Ch. 5 is called ‘Topics in definability’ and, unlike the previous rather tightly organised chapter, is something of a grab-bag of topics. §5.1 says something about the arithmetical hierarchy; §5.2 discusses inter alia the indexing of recursive functions and the halting problem; §5.3 explains how the second incompleteness theorem is proved, and – while not attempting a full proof – there is rather more detail than usual about how you can show that the HBL derivability conditions are satisfied in PA. Then §5.4 gives more evidence for Church’s Thesis by considering a couple of other characterisations of computability (by equation manipulation and by abstract machines) and explains why they again pick out the recursive functions. §5.5 discusses ‘Applications to other languages and theories’ (e.g. the application of incompleteness to a theory like ZF which is not initially about arithmetic). These various sections are all relatively clearly done.

Pausing for breath again, we might now try to tackle Ch. 6 on set theory (whose 200 pages amount by themselves to an almost-stand-alone book). The menu covers the basics of ZF, the way we can construct mathematics inside set theory, ordinals and cardinals, then models and independence proofs, the constructible universe, models and forcing, large cardinals and determinacy. But even from the outset, this does seem quite relentlessly hard going, too short on motivation and illustrations of concepts and constructions. Dense, to say the least. The author says of the chapter that his particular mode of presentation means that ‘for each of the instances where one wants to verify that something is a class model – the intuitive universe of sets $\mathcal{V}$, the constructible universe $\mathcal{L}$ and a forcing extension $\mathcal{m}[G]$ – . . . the proofs . . . exhibit more of the underlying unity.’ So enthusiasts who know their set theory might want to do a fast read of the chapter to see if they can glean new insights. But I can’t recommend this as a way into set theory when compared with the standard set theory texts mentioned in §3.4 and §5.8.1 of this Guide.

Ch. 7 returns to more advanced model theory for another 80 pages, getting as far Morley’s theorem. Again, if you want a more accessible initial treatment, you’ll go for Hodges’s Shorter Model Theory. And then why not tackle Marker’s book if you are a graduate mathematician?

Finally, there’s another equally long chapter on recursion theory. The opening sections on degrees and Turing reducibility are pretty approachable. The rest of the chapter gets more challenging but (at least compared with the material on model theory and set theory) should still be tolerably accessible to those willing to put in the work.

**Summary verdict** It is very ambitious to write a book with this range and depth of coverage (as it were, an expanded version of Shoenfield, forty years on – but now when there is already a wealth of textbooks on the various areas covered, at various levels of sophistication). After such a considerable labour from a good logician, it seems very churlish to say it, but the treatments of, respectively, (i) first-order logic, (ii) model theory, (iii) computability theory and incompleteness, and (iv) set theory aren’t as good as the best of the familiar stand-alone textbooks on the four areas. And I can’t see that these shortcomings are balanced by any conspicuous advantage in having the accounts in a single
text, rather than a handful of different ones. Still, the text should be in any university library, as enthusiasts might well find parts of it quite useful supplementary/reference material. Chapters 4, 5 and 8 on computability and recursion work the best.

A.17 Chiswell and Hodges, 2007

Ian Chiswell and Wilfrid Hodges’s Mathematical Logic (OUP, 2007: pp. 249) is very largely focused on first-order logic, only touching on Church’s undecidability theorem and Gödel’s first incompleteness theorem in a Postlude after the main chapters. So it is perhaps stretching a point to include it in a list of texts which cover more than one core area of the mathematical logic curriculum. Still, I wanted to comment on this book at some length, without breaking the flow of §2.2 (where it is warmly recommended in headline terms), and this is the obvious place to do so.

Let me highlight three key features of the book, the first one not particularly unusual (though it still marks out this text from quite a few of the older, and not so old, competitors), the second very unusual but extremely welcome, the third a beautifully neat touch:

1. Chiswell and Hodges (henceforth C&H) present natural deduction proof systems and spend quite a bit of time showing how such formal systems reflect the natural informal reasoning of mathematicians in particular.

2. Instead of dividing the treatment of logic into two stages, propositional logic and quantificational logic, C&H take things in three stages. First, propositional logic. Then we get the quantifier-free part of first-order logic, dealing with properties and relations, functions, and identity. So at this second stage we get the idea of an interpretation, of truth-in-a-structure, and we get added natural deduction rules for identity and the handling of the substitution of terms. At both these first two stages we get a Hintikka-style completeness proof for the given natural deduction rules. Only at the third stage do quantifiers get added to the logic and satisfaction-by-a-sequence to the semantic apparatus. Dividing the treatment of first order logic into stages like this means that a lot of key notions get first introduced in the less cluttered contexts of propositional and/or quantifier-free logic, and the novelties at the third stage are easier to keep under control. This does make for a great gain in accessibility.

3. The really cute touch is to introduce the idea of polynomials and diophantine equations early – in fact, while discussing quantifier-free arithmetic – and to state (without proof!) Matiyasevich’s Theorem. Then, in the Postlude, this can be appealed to for quick proofs of Church’s Theorem and Gödel’s Theorem.

This is all done with elegance and a light touch – not to mention photos of major logicians and some nice asides – making an admirably attractive introduction to the material.

Some details C&H start with almost 100 pages on the propositional calculus. Rather too much of a good thing? Perhaps, if you have already done a logic course at the level of my intro book or Paul Teller’s. Still, you can easily skim and skip. After Ch. 2 which talks about informal natural deductions in mathematical reasoning, Ch. 3 covers propositional logic, giving a natural deduction system (with some mathematical bells and whistles along the way, being careful about trees, proving unique parsing, etc.). The presentation of the formal natural deduction system is not exactly my favourite in its way representing discharge of assumptions (I fear that some
readers might be puzzled about vacuous discharge and balk at Ex. 2.4.4 at the top of p. 19): but apart from this little glitch, this is done well. The ensuing completeness proof is done by Hintikka’s method rather than Henkin’s.

After a short interlude, Ch. 5 treats quantifier-free logic. The treatment of the semantics without quantifiers in the mix to cause trouble is very nice and natural; likewise at the syntactic level, treatment of substitution goes nicely in this simple context. Again we get a soundness and Hintikka-style completeness proof for an appropriate natural deduction system.

Then, after another interlude, Ch. 7 covers full first-order logic with identity. Adding natural deduction rules (on the syntactic side) and a treatment of satisfaction-by-finite-n-tuples (on the semantic side) all now comes very smoothly after the preparatory work in Ch. 5. The Hintikka-style completeness proof for the new logic builds very nicely on the two earlier such proofs: this is about as accessible as it gets in the literature, I think. The chapter ends with a look at the Löwenheim-Skolem theorems and ‘Things that first-order logic cannot do’.

Finally, as explained earlier, material about diophantine equations introduced naturally by way of examples in earlier chapters is used in a final Postlude to give us undecidability and incompleteness results very quickly (albeit assuming Matiyasevich’s Theorem).

\textit{Summary verdict} C&H have written a very admirably readable and nicely structured introductory treatment of first-order logic that can be warmly recommended. The presentation of the syntax of their type of (Gentzen-Prawitz) natural deduction system is perhaps done a trifle better elsewhere (Tennant’s freely available \textit{Natural Logic} mentioned in §2.3 gives a full dress version). But the core key sections on soundness and completeness proofs and associated metalogical results are second to none for their clarity and accessibility.