Pass it on, . . . . That’s the game I want you to learn. Pass it on.

Alan Bennett, The History Boys
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A very quick introduction

I retired from the University of Cambridge in 2011. Previously, it was my greatest good fortune to have secure, decently paid, university posts for forty years in leisurely times, with almost total freedom to follow my interests wherever they led. Looking back, I am quite sure that, like many of my contemporaries, I didn’t really appreciate just how lucky I was. This Guide to logic textbooks for students is one very small effort to give a little back by way of heartfelt thanks.

Don’t be scared off by the Guide’s length! This is due to its starting just one step above the level of ‘baby logic’ and then going quite a long way down the road towards pretty advanced stuff. Different readers will therefore want to jump on and off the bus at different stops. Simply pick and choose the sections which are relevant to your interests.

- If you are impatient and want a quick start guide to the Guide, then just read §2.1. Then glance through §3.1 to see whether you already know enough about first-order logic to skip the rest of the chapter. If you don’t, continue from §3.2. Otherwise, start in earnest with Ch. 4.

- For a slower introduction, read through Chs. 1 and 2 over a cup of coffee to get a sense of the purpose, shape and coverage of the Guide, and to get some advice about how to structure your reading. That should enable you to pick and choose your way through the remaining chapters.

- However, if you are hoping for help with very elementary logic (e.g. as typically encountered by philosophers in their first-year
courses), then I am afraid that this Guide isn’t designed for you. The only section that directly pertains to ‘baby logic’ is §2.2; all the rest is about rather more advanced – and eventually very much more advanced – material.
Chapter 1

Preliminaries

This chapter and the next are for new readers, and include a gentle guide to the Guide, explaining its aims and how it is structured.

1.1 Why this Guide for philosophers?

It is an odd phenomenon, and a rather depressing one too. Logic, beyond the first baby steps, is seemingly taught less and less, at least in UK philosophy departments. Fewer and fewer serious logicians get appointed to teaching posts. Yet logic itself is, of course, no less exciting and rewarding a subject than it ever was, and the amount of good formally-informed work in philosophy is ever greater as time goes on. Moreover, logic is far too important to be left entirely to the mercies of technicians from maths or computer science departments with different agendas (who often reveal an insouciant casualness about conceptual details that will matter to the philosophical reader).

So how is a competence in logic to be passed on if there are not enough courses, or are none at all?

It seems that many beginning graduate students in philosophy – if they are not to be quite dismally uneducated in logic and therefore cut off from working in some of the most exciting areas of their discipline – will need to teach themselves from books, either solo or (much better) by organizing their own study groups.
In a way, that’s perhaps no real hardship, as there are some wonderful books written by great expositors out there. But what to read? Logic books can have a very long shelf life, and one shouldn’t at all dismiss older texts when starting out on some topic area: so there’s more than a fifty year span of publications to select from. Without having tried very hard, I seem to have accumulated on my own shelves some three hundred formal logic books at roughly the level that might feature in a Guide such as this – and of course these are still only a selection of what’s available.

Philosophy students evidently need a Study Guide if they are to find their way around the available literature old and new: this is my (on-going, still developing) attempt to provide one.

1.2 Why this Guide for mathematicians too?

The situation of logic teaching in mathematics departments can also be pretty dire. Indeed there are full university maths courses in good UK universities with precisely zero courses on logic or set theory (maybe a few beginning ideas are touched on in a discrete maths course, but that is all). And I believe that the situation is equally patchy in many other places.

So again, if you want to teach yourself some logic, where should you start? What are the topics you might want to cover? What textbooks are likely to prove accessible and tolerably enjoyable and rewarding to work through? Again, this Guide – or at least, the sections on the core mathematical logic curriculum – will give you some pointers.

True, this is written by someone who has – apart from a few guest mini-courses – taught in a philosophy department, and who is no research mathematician. Which probably gives a distinctive tone to the Guide (and certainly explains why it also ranges into areas of logic of more special interest to philosophers). Still, mathematics remains my first love, and these days it is mathematicians whom I mostly get to hang out with. A large number of the books I recommend are very definitely paradigm mathematics texts. So I shouldn’t be leading you astray.
I didn’t want to try to write two overlapping Guides, one primarily for philosophers and one aimed primarily at mathematicians. This was not just to avoid multiplying work. Areas of likely interest don’t so neatly categorize. Anyway, a number of philosophers develop serious interests in more mathematical corners of the broad field of logic, and a number of mathematicians find themselves interested in more foundational/conceptual issues. So there’s a single but wide-ranging menu here for everyone to choose from as their interests dictate.

1.3 Choices, choices

So what has guided my choices of what to recommend in this Guide? Different people find different expository styles congenial. For example, what is agreeably discursive for one reader is irritatingly verbose and slow-moving for another. For myself, I do particularly like books that are good on conceptual details and good at explaining the motivation for the technicalities while avoiding needless complications or misplaced ‘rigour’, though I do like elegance too. Given the choice, I tend to prefer a treatment that doesn’t rush too fast to become too general, too abstract, and thereby obscures intuitive motivation. (There’s a certain tradition of masochism in maths writing, of going for brusque formal abstraction from the outset: that is unnecessary in other areas, and just because logic is all about formal theories, that doesn’t make it any more necessary here.)

The selection of books in later chapters no doubt reflects these tastes. But overall, I don’t think that I have been downright idiosyncratic. Nearly all the books I recommend will very widely be agreed to have significant virtues (even if some logicians would have different preference-orderings).

1.4 A strategy for reading logic books (and why this Guide is so long)

We cover a lot in this Guide, which is one reason for its size. But there is another reason.
I very strongly recommend tackling an area of logic (or indeed any new area of mathematics) by reading a series of books which overlap in level (with the next one covering some of the same ground and then pushing on from the previous one), rather than trying to proceed by big leaps.

In fact, I probably can’t stress this advice too much, which is why I am highlighting it here in a separate section. For this approach will really help to reinforce and deepen understanding as you re-encounter the same material from different angles, with different emphases.

The multiple overlaps in coverage in the reading lists below are therefore fully intended, and this explains why the lists are always longer rather than shorter (and also means that you should more often be able to find options that suit your degree of mathematical competence). You will certainly miss a lot if you concentrate on just one text in a given area, especially at the outset. Yes, do very carefully read one or two central texts, at a level that appeals to you. But do also cultivate the additional habit of judiciously skipping and skimming through a number of other works so that you can build up a good overall picture of an area seen from various somewhat different angles of approach.

1.5 Other preliminary points

(a) Mathematics is not merely a spectator sport: you should try some of the exercises in the books as you read along to check and reinforce comprehension. On the other hand, don’t obsess about doing exercises if you are a philosopher – understanding proof ideas is very much the crucial thing, not the ability to roll-your-own proofs. And even mathematicians shouldn’t get too hung up on routine exercises (unless you have specific exams to prepare for!): concentrate on the exercises that look interesting and/or might deepen understanding.

Do note however that some authors have the irritating(?) habit of burying important results in the exercises, mixed in with routine homework, so it is often worth skimming through the exercises even if you don’t plan to tackle very many of them.
(b) Nearly all the books mentioned here should be held by any university library which has been paying reasonable attention to maintaining a core collection in the area of logic (and any book should be borrowable through your interlibrary loans system). If some books you particularly want to read are missing from the shelves then, in my experience, university librarians are quite happy to get informed recommendations of books which are reliably warranted actually to be good of their kind.

Since I’m not assuming you will be buying personal copies, I have not made cost or even being currently in print a significant consideration: indeed it has to be said that the list price of some of the books is just ridiculous (though second-hand copies of some books at better prices might well be available via Amazon sellers or from abebooks.com). However, I have marked with one star* books that are available new at a reasonable price (or at least are unusually good value for the length and/or importance of the book). And I’ve marked with two stars** those books for which e-copies are freely (and legally!) available and links are provided.¹ Most articles, encyclopaedia entries, etc., can also be downloaded, again with links supplied.

(c) And yes, the references here are very largely to published books and articles rather than to on-line lecture notes etc. Many such notes are excellent, but they do tend to be a bit terse (as befits material intended to support a lecture course) and so perhaps not as helpful as fully-worked-out book-length treatments for students needing to teach themselves. But I’m sure that there is an increasing number of excellent e-resources out there which do amount, more or less, to free stand-alone textbooks. I’d be very happy indeed to get recommendations about the best.

(d) Finally, the earliest versions of this Guide kept largely to positive recommendations: I didn’t pause to explain the reasons for the then absence of some well-known books. This was partly due to considerations of length which have now quite gone by the wayside; but also I wanted to

¹We will have to pass over in silence the issue of illegal file-sharing of PDFs of e.g. out-of-print books: most students will know the possibilities here.
keep the tone enthusiastic, rather than to start criticizing or carping.

However, enough people have asked what I think about the classic $X$, or asked why the old warhorse $Y$ wasn’t mentioned, to change my mind. So I have very occasionally added some reasons why I don’t particularly recommended certain books.
Chapter 2

Logical geography

This chapter starts by explaining how the field of logic is being carved up in this Guide. I then add something about the (limited!) background that you are going to be assumed to have.

‘Logic’ is a big field. Its technical development is of concern to philosophers, mathematicians and computer scientists. Different constituencies will be particularly interested in different areas and give different emphases. For example, modal logic is of considerable interest to some philosophers, but also parts of this sub-discipline are of concern to computer scientists too. Set theory (which falls within the purview of mathematical logic, broadly understood) is an active area of research interest in mathematics, but – because of its (supposed) foundational status – even quite advanced results can be of interest to philosophers too. Type theory started off as a device of philosophy-minded logicians looking to avoid the paradoxes: it has become primarily the playground of computer scientists. The incompleteness theorems are relatively elementary results of the theory of computable functions, but are of particular conceptual interest to philosophers. Finite model theory is of interest to mathematicians and computer scientists, but perhaps not so much to philosophers. And so it goes.

In this Guide, I’m going to have to let the computer scientists largely look after themselves. Our focus is going to be the areas of logic of most interest to philosophers and mathematicians, if only because that’s what I know a little about. So what’s the geography of these areas?
2.1 Mapping the field

Here’s an overview of the territory we cover in the Guide:

- **First-order logic (Ch. 3).** The serious study of logic always starts with a reasonably rigorous treatment of quantification theory, covering both a proof-system for classical FOL, and the standard classical semantics, getting at least as far as a soundness and completeness proof for your favourite proof system. I have both given an outline of the topics that need to be covered and also suggested a number of different ways of covering them (I’m not suggesting you read all the texts mentioned!). This part of the Guide is therefore quite long even though it doesn’t cover a lot of ground: it does, however, provide the essential foundation for getting to grips with . . .

- **The basic ‘Mathematical Logic’ curriculum (Ch. 4)** Mathematical logic courses typically comprise – in one order or another and in various proportions – four elements:

1. **A little model theory**, i.e. a little more exploration of the fit between theories cast framed in formal languages and the structures they are supposed to be ‘about’. This will start with the compactness theorem and Löwenheim-Skolem theorems (if these aren’t already covered in your basic FOL reading), and then will push on just a bit further. You will need to know a very little set theory as background (mainly ideas about cardinality), so you might want to interweave beginning model theory with the very beginnings of your work on set theory if those cardinality ideas are new to you.

2. **Computability and decidability**, and proofs of epochal results such as Gödel’s incompleteness theorems. This is perhaps the most readily approachable area of mathematical logic.

3. **Some introductory set theory.** Informal set theory, basic notions of cardinals and ordinals, constructions in set theory,
the role of the axiom of choice, etc. The formal axiomatization of ZFC.

4. **Extras: variants of standard FOL** The usual additional material that you meet in a first serious mathematical logic course covers Second Order Logic (where we can quantify over properties as well as objects) and Intuitionistic Logic (which drops the law of excluded middle, motivated by a non-classical understanding of the significance of the logical operators). Both are of some technical and philosophical interest.

- **Modal and other logics (Ch. 5)** Now we move on in the Guide to consider a number of logical topics of particular concern to philosophers. Some of these topics are actually quite approachable even if you know little other logic. So the fact this material is mentioned after the heavier-duty mathematical topics in the previous two chapters doesn’t mean that there is a jump in difficulty. We look at:

  1. **Modal logic** Even before encountering a full-on-treatment of first-order logic, philosophers are often introduced to modal logic (the logic of necessity and possibility) and to its ‘possible world semantics’. You can indeed do an amount of propositional modal logic with no more than ‘baby logic’ as background.

  2. **Other classical variations** and extensions of standard logic which are of conceptual interest but still classical in spirit, e.g. free logic, plural logic.

  3. **Non-classical variations** The most important non-classical logic – and the one of real interest to mathematicians too – is intuitionist logic which we’ve already mentioned. But here we consider e.g. relevant logics which drop the classical rule that a contradiction entails anything.

- **Exploring further (Chs 6, 7)** The introductory books mentioned in Chapters 3, 4 and 5 will already contain numerous pointers to
further reading, more than enough to put you in a position to continue exploring solo. However, Chapter 6 does offer my suggestions for more advanced reading on the core topics in mathematical logic. This is mostly for specialist graduate students (among philosophers) and final year undergraduate/beginning graduate students (among mathematicians). And then Chapter 7 very briefly gestures at some topics outside the traditional core.

Don’t be alarmed if (some of) those descriptions above are at the moment pretty opaque to you: we will be explaining things a little more as we go through the Guide.

Three comments on all this:

i. We are dividing up the broad field of logic into subfields in a fairly conventional way. But of course, even the horizontal divisions into different areas can in places be a little arbitrary. And the vertical division between the entry-level readings in earlier chapters and the further explorations in Chapter 6 is evidently going to be a lot more arbitrary. I think that everyone will agree (at least in retrospect!) that e.g. the elementary theory of ordinals and cardinals belongs to the basics of set theory, while explorations of ‘large cardinals’ or independence proofs via forcing are decidedly advanced. But in most areas, there are far fewer natural demarcation lines between the basics and more advanced work. Still, it is surely very much better to have some such structuring than to heap everything together.

ii. Within sections in Chapters 3–6, I have put the main recommendations into what strikes me as a sensible reading order of increasing difficulty (without of course supposing you will want to read everything – those with stronger mathematical backgrounds might sometimes want to try starting in the middle of a list). Some further books are listed in small-print asides or postscripts.

iii. The Guide used also to have a substantial Appendix considering some of ‘The Big Books on mathematical logic’ (meaning typically broader-focus books that cover first-order logic together with one or
more subfields from the further menu of mathematical logic). These books vary a lot in level and coverage, but can provide very useful consolidating/amplifying reading. This supplement to the main Guide is now available online as an Appendix. Alternatively, for separate webpages on those texts and a number of additional reviews, visit the Logic Matters Book Notes.

2.2 Assumed background: ‘baby logic’

If you are a mathematician, you will have picked up a smattering of logical ideas and notations. But there is no specific assumed background you really need before tackling the books recommended in the next chapter. So you can just skip on to §2.4.

If you are a philosopher without any mathematical background, however, you will almost certainly need to have already done some formal logic if you are going to cope with what are, in the end, mathematics books on areas of mathematical logic.

- So, if you have only done an ‘informal logic’ or ‘critical reasoning course’, then you’ll surely need to read a good introductory formal logic text before tackling more advanced work. See below.

- If you have taken a logic course with a formal element, but it was based on some really, really, elementary text book like Sam Guttenplan’s The Languages of Logic, Howard Kahane’s Logic and Philosophy, or Patrick Hurley’s Concise Introduction to Logic, then you might still struggle with the initial suggestions in §3.2 (though these things are very hard to predict). If you do, one possibility would be to use Intermediate Logic by Bostock mentioned in §3.3 to bridge the gap between what you know and the beginnings of mathematical logic. Or again, try one of the books below, skipping fast over what you already know.

- If you have taken a formal ‘baby logic’ course based on a substantial text like those mentioned below, then you should be well prepared.
Here then, for those that need them, are couple of initial suggestions of formal logic books that start from scratch and go far enough to provide a good foundation for further work – the core chapters of these cover the ‘baby logic’ that it would be ideal to have under your belt:

1. My *Introduction to Formal Logic* (CUP 2003, corrected reprint 2013): for more details see the [IFL pages](#), where there are also answers to the exercises). This is intended for beginners, and was the first year text in Cambridge for a decade. It was written as an accessible teach-yourself book, covering propositional and predicate logic ‘by trees’. It in fact gets as far as a completeness proof for the tree system of predicate logic without identity, though for a beginner’s book that’s very much an optional extra.

2. Paul Teller’s *A Modern Formal Logic Primer* (Prentice Hall 1989) predates my book, is now out of print, but is freely available online at the book’s [website](#), which makes it unbeatable value! It is in many ways quite excellent, and had I known about it at the time (or listened to Paul’s good advice, when I got to know him, about how long it takes to write an intro book), I’m not sure that I’d have written my own book, full of good things though it is! As well as introducing trees, Teller also covers a version of ‘Fitch-style’ natural deduction – I didn’t have the page allowance to do this, regrettably. (Like me, he also goes beyond the really elementary by getting as far as a completeness proof.) Notably user-friendly. Answers to exercises are available at the website.

Of course, those are just two possibilities from very many. I have not latterly kept up with the seemingly never-ending flow of books of entry-level introductory logic books, some of which are no doubt excellent too, though there are also some rather poor books out there (and also there are more books like Guttenplan’s that will probably not get readers to the starting line as far as this Guide is concerned). Mathematicians should also be warned that some of the books on ‘discrete mathematics’ cover elementary logic pretty badly. This is not the place, however, to discuss
lots more options for elementary texts. I’ll mention here just two other books:

3. I have been asked a number of times about Jon Barwise and John Etchemendy’s *Language, Proof and Logic* (CSLI Publications, 1999; 2nd edition 2011 – for more, see the book’s website). The unique selling point for this widely used book is that it comes with a CD of programs, including a famous one by the authors called ‘Tarski’s World’ in which you build model worlds and can query whether various first-order sentences are true of them. Some students really like it, and but at least equally many don’t find this kind of thing particularly useful. And I believe that the CD can’t be registered to a second owner, so you have to buy the book new to get the full advantage.

   On the positive side, this is another book which is in many respects user-friendly, goes slowly, and does Fitch-style natural deduction. It is a respectable option. But Teller is rather snappier, and I think no less clear, and certainly wins on price!

4. Nicholas Smith’s recent *Logic: The Laws of Truth* (Princeton UP 2012) is very clearly written and seems to have many virtues. The first two parts of the book overlap very largely with mine (it too introduces logic by trees). But the third part ranges wider, including a brisk foray into natural deduction – indeed the logical coverage goes almost as far as Bostock’s book, mentioned below in §3.3, and are there some extras too. It seems a particularly readable addition to the introductory literature. I have commented further here. Answers to exercises can be found at the book’s website.

### 2.3 Do you *really* need more logic?

This section is for philosophers. It is perhaps worth pausing to ask such readers if they are sure – especially if they already worked through a book like mine or Paul Teller’s on Nick Smith’s – whether they really *do* want
or need to pursue things much further, at least in a detailed, formal, way. Far be it from me to put people off doing more logic: perish the thought! But for many purposes, you might well survive by just reading the likes of

1. David Papineau, *Philosophical Devices: Proofs, Probabilities, Possibilities, and Sets* (OUP 2012). From the publisher’s description: ‘This book is designed to explain the technical ideas that are taken for granted in much contemporary philosophical writing. . . . The first section is about sets and numbers, starting with the membership relation and ending with the generalized continuum hypothesis. The second is about analyticity, a prioricity, and necessity. The third is about probability, outlining the difference between objective and subjective probability and exploring aspects of conditionalization and correlation. The fourth deals with metalogic, focusing on the contrast between syntax and semantics, and finishing with a sketch of Gödel’s theorem.’

Or better – since perhaps Papineau arguably gives rather *too* brisk an overview – you could rely on

2. Eric Steinhart, *More Precisely: The Math You Need to Do Philosophy* (Broadview 2009) The author writes: ‘The topics presented . . . include: basic set theory; relations and functions; machines; probability; formal semantics; utilitarianism; and infinity. The chapters on sets, relations, and functions provide you with all you need to know to apply set theory in any branch of philosophy. The chapter of machines includes finite state machines, networks of machines, the game of life, and Turing machines. The chapter on formal semantics includes both extensional semantics, Kripkean possible worlds semantics, and Lewisian counterpart theory. The chapter on probability covers basic probability, conditional probability, Bayes theorem, and various applications of Bayes theorem in philosophy. The chapter on utilitarianism covers act utilitarianism, applications involving utility and probability (expected utility), and applications involving
possible worlds and utility. The chapters on infinity cover recursive
definitions, limits, countable infinity, Cantor’s diagonal and power
set arguments, uncountable infinities, the aleph and beth numbers,
and definitions by transfinite recursion. More Precisely is designed
both as a text book and reference book to meet the needs of upper
type undergraduates and graduate students. It is also useful as a
reference book for any philosopher working today.’

Steinhart’s book is admirable, and will give many philosophers a handle
on some technical ideas going well beyond baby logic and which they
really should know just a little about, without the hard work of doing a
‘real’ logic course. What’s not to like? And if there indeed turns out to
be some particular area (modal logic, for example) that seems especially
germane to your philosophical interests, then you can go to the relevant
section of this Guide for more.

2.4 How to prove it

Assuming, however, that you do want to learn more serious logic, before
getting down to business in the next chapter, let me mention one other –
rather different and often-recommended – book:

Daniel J. Velleman, How to Prove It: A Structured Approach (CUP,
often have trouble the first time that they’re asked to work seriously
with mathematical proofs, because they don’t know ‘the rules of the
game’. What is expected of you if you are asked to prove something?
What distinguishes a correct proof from an incorrect one? This book
is intended to help students learn the answers to these questions by
spelling out the underlying principles involved in the construction
of proofs.’

There are chapters on the propositional connectives and quantifiers, and
informal proof-strategies for using them, and chapters on relations and
functions, a chapter on mathematical induction, and a final chapter on
infinite sets (countable vs. uncountable sets). This truly excellent student text could well be of use both to many philosophers and to some mathematicians reading this Guide. By the way, if you want to check your answers to exercises, here’s a long series of blog posts (in reverse order).

True, if you are a mathematics student who has got to the point of embarking on an upper level undergraduate course in some area of mathematical logic, you should certainly have already mastered nearly all the content of Velleman’s splendidly clear book. However, an afternoon or two skimming through this text (except perhaps for the very final section), pausing over anything that doesn’t look very comfortably familiar, could still be time extremely well spent.

What if you are a philosophy student who (as we are assuming) has done some baby logic? Well, experience shows that being able to handle e.g. natural deduction proofs in a formal system doesn’t always translate into being able to construct good informal proofs. For example, one of the few meta-theoretic results usually met in a baby logic course is the expressive completeness of the set of formal connectives \{\land, \lor, \neg\}. The proof of this result is really easy, based on a simple proof-idea. But many students who can ace the part of the end-of-course exam asking for quite complex formal proofs inside a deductive system find themselves all at sea when asked to replicate this informal bookwork proof about a formal system. Another example: it is only too familiar to find philosophy students introduced to set notation not even being able to make a start on a good informal proof that \{\{a\}, \{a, b\}\} = \{\{a\}', \{a\}', \{b\}'\}\} if and only if \(a = a'\) and \(b = b'\).

Well, if you are one of those students who jumped through the formal hoops but were unclear about how to set out elementary mathematical proofs (e.g. from the ‘meta-theory’ theory of baby logic, or from baby set theory), then again working through Velleman’s book from the beginning could be just what you need to get you prepared for the serious study of logic. And even if you were one of those comfortable with the informal proofs, you will probably still profit from skipping and skimming through (perhaps paying especial attention to the chapter on mathematical induction).
Chapter 3

First order logic

This chapter gives a checklist of the topics we will be treating as belonging to the basics of first-order logic (FOL). And then gives some reading recommendations.

3.1 FOL: the basic topics

For those who already have some significant background in formal logic, this list will enable you to determine whether you can skip forward in the Guide. If you find that you are indeed familiar with these topics, then go straight to the next chapter. While for those who know only (some fragments of) baby logic, then this list might provide some preliminary orientation – though don’t worry, of course, if at present you don’t fully grasp the import of every point. The next section gives recommended reading to help you reach enlightenment!

So, without further ado, here’s what you need to get to know about:

1. Starting with syntax, you need to know how first-order languages are constructed. And now you need to understand how to prove various things that seem obvious and that you previously took for granted, e.g. that ‘bracketing works’ to avoid ambiguities, meaning that every well-formed formula has a unique parsing.

   Baby logic courses for philosophers very often ignore functions; but given that FOL is deployed to regiment everyday mathematical
reasoning, and that functions are of course crucial to mathematics, function expressions now become centrally important (even if there are tricks that make them in principle eliminable).

2. On the *semantic* side, you need to understand the idea of a structure (a domain of objects equipped with some relations and/or functions, and perhaps having some particular objects especially picked out, e.g. as a zero or unit), and you need to grasp the idea of an interpretation of a language in such a structure. You’ll also need to understand how an interpretation generates an assignment of truth-values to every sentence of the interpreted language – this means grasping a proper formal semantic story with the bells and whistles required to cope with quantifiers adequately.

You now can define the notion of semantic entailment (where $\Gamma$ semantically entails $\varphi$ when no interpretation in any structure can make all the sentences among $\Gamma$ true without making $\varphi$ true too.)

3. Back to syntax: you need to get to know a deductive proof-system for FOL reasonably well. But what sort of system should you explore first? It is surely natural to give centre stage, at least at the outset, to so-called *natural deduction* systems.

The key feature of natural deduction systems is that they allow us to make temporary assumptions ‘for the sake of argument’ and then later discharge the temporary assumptions. This is, of course, exactly what we do all the time in everyday reasoning, mathematical or otherwise – as, for example, when we temporarily suppose $A$, show it leads to absurdity, and then drop the supposition and conclude $\neg A$. So surely it is indeed natural to seek to formalize these key informal ways of reasoning.

Different formal natural deduction systems will offer different ways of handling temporary assumptions, keeping track of them while they stay ‘live’, and then showing where in the argument they get discharged. Suppose, for example, we want to show that from $\neg(P \land \neg Q)$ we can infer $(P \rightarrow Q)$ (where ‘$\neg$’, ‘$\land$’ and ‘$\rightarrow$’ are of course our symbols for, respectively, *not*, *and*, and [roughly] *implies*. )
Then one way of laying out a natural deduction proof would be like this, where we have added informal commentary on the right:

1. $\neg(P \land \neg Q)$  
   premiss
2. $P$  
   supposition for the sake of argument
3. $\neg Q$  
   supposition for the sake of argument
4. $(P \land \neg Q)$  
   from 2, 3 by and-introduction
5. $\bot$  
   1 and 4 give us a contradiction
6. $\neg \neg Q$  
   supp’n 3 must be false, by reductio
7. $Q$  
   from 6, eliminating the double negation
8. $(P \rightarrow Q)$  
   if the supposition 2 is true, so is 7.

Here, the basic layout principle for such Fitch-style proofs\(^1\) is that, whenever we make a new temporary assumption we indent the line of argument a column to the right (vertical bars marking the indented columns), and whenever we ‘discharge’ the assumption at the top of an indented column we move back to the left.

An alternative layout (going back to Genzten) would display the same proof like this, where the guiding idea is (roughly speaking) that a wff below an inference line follows from what is immediately above it, or by discharge of some earlier assumption:

\[
\frac{\frac{\frac{[P]^{(2)}}{(P \land \neg Q)}}{\frac{[\neg Q]^{(1)}}{\neg(P \land \neg Q)}}}{\bot^{(1)}}^{(1)} \quad \frac{\frac{\frac{\frac{\frac{\neg \neg Q}^{(1)}}{Q}}{(P \rightarrow Q)}^{(2)}}}{(P \rightarrow Q)}^{(2)}
\]

\(^1\)See the introductory texts by Teller or by Barwise and Etchemendy mentioned in §2.2 for more examples of natural deduction done in this style.
Here numerical labels are used to indicate where a supposition gets discharged, and the supposition to be discharged gets bracketed off (or struck through) and also given the corresponding label.

Fitch-style proofs are perhaps easier to use for beginners (indeed, we might say, especially natural, by virtue of more closely follow the basically linear style of ordinary reasoning). But Gentzen-style proofs are usually preferred for more advanced work, and that’s what the natural deduction texts that I’ll be mentioning work.

4. Next – and for philosophers this is likely to be the key move beyond even a substantial baby logic course, and for mathematicians this is probably the point at which things actually get interesting – you need to know how to prove a soundness and a completeness theorem for your favourite deductive system for first-order logic. That is to say, you need to be able to show that there’s a deduction in your chosen system of a conclusion from given premisses only if the premisses do indeed semantically entail the conclusion (the system doesn’t give false positives), and whenever an inference is semantically valid there’s a formal deduction of the conclusion from the premisses (the system captures all the semantical entailments).

5. As an optional bonus at this stage, depending on what text you read, you might catch a first glimpse of e.g. the downward Löwenheim-Skolem theorem (roughly, assuming we are dealing with an ordinary sort of first-order language $L$, if there is any infinite model that makes the set of $L$-sentences $\Gamma$ all true, then there is a model which does this whose domain is the natural numbers), and the compactness theorem (if $\Gamma$ semantically entails $\varphi$ then there is some finite subset of $\Gamma$ which already semantically entails $\varphi$). These are initial results of model theory which flow quickly from the proof of the completeness theorem.

However, we will be returning to these results in §4.1 when we turn to model theory proper, so it won’t be assumed that your knowledge of basic first-order logic covers them.
6. Finally, you of course don’t want to get confused by meeting variant versions of standard FOL _too_ soon. But you should at some fairly early stage become aware of some of the alternative ways of doing things, so you don’t become fazed when you encounter different approaches in different more advanced texts. In particular, you will need to get to know a little about old-school Hilbert-style _axiomatic_ linear proof-systems. A standard such system for e.g. the propositional calculus has a single rule of inference (modus ponens) and a bunch of axioms which can be called on at any stage in a proof. A Hilbert-style proof is just a linear sequence of wffs, each one of which is a given premiss, or a logical axiom, or follows from earlier wffs in the sequence by modus ponens. Such systems in their unadorned form are pretty horrible to work inside (proofs can be long and very unobvious), even though their Bauhaus simplicity makes them easy to theorize about from the outside. It does strike me as potentially off-putting, even a bit masochistic, to concentrate primarily on axiomatic systems when beginning serious logic – Wilfrid Hodges rightly calls them ‘barbarously unintuitive’. But for various reasons they are often used in more advanced texts, so you certainly need to get to know about them sooner or later.

3.2 The main recommendations

Let’s start with a couple of stand-out books which strike me as particularly good and, taken together, make an excellent introduction to the serious study of FOL.

1. Ian Chiswell and Wilfrid Hodges, _Mathematical Logic_ (OUP 2007). This very nicely written text is only a small notch up in actual difficulty from ‘baby logic’ texts like mine or Paul Teller’s or Nick Smith’s: but – as its title might suggest – it does have a notably more mathematical ‘look and feel’ (being indeed written by mathematicians). However, despite that, it remains particularly friendly and approachable and should be entirely manageable even at this very
early stage. It is also pleasingly short. Indeed, I’m rather tempted to say that if you can’t cope with this lovely book then serious logic might not be for you!

The headline news is that authors explore a (Gentzen-style) natural deduction system. But by building things up in stages over three chapters – so after propositional logic, they consider an interesting fragment of first-order logic before turning to the full-strength version – they make proofs of e.g. the completeness theorem for first-order logic quite unusually comprehensible. For a more detailed description see my book note on C&H.

Very warmly recommended, then. For the moment, you only need read up to and including §7.7; but having got that far, you might as well read the final couple of sections and the Postlude too! (The book has brisk solutions to some exercises. A demerit mark to OUP for not publishing C&H more cheaply.)

Next, complement C&H by reading the first half of

2. Christopher Leary’s A Friendly Introduction to Mathematical Logic (Prentice Hall 2000: currently out of print, but a slightly expanded new edition is being planned). There is a great deal to like about this book. Chs. 1–3 do make a very friendly and helpful introduction to first-order logic, this time done in axiomatic style: and indeed at this stage you could stop reading after §3.2, which means you will be reading just over 100 pages. Unusually, Leary dives straight into a full treatment of FOL without spending an introductory chapter or two on propositional logic: but that happily means (in the present context) that you won’t feel that you are labouring through the very beginnings of logic one more time than is really necessary – so this book dovetails very nicely with C&H. The book is again written by a mathematician for a mostly mathematical audience so some illustrations of ideas can presuppose a smattering of elementary background mathematical knowledge; but you will miss very little if you occasionally have to skip an example (and philosophers can always resort to Wikipedia, which is quite reliable in this area,
for explanations of mathematical terms). I like the tone very much
indeed, and say more about this admirable book in another book
note.

Next, here’s an alternative to the C&H/Leary pairing which is also
wonderfully approachable and can be warmly recommended:

3. Derek Goldrei’s *Propositional and Predicate Calculus: A Model of
Argument* (Springer, 2005) is explicitly designed for self-study.
Read Chs. 1 to 5 (you could skip §§4.4 and 4.5, leaving them un-
til you turn to elementary model theory). While C&H and Leary
together cover overlapping material twice, Goldrei – in much the
same total number of pages – covers very similar ground once. So
this is a somewhat more gently-paced book, with a lot of examples
and exercises to test comprehension along the way. A great deal of
thought has gone into making this text as helpful as possible. So
if you struggle slightly with the alternative reading, or just want a
comfortingly manageable additional text, you should find this ex-
ceptionally accessible and useful.

Like Leary, Goldrei uses an axiomatic system (which is one rea-
son why, on balance, I recommend starting with C&H instead: you’ll
need to get to know about natural deduction at some point). But
unlike Leary who goes on to deal with Gödel’s incompleteness the-
orem, Goldrei only strays beyond the basics of FOL to touch on
some model-theoretic ideas at the end of Ch. 4 and in the final Ch.
6, allowing him to be more expansive about fundamentals. As with
C&H and Leary, I like the tone and approach a great deal.

Fourthly in this section, even though it is giving a second bite to an
author we’ve already met, I must mention

4. Wilfrid Hodges’s ‘Elementary Predicate Logic’, in the *Handbook of
Philosophical Logic*, Vol. 1, ed. by D. Gabbay and F. Gunthner,
(Kluwer 2nd edition 2001). This is a slightly expanded version of
the essay in the first edition of the *Handbook* (read that earlier ver-
sion if this one isn’t available), and is written with Hodges’s usual
enviable clarity and verve. As befits an essay aimed at philosophically minded logicians, it is full of conceptual insights, historical asides, comparisons of different ways of doing things, etc., so it very nicely complements the more conventional textbook presentations of C&H and Leary (or of Goldrei). Read at this stage the first twenty sections (70 pp.): they are wonderfully illuminating.

Note (particularly to philosophers): If you have carefully read and mastered a book covering baby-logic-plus-a-little, you will actually already know about quite a few of the headline results covered in the reading mentioned in this section. However, the big change is that you will now have begun to see the familiar old material being re-presented in the sort of mathematical style and with the sort of rigorous detail that you will necessarily encounter more and more as you progress in logic, and that you very much need to start feeling entirely comfortable with at an early stage.

3.3 Some other texts for first-order logic

The material covered in the last section is so very fundamental, and the alternative options so very many, that I really need to say at least something about a few other books and note some different approaches.

So in this section I will list – in rough order of difficulty/sophistication – a small handful of recommendations that have particular virtues of one kind or another. These books will either make for parallel reading or will usefully extend your knowledge of first-order logic or both. There follows a postscript where I explain why I haven’t recommended certain texts. Some of my Book Notes will also tell you a little about how some other Big Books on mathematical logic handle the basics.

First, if you’ve read C&H and Leary you will know about natural deduction and axiomatic approaches to logic. If you are a philosopher, you may also have already encountered a (downward-branching) ‘tree’ or ‘tableau’ system of logic which is often used in baby logic courses. To illustrate, a tableau proof to warrant the same illustrative inference from
\(\neg(P \land \neg Q)\) to \((P \rightarrow Q)\) runs as follows:

\[
\begin{array}{c}
\neg(P \land \neg Q) \\
\neg(P \rightarrow Q) \\
P \\
\neg Q \\
\hline
\neg P & \neg \neg Q
\end{array}
\]

What we are doing here is starting with the given premiss and the negation of the target conclusion, and we then proceed to unpack the implications of these assumptions for their component wffs. In this case, for the second (negated-conditional) premiss to be true, it must have a true antecedent and false consequent: while for the first premiss to be true we have to consider cases (a false conjunction must have at least one false conjunct but we don’t know in advance which to blame) – which is why the tree splits. But as we pursue either branch we immediately hit contradiction with what we already have (conventionally marked with a star in tableau systems), showing that we can’t after all consistently have the premiss true and the negation of the conclusion true too, so the inference is indeed valid.

If you haven’t encountered such a system, it is worth catching up. Unsurprisingly(!), I still think the best way to do this might be quickly to read the gentle treatment in

1. Peter Smith *Introduction to Formal Logic* (CUP 2003, corrected reprint 2013). Chs 16–18 cover trees for propositional logic and then Chs. 25, 29 are on on quantifier trees. (Chs 19 and 30 give soundness and completeness proofs for the tree system, and could also be illuminating.).

Or you could jump straight to the next recommendation which covers trees but also a great deal more.

This is a text by a philosopher, aimed at philosophers, though mathematicians could still profit from a quicker browse:
2. David Bostock’s *Intermediate Logic* (OUP 1997) ranges more widely but not as deeply as Goldrei, for example, and in a more discursive style. From the preface: ‘The book is confined to . . . what is called first-order predicate logic, but it aims to treat this subject in very much more detail than a standard introductory text. In particular, whereas an introductory text will pursue just one style of semantics, just one method of proof, and so on, this book aims to create a wider and a deeper understanding by showing how several alternative approaches are possible, and by introducing comparisons between them.’ So Bostock does indeed usefully introduce you to tableaux (trees) and an Hilbert-style axiomatic proof system and natural deduction and even a so-called sequent calculus as well (as noted, it is important eventually to understand what is going on in these different kinds of proof-system). Anyone could profit from at least a quick browse of his Part II to pick up the headline news about the various approaches.

Bostock eventually touches on issues of philosophical interest such as free logic which are not often dealt with in other books at this level. Still, the discussions mostly remain at much the same level of conceptual/mathematical difficulty as the later parts of Teller’s book and my own. In particular, he proves completeness for tableaux in particular, which I always think makes the needed construction seem particularly natural. *Intermediate Logic* should therefore be, as intended, particularly accessible to philosophers who haven’t done much formal logic before and should, if required, help ease the transition to work in the more mathematical style of the books mentioned in the last section.

Next let me mention a very nice, and freely available, alternative presentation of logic via natural deduction:


   All credit to the author for writing the first textbook aimed at an introductory level which does Gentzen-style natural deduction.
Tennant thinks that this approach to logic is philosophically highly significant, and in various ways this shows through in his textbook. Although not as conventionally mathematical in look-and-feel as some alternatives, it is in fact very careful about important details. It is not always an easy read, however, despite its being intended as a first logic text for philosophers, which is why I didn’t mention it in the last section. However it is there to freely sample, and you may well find it highly illuminating parallel reading on natural deduction.

Now, I recommended *A Friendly Introduction* as a follow-up to C&H: but Leary’s book might not in every library. So as a possible alternative, I should mention an older and much used text which should certainly be very widely available:

4. Herbert Enderton’s *A Mathematical Introduction to Logic* (Academic Press 1972, 2002) also focuses on a Hilbert-style axiomatic system, and is often regarded as a classic of exposition. However, it does strike me as a little more difficult than Leary, so I’m not surprised that some students report finding it a bit challenging *if used by itself as a first text*. Still, it is an admirable and very reliable piece of work which you should be able to cope with if you take it slowly, after you have tackled C&H. Read up to and including §2.5 or §2.6 at this stage. Later, you can finish the rest of that chapter to take you a bit further into model theory. For more about this book, see this book note.

I will mention next – though with just a little hesitation – another much used text. This has gone through multiple editions and should also be in any library, making it a useful natural-deduction based alternative to C&H if the latter isn’t available (though like Enderton’s book, this is perhaps half-a-notch up in terms of mathematical style/difficulty). Later chapters of this book are also mentioned below in the Guide as possible reading for more advanced work, so it could be worth making early acquaintance with . . .
5. Dirk van Dalen, *Logic and Structure* (Springer, 1980; 5th edition 2012). The chapters up to and including §3.2 provide an introduction to FOL via natural-deduction, which *is* often approachable and written with a relatively light touch. However – and this is why I hesitate to include the book – it has to be said that the book isn’t without its quirks and flaws and inconsistencies of presentation (though maybe you have to be an alert and pernickety reader to notice them). Still the coverage and general approach is good, and mathematicians should be able to cope readily.

I suspect, however, that the book could occasionally be tougher going for philosophers if taken from a standing start – which is one reason why I have recommended beginning with C&H instead. (See my more extended review of the whole book.)

An aside: The treatments of FOL I have mentioned so far here and in the last section include exercises, of course; but for many more exercises and this time with extensive worked solutions you could also look at the ends of Chs. 1, 3 and 4 of René Cori and Daniel Lascar, *Mathematical Logic: A Course with Exercises* (OUP, 2000). I can’t, however, particularly recommend the main bodies of those chapters.

Next, here are three more books which elaborate on core FOL in different ways. In order of increasing sophistication,

6. Jan von Plato’s *Elements of Logical Reasoning* (CUP, 2014) is based on the author’s introductory lectures. A lot of material is touched on in a relatively short compass as von Plato talks about a range of different natural deduction and sequent calculi. So I suspect that, without any classroom work to round things out, this might not be easy as a first introduction. But suppose you have already met one system of natural deduction (e.g., as in C&H), and now want to know more about ‘proof-theoretic’ aspects of this and related systems. Suppose, for example, that you want to know about variant ways of setting up ND systems, about proof-search, about the relation with so-called sequent calculi, etc. Then this is a very clear, approachable and interesting book. Experts will see that there
are some novel twists, with deductive systems tweaked to have some very nice features: beginners will be put on the road towards understanding some of the initial concerns and issues in proof theory.

7. Don’t be put off by the title of Melvin Fitting’s *First-Order Logic and Automated Theorem Proving* (Springer, 1990, 2nd end. 1996). Yes, at various places in the book there are illustrations of how to implement various algorithms in Prolog. But either you can easily pick up the very small amount of background knowledge about Prolog that’s needed to follow what’s going on (and that’s a fun thing to do anyway) or you can just skip.

As anyone who has tried to work inside an axiomatic system knows, proof-discovery for such systems is often hard. Which axiom schema should we instantiate with which wffs at any given stage of a proof? Natural deduction systems are nicer: but since we can make any new temporary assumption at any stage in a proof, again we need to keep our wits about us, at least until we learn to recognize some common patterns of proof. By contrast, tableau proofs (a.k.a. tree proofs, as in my book) very often write themselves, which is why I used to introduce formal proofs to students that way – for we thereby largely separate the business of getting across the idea of formality from the task of teaching heuristics of proof-discovery. And because tableau proofs very often write themselves, they are also good for automated theorem proving.

Now (this paragraph is for cognoscenti!), if you have seen tableaux (trees) before, you’ll know that an open tableau for a single propositional calculus wff $A$ at the top of the tree in effect constructs a disjunctive normal form for $A$ – just take the conjunction of atoms or negated atoms on each open branch of a completed tree and disjoin the results from different branches to get something equivalent to $A$. And a tableau proof that $C$ is a tautology in effect works by seeking to find a disjunctive normal form for $\neg C$ and showing it to be empty. When this is pointed out, you might well think ‘Aha! Then there ought to be a dual proof-method which works with con-
junctive normal forms, rather than disjunctive normal forms!’ And you of course must be right. This alternative is called the resolution method, and indeed is the more common approach in the automated proof community.

Fitting – always a most readable author – explores both the tableau and resolution methods in this exceptionally clearly written book. The emphasis is, then, notably different from most of the other recommended books: but the fresh light thrown on first-order logic makes the detour through this book vaut le voyage, as the Michelin guides say. (By the way, if you don’t want to take the full tour, then there’s a nice introduction to proofs by resolution in Shawn Hedman, A First Course in Logic (OUP 2004): §1.8, §§3.4–3.5.)

8. Raymond Smullyan, First-Order Logic* (Springer 1968, Dover Publications 1995) is a terse classic, absolutely packed with good things. This is the most sophisticated book I’m mentioning in this chapter. But enthusiasts can try reading Parts I and II, just a hundred pages, after C&H. Those with a taste for mathematical neatness should be able to cope with these chapters and will appreciate their great elegance. This beautiful little book is the source and inspiration of many modern treatments of logic based on tree/tableau systems, such as my own.

Not always easy, especially as the book progresses, but wonderful for the mathematically minded.

Let’s finish by mentioning another, much more quirky, recent book by the last author:

9. Raymond Smullyan’s Logical Labyrinths (A. K. Peters, 2009) won’t be to everyone’s taste. From the blurb: ‘This book features a unique approach to the teaching of mathematical logic by putting it in the context of the puzzles and paradoxes of common language and rational thought. It serves as a bridge from the author’s puzzle books to his technical writing in the fascinating field of mathematical logic.
Using the logic of lying and truth-telling, the author introduces the readers to informal reasoning preparing them for the formal study of symbolic logic, from propositional logic to first-order logic, . . . The book includes a journey through the amazing labyrinths of infinity, which have stirred the imagination of mankind as much, if not more, than any other subject.’

Smullyan starts, then, with puzzles of the kind where you are visiting an island where there are Knights (truth-tellers) and Knaves (persistent liars) and then in various scenarios you have to work out what’s true from what the inhabitants say about each other and the world. And, without too many big leaps, he ends with first-order logic (using tableaux), completeness, compactness and more. Not a substitute for more conventional texts, of course, but – for those with a taste for being led up to the serious stuff via sequences of puzzles – an entertaining and illuminating supplement.

Postcript  Obviously, I have still only touched on a very small proportion of books that cover first-order logic. The Appendix covers another handful. But I end this chapter responding to some particular Frequently Asked Questions prompted by earlier versions of the Guide.

What about Mendelson?  Somewhat to my surprise, perhaps the most frequent question I have been asked in response to earlier versions of the Guide is ‘But what about Mendelson, Chs. 1 and 2’? Well, Elliot Mendelson’s Introduction to Mathematical Logic (Chapman and Hall/CRC 5th edn 2009) was first published in 1964 when I was a student and the world was a great deal younger. The book was the first modern textbook at its level (so immense credit to Mendelson for that), and I no doubt owe my career to it – it got me through tripos! And it seems that some who learnt using the book are in their turn still using it to teach from. But let’s not get sentimental! It has to be said that the book is often brisk to the point of unfriendliness, and the basic look-and-feel of the book hasn’t changed as it has run through successive editions. Mendelson’s presentation of axiomatic systems of logic are quite tough going, and as the book progresses in later chapters through formal number theory and set theory, things if anything get somewhat less reader-friendly. Which doesn’t mean the book won’t repay battling with. But unsurprisingly, nearly fifty years on, there are many
rather more accessible and more amiable alternatives for beginning serious logic. Mendelson’s book is certainly a landmark worth visiting one day, but don’t start there. For a little more about it, see here.

If you do really want an old-school introduction from roughly the same era, I’d recommend instead Geoffrey Hunter, Metalogic* (Macmillan 1971, University of California Press 1992). This is not groundbreaking in the way e.g. Smullyan’s First-Order Logic is, nor is it as comprehensive as Mendelson: but it is still an exceptionally good student textbook from a time when there were few to choose from, and I still regard it with admiration. Read Parts One to Three at this stage. And if you are enjoying it, then do eventually finish the book: it goes on to consider formal arithmetic and proves the undecidability of first-order logic, topics we revisit in §4.2. Unfortunately, the typography – from pre-L\LaTeX days – isn’t at all pretty to look at: this can make the book’s pages appear rather unappealing. But in fact the treatment of an axiomatic system of logic is extremely clear and accessible. Worth blowing the dust off your library’s copy!

What about Bell, DeVidi and Solomon? If you concentrated at the outset on a one-proof-style book, you would do well to widen your focus at an early stage to look at other logical options. And one good thing about Bostock’s book is that it tells you about different styles of proof-system. A potential alternative to Bostock at about the same level, and which initially looks promising, is John L. Bell, David DeVidi and Graham Solomon’s Logical Options: An Introduction to Classical and Alternative Logics (Broadview Press 2001). This book covers a lot pretty snappily – for the moment, just Chapters 1 and 2 are relevant – and a few years ago I used it as a text for second-year seminar for undergraduates who had used my own tree-based book for their first year course. But many students found it quite hard going, as the exposition is terse, and I found myself having to write very extensive seminar notes. For example, see my notes on Types of Proof System, which gives a brisk overview of some different proof-styles (written for those who had first done logic using by tableau-based introductory book). If you want some breadth, you’d do better sticking with Bostock.

And what about Sider? Theodore Sider – a very well-known philosopher – has written a text called Logic for Philosophy* (OUP, 2010) aimed at philosophers, which I’ve been asked to comment on. The book in fact falls into two halves. The second half (about 130 pages) is on modal logic, and I will return to that in §5.1. The first half of the book (almost exactly the same length) is on propositional and first-order logic, together with some variant logics, so is very much on the topic of this chapter. But while the coverage of modal logic is quite good, I can’t
at all recommend the first half of this book.

Sider starts with a system for propositional logic of sort-of-sequent proofs in what is pretty much the style of E. J. Lemmon’s 1965 book *Beginning Logic*. Which, as anyone who spent their youth teaching a Lemmon-based course knows, students do not find user-friendly. Why on earth do things this way? We then get shown a Hilbertian axiomatic system with a bit of reasonably clear explanation about what’s going on. But there are much better presentations for the marginally more mathematical.

Predicate logic gets only an axiomatic deductive system. Again, I can’t think this is the best way to equip *philosophers* who might have a perhaps shaky grip on formal ideas with a better understanding of how a deductive calculus for first-order logic might work, and how it relates to informal rigorous reasoning. But, as I say, if you *are* going to start with an axiomatic system, there are better alternatives. The explanation of the semantics of a first-order language is quite good, but not outstanding either.

Some of the decisions about what technical details then to cover in some depth and what to skim over are pretty inexplicable. For example, there are pages tediously proving the mathematically unexciting deduction theorem for axiomatic propositional logic (the result matters of course, but the proof is one simple idea and all the rest is tedious checking): yet later just one paragraph is given to the deep compactness theorem for first-order logic, which a philosophy student starting on the philosophy of mathematics might well need to know about and understand some applications of. Why this imbalance? By my lights, then, the first-half of Sider’s book certainly isn’t the go-to treatment for giving philosophers a grounding in core first-order logic.

True, a potentially attractive additional feature of this part of Sider’s book is that it does contain brief discussions about e.g. some non-classical propositional logics, and about descriptions and free logic. But remember all this is being done in 130 pages, which means that things are whizzing by very fast. For example, the philosophically important issue of second-order logic is dealt with far too quickly to be useful. And at the introductory treatment of intuitionistic logic is also far too fast. So the breadth of Sider’s coverage here goes with far too much superficiality. Again, if you want some breadth, Bostock is still to be preferred, plus perhaps some reading from §5.3 below.
Chapter 4

Starting Mathematical Logic

We now press on from an initial look at first-order logic to consider other core elements of mathematical logic. Recall, the three main topics we need to cover are

- Some elements of the model theory for first-order theories.
- Formal arithmetic, theory of computation, Gödel’s incompleteness theorems.
- Elements of set theory.

But at some point we’ll also need to touch briefly on two standard ‘extras’

- Second-order logic and second-order theories; intuitionistic logic

As explained in §1.4, I do very warmly recommend reading a series of books on a topic which overlap in coverage and difficulty, rather than leaping immediately from an ‘entry level’ text to a really advanced one. You don’t have to follow this excellent advice, of course. But I mention it again here to remind you of one reason why the list of recommendations in most sections is quite extensive and increments in coverage/difficulty between successive recommendations are often quite small: *this level of logic really isn’t as daunting as the overall length of this chapter might superficially suggest.* Promise!
4.1 From first-order logic to elementary model theory

The completeness theorem is the first high point – the first mathematically serious result – in a course in first-order logic; and some elementary treatments more or less stop there. Many introductory texts, however, continue just a little further with some first steps into model theory. It is clear enough what needs to come next: discussions of the so-called compactness theorem (also called the ‘finiteness theorem’), of the downward and upward Löwenheim-Skolem theorems, and of their implications. There’s less consensus about what other introductory model theoretic topics you should meet at an early stage.

As you’ll see, you’ll very quickly meet claims that involve infinite cardinalities and also occasional references to the axiom of choice. In fact, even if you haven’t yet done an official set theory course, you may well have picked up all you need to know in order to begin model theory. If you have met Cantor’s proof that infinite collections come in different sizes, and if you have been warned to take note when a proof involves making an infinite series of choices, you will probably know enough. And Goldrei’s chapter recommended in a moment in fact has a brisk section on the ‘Set theory background’ needed at this stage. (If that’s too brisk, then do a skim read of e.g. Paul Halmos’s very short *Naïve Set Theory*, or one of the other books mentioned at the beginning of §4.3 below.)

Let’s start by mentioning a book that aims precisely to bridge the gap between an introductory study of FOL and more advanced work in model theory:

1. The very first volume in the prestigious and immensely useful Oxford Logic Guides series is Jane Bridge’s very compact *Beginning Model Theory: The Completeness Theorem and Some Consequences* (Clarendon Press, 1977) which neatly takes the story onwards just a few steps from the reading on FOL mentioned in our §3.2 above. You can look at the opening chapter to remind yourself about the notion of a relational structure, then start reading Ch. 3 for a very clear
account of what it takes to prove the completeness theorem that a consistent set of sentences has a model for different sizes of sets of sentences. Though now pay special attention to the compactness theorem, and §3.5 on ‘Applications of the compactness theorem’.

Then in the forty-page Ch. 4, Bridge proves the Löwenheim-Skolem theorems, which tell us that consistent first order theories that have an infinite model at all will have models of all different infinite sizes. That shows that interesting first-order theories are never categorical – i.e. never pin down a unique model (unique ‘up to isomorphism’, as they say). But that still leaves open the possibility that a theory’s models of a particular size are all isomorphic, all ‘look the same’. So Bridge now explores some of the possibilities here, how they relate to a theory’s being complete (i.e. settling whether \( \varphi \) or \( \neg \varphi \) for any relevant sentence \( \varphi \)), and some connected issues.

The coverage is exemplary for such a very short introduction, and the writing is pretty clear. But very sadly, the book was printed in that brief period when publishers thought it a bright idea to save money by photographically printing work produced on electric typewriters. Accustomed as we now are to mathematical texts being beautifully \LaTeX ed, the look of Bridge’s book is really off-putting, and some might well find that the book’s virtues do not outweigh that sad handicap. Still, the discussion in §3.5 and the final chapter should be quite accessible given familiarity with an elementary first-order logic text like Chiswell and Hodges: just take things slowly.

Still, I am not sure that – almost forty years on from publication – this is still the best place for beginners to start: but what are the alternatives?

2. I have already sung the praises of Derek Goldrei’s Propositional and Predicate Calculus: A Model of Argument* (Springer, 2005) for the accessibility of its treatment of FOL in the first five chapters. Now you can read his §§4.4 and 4.5 (which I previously said you could skip) and then Ch. 6 on ‘Some uses of compactness’ to get a very clear introduction to some model theoretic ideas.
In a little more detail, §4.4 introduces some axiom systems describing various mathematical structures (partial orderings, groups, rings, etc.): this section could be particularly useful to philosophers who haven’t really met the notions before. Then §4.5 introduces the notions of substructures and structure-preserving mappings. After proving the compactness theorem in §6.1 (as a corollary of his completeness proof), Goldrei proceeds to use it in §§6.2 and 6.3 to show various theories can’t be finitely axiomatized, or can’t be axiomatized at all. §6.4 introduces the Löwenheim-Skolem theorems and some consequences, and the following section introduces the notion of diagrams and puts it to work. The final section, §6.6 considers issues about categoricity, completeness and decidability. All this is done with the same admirable clarity as marked out Goldrei’s earlier chapters.

Two other introductions to FOL that I mentioned in the previous Chapter also have treatments of some elementary model theory. Thus there are fragments of model theory in §2.6 of Herbert Enderton’s A Mathematical Introduction to Logic (Academic Press 1972, 2002), followed by a discussion in §2.8 of non-standard analysis: but this, for our purposes here, is too little done too fast. Dirk van Dalen’s Logic and Structure* (Springer, 1980; 5th edition 2012) covers more model-theoretic material in more detail in his Ch. 3. You could read the first section for revision on the completeness theorem, then §3.2 on compactness, the Löwenheim-Skolem theorems and their implications, before moving on to the action-packed §3.3 which covers more model theory including non-standard analysis again, and indeed touches on topics like ‘quantifier elimination’ which Goldrei doesn’t get round to.

However, after Bridge, Goldrei is probably clearest on the material he covers: and if you want to go a bit further then what you need is the more expansive

3. Maria Manzano, Model Theory, Oxford Logic Guides 37 (OUP, 1999). This book aims to be an introduction at the kind of intermediate level we are currently concerned with. And standing back
from the details, I do very much like the way that Manzano structures her book. The sequencing of chapters makes for a very natural path through her material, and the coverage seems very appropriate for a book at her intended level. After chapters about structures (and mappings between them) and about first-order languages, she proves the completeness theorem again, and then has a sequence of chapters on core model-theoretic notions and proofs. This is all done tolerably accessibly (just half a step up from Goldrei, perhaps). True, the discussions at some points would have benefitted from rather more informal commentary, motivating various choices. But overall, Manzano’s text should work at its intended level.

See this Book Note on Manzano for more details.

And this might already be about as far as philosophers may want or need to go. Many mathematicians, however, will be eager to take up the story about model theory again in §6.2.

Postscript Thanks to the efforts of the respective authors to be accessible, the path through Chiswell & Hodges → Leary → Manzano is not a hard road to follow, though we end up at least in the foothills of model theory.

We can climb up to the same foothills by routes involving slightly tougher scrambles, taking in some additional side-paths and new views along the way. Here are two suggestions:

Shawn Hedman’s A First Course in Logic (OUP, 2004) covers a surprising amount of model theory. Ch. 2 tells you about structures and relations between structures. Ch. 4 starts with a nice presentation of a Henkin completeness proof, and then pauses (as Goldrei does) to fill in some background about infinite cardinals etc., before going on to prove the Löwenheim-Skolem theorems and compactness theorems. Then the rest of Ch. 4 and the next chapter covers more introductory model theory, already touching on some topics beyond the scope of Manzano’s book, and could serve as an alternative to hers. (Then Ch. 6 takes the story on a lot further, quite a way beyond what I’d regard as ‘entry level’ model theory.) For more, see this Book Note on Hedman.

Peter Hinman’s weighty Fundamentals of Mathematical Logic (A. K. Peters, 2005) is not for the faint-hearted, and I perhaps wouldn’t recommend us-
ing this book as your guide in your first outing into this territory. But if you are mathematically minded and have already made a first foray along gentler routes, you could now try reading Ch. 1 – skipping material that is familiar – and then carefully working through Ch. 2 and Ch. 3 (leaving the last two sections, along with a further chapter on model theory, for later). This should significantly deepen your knowledge of FOL, or at least of its semantic features, and of the beginnings of model theory. For more, see this Book Note on Hinman.

4.2 Computability and Gödelian incompleteness

The standard mathematical logic curriculum, as well as looking at some elementary general results about formalized theories and their models, looks at two particular instances of non-trivial, rigorously formalized, axiomatic systems – arithmetic (a paradigm theory about finite whatnots) and set theory (a paradigm theory about infinite whatnots). We’ll take arithmetic first.

In more detail, there are three inter-related topics here: (a) the elementary (informal) theory of arithmetic computations and of computability more generally, (b) an introduction to formal theories of arithmetic, leading up to (c) Gödel’s epoch-making proof of the incompleteness of any sufficiently nice formal theory that can ‘do’ enough arithmetical computations (a result of profound interest to philosophers).

Now, Gödel’s 1931 proof of his incompleteness theorem uses facts in particular about so-called primitive recursive functions: these functions are a subclass (but only a subclass) of the computable numerical functions, i.e. a subclass of the functions which a suitably programmed computer could evaluate (abstracting from practical considerations of time and available memory). A more general treatment of the effectively computable functions (arguably capturing all of them) was developed a few years later, and this in turn throws more light on the incompleteness phenomenon.

So, if you are going to take your first steps into this area, there’s a choice to be made. Do you look at things in roughly the historical
order, introducing just the primitive recursive functions and theories of formal arithmetic and learning how to prove initial versions of Gödel’s incompleteness theorem before moving on to look at the general treatment of computable functions? Or do you do some of the general theory of computation first, turning to the incompleteness theorems later?

Here then are two introductory books which take the two different routes:

1. Peter Smith, *An Introduction to Gödel’s Theorems* (CUP 2007, 2nd edition 2013) takes things in something like the historical order. Mathematicians: please don’t be put off by the series title ‘Cambridge Introductions to Philosophy’ – putting it in that series was the price I happily paid for cheap paperback publication. This is still quite a meaty logic book, with a lot of theorems and a lot of proofs, but I hope rendered very accessibly. The book’s website is at http://godelbook.net, where there are supplementary materials of various kinds, including a freely available cut-down version of a large part of the book, *Gödel Without (Too Many) Tears*.


As you’ll already see from making a start on either of these two books, this really is a delightful area. Elementary computability theory is conceptually very neat and natural, and the early Big Results are proved in quite remarkably straightforward ways (just get the hang of the basic ‘diagonalization’ construction, the idea of Gödel-style coding, and one or two other tricks and off you go . . . ).

Perhaps half a notch up in difficulty, here’s another book that focuses first on the general theory of computation before turning to questions of logic and arithmetic:
3. George Boolos, John Burgess, Richard Jeffrey, *Computability and Logic* (CUP 5th edn. 2007). This is the latest edition of an absolute classic. The first version (just by Boolos and Jeffrey) was published in 1974; and there’s in fact a great deal to be said for regarding their 1990 third edition as being the best. The last two versions have been done by Burgess and have grown considerably and perhaps in the process lost elegance and some of the individuality. But whichever edition you get hold of, this is still great stuff. Taking the divisions in the last two editions, you will want to read the first two parts of the book at an early stage, perhaps being more selective when it comes to the last part, ‘Further Topics’.

We’ll return to say a quite a lot more about computability and Gödelian incompleteness in Ch. 6.

*Postscript*  There are many other introductory treatments covering computability and/or incompleteness. For something a little different, let me mention

A. Shen and N. K. Vereshchagin, *Computable Functions*, (American Math. Soc., 2003). This is a lovely, elegant, little book in the AMA’s ‘Student Mathematical Library’ – the opening chapters can be recommended for giving a differently-structured quick tour through some of the Big Ideas, and hinting at ideas to come.

Then various of the Big Books on mathematical logic have treatments of incompleteness. For the moment, here are three:

I have already very warmly recommended Christopher Leary’s *A Friendly Introduction to Mathematical Logic* (Prentice Hall, 2000) for its coverage of first-order logic. The final chapter has a nice treatment of incompleteness, and one that doesn’t overtly go via computability. (In headline terms you will only understand in retrospect, instead of showing that certain syntactic properties are (primitive) recursive and showing that all primitive recursive properties can be ‘represented’ in theories like $PA$, Leary relies on more directly showing that syntactic properties can be represented.)

Herbert Enderton’s *A Mathematical Introduction to Logic* (Academic Press 1972, 2002), Ch. 3 is very good on different strengths of formal theories of arithmetic, and then proves the incompleteness theorem first for a formal
arithmet with exponentiation and then – after touching on other issues – shows how to use the \( \beta \)-function trick to extend the theorem to apply to Robinson arithmetic without exponentiation. Well worth reading after e.g. my book for consolidation.

Peter G. Hinman’s *Fundamentals of Mathematical Logic* (A. K. Peters, 2005), Chs. 4 and 5, which are perhaps the best written in this very substantial book, could be read after my *IGT* as somewhat terse revision for mathematicians, and as sharpening the story in various ways.

Finally, if only because I’ve been asked about it a good number of times, I suppose I should also mention

Douglas Hofstadter, *Gödel, Escher, Bach* (Penguin, first published 1979). When students enquire about this, I helpfully say that it is the sort of book that you might well like if you like that kind of book, and you won’t if you don’t. It is, to say the least, quirky and distinctive. As I far as I recall, though, the parts of the book which touch on techie logical things are pretty reliable and won’t lead you astray. Which is a great deal more than can be said about many popularizing treatments of Gödel’s theorems.

### 4.3 Beginning set theory

Let’s say that the *elements of set theory* – the beginnings that any logician really ought to know about – will comprise enough to explain how numbers (natural, rational, real) are constructed in set theory (so enough to give us a glimmer of understanding about why it is said that set theory provides a foundation for mathematics). The elements also include the development of ordinal numbers and transfinite induction over ordinals, ordinal arithmetic, and something about the role of the axiom(s) of choice and its role in the arithmetic of cardinals. These initial ideas and constructions can (and perhaps should) be presented fairly informally: but something else that also belongs here at the beginning is an account of the development of ZFC as the now standard way of formally encapsulating and regimenting the key principles involved in the informal development of set theory.
Going beyond these elements we then have e.g. the exploration of ‘large cardinals’, proofs of the consistency and independence of e.g. the Continuum Hypothesis, and a lot more besides. But readings on these further delights are for §6.6: this present section is, as advertised, about the first steps for beginners in set theory. Even here, however, there are many books to choose from, so an annotated Guide should be particularly welcome.

I’ll start by mentioning again a famous ‘bare minimum’ book (only 104 pp. long), which could well be very useful for someone who just wants to get to know their way around basic set-theoretic notation and some fundamental concepts:


However, most readers will want to look at one or other of the following equally admirable ‘entry level’ treatments which cover a little more in a bit more depth:

2. Herbert B. Enderton, *The Elements of Set Theory* (Academic Press, 1977) has exactly the right coverage. But more than that, it is particularly clear in marking off the informal development of the theory of sets, cardinals, ordinals etc. (guided by the conception of sets as constructed in a cumulative hierarchy) and the formal axiomatization of ZFC. It is also particularly good and non-confusing about what is involved in (apparent) talk of classes which are too big to be sets – something that can mystify beginners. It is written with a certain lightness of touch and proofs are often presented in particularly well-signposted stages. The last couple of chapters or so perhaps do get a bit tougher, but overall this really is quite exemplary exposition.

3. Derek Goldrei, *Classic Set Theory* (Chapman & Hall/CRC 1996) has the subtitle ‘For guided independent study’. It is as you might expect – especially if you looked at Goldrei’s FOL text mentioned
in §3.2 – extremely clear, and is indeed very well-structured for independent reading. And moreover, it is quite attractively written (as set theory books go!). The coverage is very similar to Enderton’s, and either book makes a fine introduction (for what little it is worth, I slightly prefer Enderton).

Still starting from scratch, and initially also only half a notch or so up in sophistication from Enderton and Goldrei, we find two more really nice books:

4. Karel Hrbacek and Thomas Jech, *Introduction to Set Theory* (Marcel Dekker, 3rd edition 1999). This eventually goes a bit further than Enderton or Goldrei (more so in the 3rd edition than earlier ones), and you could – on a first reading – skip some of the later material. Though do look at the final chapter which gives a remarkably accessible glimpse ahead towards large cardinal axioms and independence proofs. Again this is a very nicely put together book, and recommended if you want to consolidate your understanding by reading a second presentation of the basics and want then to push on just a bit. (Jech is of course a major author on set theory, and Hrbacek once won a AMA prize for maths writing.)

5. Yiannis Moschovakis, *Notes on Set Theory* (Springer, 2nd edition 2006). A slightly more individual path through the material than the previously books mentioned, again with glimpses ahead and again attractively written.

These last two pairs of books are all in print: make sure that your university library has them – though none is cheap (indeed, Enderton’s is quite absurdly expensive). I’d strongly advise reading one of the first pair and then one of the second pair.

My next recommendation might come as a bit of surprise, as it is something of a ‘blast from the past’. But we shouldn’t ignore old classics – they can have a lot to teach us even if we have read the modern books.

phers and mathematicians should appreciate the way this puts the development of our canonical ZFC set theory into some context, and also discusses alternative approaches. It really is attractively readable, and should be very largely accessible at this early stage. I’m not an enthusiast for history for history’s sake: but it is very much worth knowing the stories that unfold here.

One intriguing feature of that last book is that it doesn’t emphasize the ‘cumulative hierarchy’ – the picture of the universe of sets as built up in a hierarchy of stages or levels, each level containing all the sets at previous levels plus new ones (so the levels are cumulative). This picture is nowadays familiar to every beginner: you will find it e.g. in the opening pages of Joseph Shoenfield ‘The axioms of set theory’, *Handbook of mathematical logic*, ed. J. Barwise, (North-Holland, 1977) pp. 321–344: it is interesting to find that the picture wasn’t firmly in place from the beginning.

The hierarchical conception of the universe of sets is brought to the foreground again in

7. Michael Potter, *Set Theory and Its Philosophy* (OUP, 2004). For philosophers (and for mathematicians concerned with foundational issues) this surely is a ‘must read’, a unique blend of mathematical exposition (mostly about the level of Enderton, with a few glimpses beyond) and extensive conceptual commentary. Potter is presenting not straight ZFC but a very attractive variant due to Dana Scott whose axioms more directly encapsulate the idea of the cumulative hierarchy of sets. However, it has to be said that there are passages which are harder going, sometimes because of the philosophical ideas involved, but sometimes because of unnecessary expositional compression. In particular, at the key point at p. 41 where a trick is used to avoid treating the notion of a level (i.e. a level in the hierarchy) as a primitive, the definitions are presented too quickly, and I know that some relative beginners can get lost. However, if you have already read one or two set theory books from earlier in the list, you should be fairly easily be able to work out what is going on and read on past this stumbling block.

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It is a nice question how much more technical knowledge of results in
set theory a philosophy student interested in logic and the philosophy of
maths needs (if she is not specializing in the technical philosophy of set
theory). But getting this far will certainly be a useful start, so let’s pause
here.

Postscript Books by Ciesielski and by Hajnal and Hamburger, although in LMS
Student Text series and starting from scratch, are not really suitable for the
present list (they go too far and probably too fast). But the following four(!)
still-introductory books, listed in order of publication, each have things to rec-
ommend them for beginners. One is freely available online, and good libraries
will have the others: so browse through and see which might suit your interests
and mathematical level.

D. van Dalen, H.C. Doets and H. de Swart, Sets: Naive, Axiomatic and
Applied (Pergamon, 1978). The first chapter covers the sort of element-
tary (semi)-naive set theory that any mathematician needs to know, up
to an account of cardinal numbers, and then a first look at the paradox-
avoiding ZF axiomatization. This is attractively and illuminatingly done
(or at least, the conceptual presentation is attractive – sadly, and a sign of
its time of publication, the book seems to have been photo-typeset from
original pages produced on electric typewriter, and the result is visually
not attractive at all).

The second chapter carries on the presentation axiomatic set theory,
with a lot about ordinals, and getting as far as talking about higher infini-
ties, measurable cardinals and the like. The final chapter considers some
applications of various set theoretic notions and principles. Well worth
seeking out, if you don’t find the typography off-putting..

chapters of this book are remarkably lucid and attractively written (as
you would expect from this author). The opening chapter explores ‘naive’
ideas about sets and some set-theoretic constructions, and the next chapter
introducing axioms for ZFC pretty gently (indeed, non-mathematicians
could particularly like Chs 1 and 2, omitting §2.6). Things then speed up
a bit, and by the end of Ch. 3 – some 100 pages into the book – we are
pretty much up to the coverage of Goldrei’s much longer first six chapters,
though Goldrei says more about (re)constructing classical maths in set
theory. Some will prefer Devlin’s fast-track version. (The rest of the book then covers non-introductory topics in set theory, of the kind we take up again in §6.6.)

Judith Roitman, *Introduction to Modern Set Theory** (Wiley, 1990: now freely downloadable, or available as an inexpensive paperback via Amazon). This relatively short, and very engagingly written, book manages to cover quite a bit of ground – we’ve reached the constructible universe by p. 90 of the downloadable pdf version, and there’s even room for a concluding chapter on ‘Semi-advanced set theory’ which says something about large cardinals and infinite combinatorics. A few quibbles aside, this could make excellent revision material as Roitman is particularly good at highlighting key ideas without getting bogged down in too many details.

Winfried Just and Martin Weese, *Discovering Modern Set Theory I: The Basics* (American Mathematical Society, 1996). This covers overlapping ground to Enderton, patchily but perhaps more zestfully and with a little more discussion of conceptually interesting issues. It is at some places more challenging – the pace can be uneven. But this is evidently written by enthusiastic teachers, and the book is very engaging. (The story continues in a second volume.)

I like the style a lot, and think it works very well. I don’t mean the occasional (slightly laboured?) jokes: I mean the in-the-classroom feel of the way that proofs are explored and motivated, and also the way that teach-yourself exercises are integrated into the text. For instance there are exercises that encourage you to produce proofs that are in fact non-fully-justified, and then the discussion explores what goes wrong and how to plug the gaps.

Those four books from the last millenium(!) have stood the test of time. Here, to conclude, are just two more short books aimed at beginners that are excellent and could also be very useful:

A. Shen and N. K. Vereshchagin, *Basic Set Theory* (American Mathematical Society, 2002), just over 100 pages, and mostly about ordinals. But very readable, with 151 ‘Problems’ as you go along to test your understanding. Could be very helpful by way of revision/consolidation.

Ernest Schimmerling, *A Course on Set Theory* (CUP, 2011) is slightly mistitled: it is just 160 pages, again introductory but with some rather different emphases and occasional forays into what Roitman would call
'semi-advanced' material. Quite an engaging supplementary read at this level.

Finally, what about the introductory chapters on set theory in those Big Books on Mathematical Logic? I’m not convinced that any are to be particularly recommended compared with the stand-alone treatments.

4.4 Extras: variant logics

4.4.1 Second-order logic

At some fairly early point we must look at a familiar extension of first-order classical logic, namely second-order logic, where we also allow generalizations which quantify into predicate position.

Consider, for example, the intuitive principle of arithmetical induction. Take any property $X$; if 0 has it, and for any $n$ it is passed down from $n$ to $n+1$, then all numbers must have $X$. It is tempting to regiment this as follows:

$$\forall X[(0 \in X \land \forall n(Xn \rightarrow X(n + 1))] \rightarrow \forall n Xn$$

where the second-order quantifier $\forall X$ quantifies ‘into predicate position’ and supposedly runs over all properties of numbers. But this is illegitimate in standard first-order logic. [Historical aside: note that the earliest presentations of quantificational logic, in Frege and in Principia Mathematica, were of logics that did allow this kind of higher-order quantification: the concentration on first-order logic which has become standard was a later development.]

What to do? One option is to keep your logic first-order but go set-theoretic and write the induction principle instead as

$$\forall X[(0 \in X \land \forall n(n \in X \rightarrow (n + 1) \in X)] \rightarrow \forall n n \in X]$$

where the variable ‘$X$’ is now a sorted first-order variable running over sets. But arguably this changes the subject (our ordinary principle of arithmetical induction doesn’t seem to be about sets), and there are other
issues too. So why not take things at face value and allow that the ‘natural’ logic of informal mathematical discourse often deploys second-order quantifiers that range over properties (expressed by predicates) as well as first-order quantifiers that range over objects (denoted by names), i.e. why not allow quantification into predicate position as well as into name position?

For a brief but informative overview, see the article


You could then try one or both of


3. Dirk van Dalen, Logic and Structure, Ch. 4,

That will be as much as many readers will need, at least at the outset: you will learn, inter alia, why the compactness and Löwenheim-Skolem theorems fail for second-order logic (with so-called ‘standard’ or ‘full’ semantics). But having got this far, some will want to dive into


And indeed it would be a pity, while you have Shapiro’s wonderfully illuminating book in your hands, to skip the initial philosophical/methodological discussion in the first two chapters here. This whole book is a modern classic, remarkably accessible, and important too for the contrasting side-light it throws on FOL.

4.4.2 Intuitionist logic

(a) Could there be domains (mathematics, for example) where truth is in some good sense a matter of provability-in-principle, and falsehood
a matter of refutability-in-principle? And if so, would every proposition from such a domain be either true or false, i.e. provable-in-principle or refutable-in-principle? Why so? Perhaps we shouldn’t endorse the principle that \( \varphi \lor \neg \varphi \) is always true no matter the domain. (Maybe the principle, even when it does hold for some domain, doesn’t hold as a matter of logic but e.g. as a matter of metaphysics.)

Thoughts like this give rise to one kind of challenge to classical two-valued logic, which of course does assume excluded middle across the board. For more on this intuitionist challenge, see e.g. the brief remarks in Theodore Sider,’s *Logic for Philosophy* (OUP, 2010), §3.5.

Now, particularly in a natural deduction framework, the syntax/proof theory of intuitionist logic is straightforward: putting it crudely, it is a matter of suitably tinkering with the rules of your favourite proof system so as to block the derivation of excluded middle while otherwise keeping as much as possible. Things get interesting when we look at the semantics. The now-standard version due to Kripke is a brand of ‘possible-world semantics’ of a kind that is also used in modal logic. Philosophers might like, therefore, to cover intuitionism after first looking at modal logic more generally. But mathematicians should have no problem diving straight into


2. M. Fitting, *Intuitionistic Logic, Model Theory, and Forcing* (North Holland, 1969). The first part of the scarily-titled book is a particularly attractive stand-alone introduction to the semantics and a proof-system for intuitionist logic (the second part of the book concerns an application of this to a construction in set theory, but don’t let that put you off the brilliantly clear Chs 1–6!).

If, however, you want to approach intuitionistic logic after looking at some modal logic, then you could instead start with

3. Graham Priest, *An Introduction to Non-Classical Logic* (CUP, much expanded 2nd edition 2008), Chs. 6, 20. These chapters of
course flow on naturally from Priest’s treatment in that book of modal logics, first propositional and then predicate.

And then, if you want to pursue things further (though this is ratcheting up the difficulty level), you could try the following wide-ranging essay, which is replete with further references:


(b) One theme not highlighted in these initial readings is that intuitionistic logic seemingly has a certain naturalness compared with classical logic, from a more proof-theoretic point of view. Suppose we think of the natural deduction introduction rule for a logical operator as fixing the meaning of the operator (rather than a prior semantics fixing what is the appropriate rule). Then the corresponding elimination rules surely ought to be in harmony with the introduction rule, in the sense of just ‘undoing’ its effect, i.e. giving us back from a wff $\varphi$ with $O$ as its main operator no more than what an application of the $O$-introduction rule to justify $\varphi$ would have to be based on. For this idea of harmony see e.g. Neil Tennant’s Natural Logic, §4.12. From this perspective the characteristic classical excluded middle rule is seemingly not ‘harmonious’. There’s a significant literature on this idea: for some discussion, and pointers to other discussions, you could start with Peter Milne, ‘Classical harmony: rules of inference and the meaning of the logical constants’, Synthese vol. 100 (1994), pp. 49–94.

For an introduction to intuitionistic logic in a related spirit, see


Postscript Note, by the way, that what we’ve been talking about is intuitionist logic not intuitionist mathematics. For something on this, see e.g.

which rather runs together the topics. Also see


and then the stand-out recommendation is

Michael Dummett, *Elements of Intuitionism*, Oxford Logic Guides 39 (OUP 2nd edn. 2000). A classic – but (it has to be said) quite tough. The final chapter, ‘Concluding philosophical remarks’, is very well worth looking at, even if you bale out from reading all the formal work that precedes it.

But perhaps this really needs to be set in the context of a wider engagement with varieties of constructive mathematics.
Chapter 5  

Modal and other logics

Here’s the menu for this chapter, which is probably of more particular interest to philosophers:

5.1 We start with modal logic – like second-order logic, an extension of classical logic – for two reasons. First, the basics of modal logic don’t involve anything mathematically more sophisticated than the elementary first-order logic covered in Chiswell and Hodges (indeed to make a start on modal logic you don’t even need as much as that). Second, and much more importantly, philosophers working in many areas surely ought to know a little modal logic.

5.2 Classical logic demands that all terms denote one and one thing – i.e. it doesn’t countenance empty terms which denote nothing, or plural terms which may denote more than one thing. In this section, we look at logics which remain classical in spirit but which do allow empty and/or plural terms.

5.3 Among variant logics which are non-classical in spirit, we have already mentioned intuitionist logic. Here we consider some other (wilder?) deviations from the classical paradigm.

5.1 Getting started with modal logic

Basic modal logic is the logic of the operators ‘□’ and ‘◇’ (read ‘it is necessarily true that’ and ‘it is possibly true that’); it adopts new principles
like □φ → φ and φ → ◇φ, and investigates more disputable principles like ◇φ → □◇φ. The place to start is clear:

1. Rod Girle, *Modal Logics and Philosophy* (Acumen 2000; 2nd edn. 2009). Girle’s logic courses in Auckland, his enthusiasm and abilities as a teacher, are justly famous. Part I of this book provides a particularly lucid introduction, which in 136 pages explains the basics, covering both trees and natural deduction for some propositional modal logics, and extending to the beginnings of quantified modal logic. Philosophers may well want to go on to read Part II of the book, on applications of modal logic.

Also pretty introductory, though perhaps a little brisker than Girle at the outset, is

2. Graham Priest, *An Introduction to Non-Classical Logic* (CUP, much expanded 2nd edition 2008): read Chs 2–4 for propositional modal logics, Chs 14–18 for quantified logics. This book – which is a terrific achievement and enviably clear and well-organized – systematically explores logics of a wide variety of kinds, using trees throughout in a way that can be very illuminating indeed. Although it starts from scratch, however, it would be better to come to the book with a prior familiarity with logic via trees, as in my *IFL*. We will be mentioning this book again in later sections for its excellent coverage of other non-classical themes.

If you do start with Priest’s book, then at some point you will want to supplement it by looking at a treatment of natural deduction proof systems for modal logics. One option is to dip into Tony Roy’s comprehensive ‘Natural Derivations for Priest, *An Introduction to Non-Classical Logic*’ which presents natural deduction systems corresponding to the propositional logics presented in tree form in the first edition of Priest (so the first half of the new edition). Another possible way in to ND modal systems would be via the opening chapters of

3. James Garson, *Modal Logic for Philosophers* (CUP, 2006). This again is certainly intended as a gentle introductory book: it deals
with both ND and semantic tableaux (trees), and covers quantified modal logic. It is reasonably accessible, but not – I think – as attractive as Girle.

We now go a step up in sophistication:

4. Melvin Fitting and Richard L. Mendelsohn, *First-Order Modal Logic* (Kluwer 1998). This book starts again from scratch, but then does go rather more snappily, with greater mathematical elegance (though it should certainly be accessible to anyone who is modestly on top of non-modal first-order logic). It still also includes a good amount of philosophically interesting material. Recommended.

A few years ago, I would have said that getting as far as Fitting and Mendelsohn will give most philosophers a good enough grounding in basic modal logic. But e.g. Timothy Williamson’s book *Modal Logic as Metaphysics* (OUP, 2013) calls on rather more, including second-order modal logics. If you want to sharpen your knowledge of the technical background here, I guess there is nothing for it but to tackle

5. Nino B. Cocchiarella and Max A. Freund, *Modal Logic: An Introduction to its Syntax and Semantics* (OUP, 2008). The blurb announces that ‘a variety of modal logics at the sentential, first-order, and second-order levels are developed with clarity, precision and philosophical insight’. However, when I looked at this book with an eye to using it for a graduate seminar a couple of years back, I confess I didn’t find it very appealing: so I do suspect that many philosophical readers will indeed find the treatments in this book rather relentless. However, the promised wide coverage could make the book of particular interest to determined philosophers concerned with the kind of issues that Williamson discusses.

Finally, I should certainly draw your attention to the classic book by Boolos mentioned at the end of §6.4.
Postscript for philosophers  Old hands learnt their modal logic from G. E. Hughes and M. J. Cresswell *An Introduction to Modal Logic* (Methuen, 1968). This was at the time of original publication a unique book, enormously helpfully bringing together a wealth of early work on modal logic in an approachable way. Nearly thirty years later, the authors wrote a heavily revised and updated version, *A New Introduction to Modal Logic* (Routledge, 1996). This newer version like the original one concentrates on axiomatic versions of modal logic, which doesn’t make it always the most attractive introduction from a modern point of view. But it is still an admirable book at an introductory level (and going beyond), that enthusiasts can learn from.

I didn’t recommend the first part of Theodore Sider’s *Logic for Philosophy* (OUP, 2010). However, the second part of the book which is entirely devoted to modal logic (including quantified modal logic) and related topics like Kripke semantics for intuitionistic logic is significantly better. Compared with the early chapters with their inconsistent levels of coverage and sophistication, the discussion here develops more systematically and at a reasonably steady level of exposition. There is, however, a lot of (acknowledged) straight borrowing from Hughes and Cresswell, and – like those earlier authors – Sider also gives axiomatic systems. But if you just want a brisk and pretty clear explanation of Kripke semantics, and want to learn e.g. how to search systematically for countermodels, Sider’s treatment in his Ch. 6 could well work as a basis. And then the later treatments of quantified modal logic in Chs 9 and 10 (and some of the conceptual issues they raise) are also lucid and approachable.

Postscript for the more mathematical  Here are a couple of good introductory modal logic books with a mathematical flavour:

Sally Popkorn, *First Steps in Modal Logic* (CUP, 1994). The author is, at least in this possible world, identical with the mathematician Harold Simmons. This book, which entirely on propositional modal logics, is written for computer scientists. The Introduction rather boldly says ‘There are few books on this subject and even fewer books worth looking at. None of these give an acceptable mathematically correct account of the subject. This book is a first attempt to fill that gap.’ This considerably oversells the case: but the result is illuminating and readable.

Also just on propositional logic, I’d recommend Patrick Blackburn, Maarten de Rijke and Yde Venema’s *Modal Logic* (CUP, 2001). This is one of the Cambridge Tracts in Theoretical Computer Science: but again don’t let that provenance put you off – it is (relatively) accessibly and agreeably
written, with a lot of signposting to the reader of possible routes through the book, and interesting historical notes. I think it works pretty well, and will certainly give you an idea about how non-philosophers approach modal logic.

Going in a different direction, if you are interested in the relation between modal logic and intuitionistic logic (see §4.4.2), then you might want to look at Alexander Chagrov and Michael Zakharyaschev Modal Logic (OUP, 1997). This is a volume in the Oxford Logic Guides series and again concentrates on propositional modal logics. Written for the more mathematically minded reader, it tackles things in an unusual order, starting with an extended discussion of intuitionistic logic, and is pretty demanding. But enthusiasts should take a look.

Finally, if you want to explore even more, there’s the giant Handbook of Modal Logic, edited by van Bentham et al. (Elsevier, 2005). You can get an idea of what’s in the volume by looking at this page of links to the opening pages of the various contributions.

5.2 Other classical extensions and variants

We next look at what happens if you stay first-order in the sense of keeping your variables running over objects, but allow terms that fail to denote (free logic) or which allow terms that refer to more than one thing (plural logic).

5.2.1 Free Logic

Classical logic assumes that any term denotes an object in the domain of quantification, and in particular assumes that all functions are total, i.e. defined for every argument – so an expression like ‘\( f(c) \)’ always denotes. But mathematics cheerfully countenances partial functions, which may lack a value for some arguments. Should our logic accommodate this, by allowing terms to be free of existential commitment? In which case, what would such a ‘free’ logic look like?

For some background and motivation, see
1. David Bostock, *Intermediate Logic* (OUP 1997), Ch. 8, and also look at a useful and quite detailed overview article from the Stanford Encyclopedia (what would philosophers do without that?):


Then for another very accessible brief formal treatment, this time in the framework of logic-by-trees, see


For more details (though going rather beyond the basics), you could make a start on


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**Postscript**

Rolf Schock’s *Logics without Existence Assumptions* (Almqvist & Wiskell, Stockholm 1968) is still well worth looking at on free logic after all this time. And for a much more recent collection of articles around and about the topic of free logic, see Karel Lambert, *Free Logic: Selected Essays* (CUP 2003).

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### 5.2.2 Plural logic

In ordinary mathematical English we cheerfully use plural denoting terms such as ‘2, 4, 6, and 8’, ‘the natural numbers’, ‘the real numbers between 0 and 1’, ‘the complex solutions of $z^2 + z + 1 = 0$', ‘the points where line $L$ intersects curve $C$’, ‘the sets that are not members of themselves’, and the like. Such locutions are entirely familiar, and we use them all the time without any sense of strain or logical impropriety. We also often generalize by using plural quantifiers like ‘any natural numbers’ or ‘some
reals’ together with linked plural pronouns such as ‘they’ and ‘them’. For example, here is a version of the Least Number Principle: given any natural numbers, one of them must be the least. By contrast, there are some reals – e.g. those strictly between 0 and 1 – such that no one of them is the least.

Plural terms and plural quantifications appear all over the place in mathematical argument. It is surely a project of interest to logicians to regiment and evaluate the informal modes of argument involving such constructions. Hence the business of plural logic, a topic of much recent discussion. For an introduction, see


And do read at least two of the key papers listed in Linnebo’s expansive bibliography:


( Oliver and Smiley give reasons why there is indeed a real subject here: you can’t readily eliminate all plural talk in favour e.g. of singular talk about sets. Boolos’s classic will tell you something about the possible relation between plural logic and second-order logic.) Then, for much more about plurals, you could look at


which is clear and approachable. Real enthusiasts for plural logic will want to dive into the long-awaited (though occasionally rather idiosyncratic)

5.3 Relevance logics (and wilder logics too)

Classically, if \( \varphi \vdash \psi \), then \( \varphi, \chi \vdash \psi \) (read ‘\( \vdash \)’ as ‘entails’: irrelevant premisses can be added without making a valid entailment invalid). And if \( \varphi, \chi \vdash \psi \) then \( \varphi \vdash \chi \rightarrow \psi \) (that’s the Conditional Proof rule in action). Presumably we have \( P \vdash P \). So we have \( P, Q \vdash P \), whence \( P \vdash Q \rightarrow P \). It seems then that classical logic’s carefree attitude to questions of relevance in deduction and its dubious version of the conditional are tied closely together.

Classically, we also have \( \varphi, \neg \varphi \vdash \psi \). But doesn’t the inference from \( P \) and \( \neg P \) to \( Q \) commit another fallacy of relevance? And again, if we allow it and also allow conditional proof, we will have \( P \vdash \neg P \rightarrow Q \), another seemingly unhappy result about the conditional.

Can we do better? What does a more relevance-aware logic look like? For useful introductory reading, see


These two articles have many pointers for further reading across a range of topics, so I’ll be brief here. But I will mention, first, two wide-ranging introductory texts:

3. Graham Priest, *An Introduction to Non-Classical Logic* (CUP, much expanded 2nd edition 2008). Look now at Chs. 7–10 for a treatment of propositional logics of various deviant kinds. Priest starts with relevance logic and goes on to also treat logics where there are truth-value gaps, and – more wildly – logics where a proposition can be both true and false (there’s a truth-value glut). Then, if this excites you, carry on to look at Chs. 21–24 where the corresponding quantificational logics are presented. This book really is a wonderful resource.
4. J. C. Beall and Bas van Fraassen’s *Possibilities and Paradox* (OUP 2003), also covers a range of logics. In particular, Part III of the book covers relevance logic and also non-standard logics involving truth-value gaps and truth-value gluts. (It is worth looking too at the earlier parts of the book on logical frameworks generally and on modal logic.)

Note, however, that although they get discussed in close proximity in those books, there’s no tight connection between the reasonable desire to have a more relevance-aware logic (e.g. without the principle that a contradiction implies everything) and the immodest proposal that there can be propositions which are both true and false at the same time. (At the risk of corrupting the youth, if you are interested in exploring the latter immodest proposal further, then I can point you to

5. Graham Priest, ‘*Dialetheism*’, *The Stanford Encyclopedia of Philosophy*.

But retreating from those wilder shores, let’s here concentrate on less radical relevance logics.)

The obvious next place to go is the very lucid

6. Edwin Mares, *Relevant Logic: A Philosophical Interpretation* (CUP 2004). As the title suggests, this book has very extensive conceptual discussion alongside the more formal parts elaborating what might be called the mainstream tradition in relevance logics.

There is however a minority tradition that I myself find extremely appealing, developed by Neil Tennant in very scattered papers and episodes in books. You will need to know a little proof theory to appreciate it – though you can get a flavour of the approach from the early programmatic paper

You can then follow this up by looking at papers on Tennant’s website with titles involving key words and phrases like ‘entailment’ ‘relevant logic’ and ‘core logic’: but I regret that there is (as far as I know) still no single, one-stop, account of Tennant’s style of relevance logic bringing things neatly together.
Chapter 6

More advanced reading on core topics

In this chapter, there are some suggestions for more advanced reading on a selection of topics in and around the core mathematical logic curriculum we looked at in Chs. 3 and 4. Two points before we begin:

• Before tackling this often significantly more difficult material, it could be very well worth first taking the time to look at one or two of the wider-ranging Big Books on mathematical logic which will help consolidate your grip on the basics at the level of Chapter 4 and/or push things on just a bit. See the Book Notes for some guidance on what’s available.

• I did try to be fairly systematic in Chapter 4, aiming to cover the different core areas at comparable levels of detail, depth and difficulty. The coverage of various topics from here on is more varied: the recommendations can be many or few (or non-existent!) depending on my own personal interests and knowledge.

And a warning to those philosophers still reading: some of the material I point to is inevitably mathematically quite demanding!
6.1 Proof theory

Proof theory has been (and continues to be) something of a poor relation in the standard Mathematical Logic curriculum: the usual survey textbooks don’t discuss it. Yet this is a fascinating area, of interest to philosophers, mathematicians, and computer scientists who, after all, ought to be concerned with the notion of proof! So let’s start to fill this gap next.

I mentioned in §3.3 the introductory book by Jan von Plato, *Elements of Logical Reasoning* (CUP, 2014), which approaches elementary logic with more of an eye on proof theory than is at all usual: you might want to take a look at that book if you didn’t before. But you should start serious work by reading the same author’s extremely useful encyclopaedia entry:


This will give you orientation and introduce you to some main ideas: there is also an excellent bibliography which you can use to guide further exploration. So here I can be brief.

But I certainly think you should read the little hundred-page classic


And if you want to follow up in more depth Prawitz’s investigations of the proof theory of various systems of logic, the next place to look is surely

3. Sara Negri and Jan von Plato, *Structural Proof Theory* (CUP 2001). This is a modern text which is neither too terse, nor too laboured, and is generally very clear. When we read it in a graduate-level reading group, we did find we needed to pause sometimes to stand back and think a little about the motivations for various technical developments. So perhaps a few more ‘classroom asides’ in the text would have made a rather good text even better. But this is still extremely helpful.
Then in a more mathematical style, there is the editor’s own first contribution to

4. Samuel R. Buss, ed., *Handbook of Proof Theory* (North-Holland, 1998). Later chapters of this very substantial handbook do get pretty hard-core; but the 78 pp. opening chapter by Buss himself, a ‘Introduction to Proof Theory’**, is readable, and freely downloadable. (Student health warning: there are, I am told, some confusing misprints in the cut-elimination proof.)

And now perhaps the path forks. In one direction, the path cleaves to what we might call classical themes (I don’t mean themes simply concerning classical logic, as intuitionistic logic was also central from the start: I mean themes explicit in the early classic papers in proof theory, in particular in Gentzen’s work). It is along this path that we find e.g. Gentzen’s famous proof of the consistency of first-order Peano Arithmetic using proof-theoretic ideas. One obvious next text to read, at least in part, remains

5. Gaisi Takeuti, *Proof Theory* (North-Holland 1975, 2nd edn. 1987: reprinted Dover Publications 2013). This is a true classic – if only because for a while it was about the only available book on most of its topics. Later chapters won’t really be accessible to beginners. But you could/should try reading Ch. 1 on logic, §§1–7 (and perhaps the beginnings of §8, pp. 40–45, which is easier than it looks if you compare how you prove the completeness of a tree system of logic). Then on Gentzen’s proof, read Ch. 2, §§9–11 and §12 up to at least p. 114. This isn’t exactly plain sailing – but if you skip and skim over some of the more tedious proof-details you can pick up a very good basic sense of what happens in the consistency proof.

Gentzen’s proof of the consistency of depends on transfinite induction along ordinals up to $\varepsilon_0$; and the fact that it requires just so much transfinite induction to prove the consistency of first-order PA is an important characterization of the strength of the theory. The project of ‘ordinal analysis’ in proof theory aims to provide comparable characterizations of other
theories in terms of the amount of transfinite induction that is needed to prove their consistency. Things do get quite hairy quite quickly, however. For a glimpse ahead, you could look at (initial segments of) these useful notes for mini-courses by Michael Rathjen, on ‘The Realm of Ordinal Analysis’ and ‘Proof Theory: From Arithmetic to Set Theory’.

Turning back from these complications, however, let’s now glance down the other path from the fork, where we investigate not the proof theory of theories constructed in familiar logics but rather investigate non-standard logics themselves. Reflection on the structural rules of classical and intuitionistic proof systems naturally raises the question of what happens when we tinker with these rules. We noted before the inference which takes us from the trivial $P \vdash P$ by ‘weakening’ to $P, Q \vdash P$ and on, via ‘conditional proof’, to $P \vdash Q \rightarrow P$. If we want a conditional that conforms better to intuitive constraints of relevance, then we need to block that proof: is ‘weakening’ the culprit? The investigation of what happens if we tinker with standard structural rules such as weakening belongs to substructural logic, outlined in


(which again has an admirable bibliography). And the place to continue exploring these themes at length is the same author’s splendid

7. Greg Restall, An Introduction to Substructural Logics (Routledge, 2000), which will also teach you a lot more about proof theory generally in a very accessible way. Do read at least the first seven chapters. (And note again too the work on Neil Tennant mentioned at the very end of §5.3.)

Postscript Philosophers will want to take an early look at the two chapters of a – regrettably, very partial – draft book by the last mentioned author (again!), which with a bit of judicious skipping will help explain more about the conceptual interest of proof theory: Greg Restall, Proof Theory and Philosophy.

For the more mathematically minded, here are a few more books of interest. I’ll start with a couple that aim to be accessible to beginners in proof theory:
Jean-Yves Girard, *Proof Theory and Logical Complexity. Vol. I* (Bibliopolis, 1987) is intended as an introduction [Vol. II was never published]. With judicious skipping, which I’ll signpost, this is readable and insightful, though some proofs are a bit arm-waving.

So: skip the ‘Foreword’, but do pause to glance over ‘Background and Notations’ as Girard’s symbolic choices need a little explanation. Then the long Ch. 1 is by way of an introduction, proving Gödel’s two incompleteness theorem and explaining ‘The Fall of Hilbert’s Program’: if you’ve read some of the recommendations in §4.2 above, you can probably skim this pretty quickly, just noting Girard’s highlighting of the notion of 1-consistency.

Ch. 2 is on the sequent calculus, proving Gentzen’s *Hauptsatz*, i.e. the crucial cut-elimination theorem, and then deriving some first consequences (you can probably initially omit the forty pages of annexes to this chapter). Then also omit Ch. 3 whose content isn’t relied on later. But Ch. 4 on ‘Applications of the *Hauptsatz*’ is crucial (again, however, at a first pass you can skip almost 60 pages of annexes to the chapter). Take the story up again with the first two sections of Ch. 6, and then tackle the opening sections of Ch. 7. A bumpy ride but very illuminating.

A. S. Troelstra and H. Schwichtenberg’s *Basic Proof Theory* (CUP 2nd ed. 2000) is a volume in the series Cambridge Tracts in Computer Science. Now, one theme that runs through the book indeed concerns the computer-science idea of formulas-as-types and invokes the lambda calculus; however, it is in fact possible to skip over those episodes in you aren’t familiar with the idea. The book, as the title indicates, is intended as a first foray into proof theory, and it is reasonably approachable. However it is perhaps a little cluttered for my tastes because it spends quite a bit of time looking at slightly different ways of doing natural deduction and slightly different ways of doing the sequent calculus, and the differences may matter more for computer scientists than others. You could, however, with a bit of skipping, very usefully read just Chs. 1–3, the first halves of Chs. 4 and 6, and then Ch. 10 on arithmetic again.

And now for three more advanced offerings:

I have already mentioned the compendium edited by Samuel R. Buss, *Handbook of Proof Theory* (North-Holland, 1998), and the fact that you can download its substantial first chapter. You can also freely access Ch. 2 on ‘First-Order Proof-Theory of Arithmetic’. Later chapters of the Hand-
book are of varying degrees of difficulty, and cover a range of topics (though there isn’t much on ordinal analysis).

Wolfram Pohlers, *Proof Theory: The First Step into Impredicativity* (Springer 2009). This book has introductory ambitions, to say something about so-called ordinal analysis in proof theory as initiated by Gentzen. But in fact I would judge that it requires quite an amount of mathematical sophistication from its reader. From the blurb: ‘As a “warm up” Gentzen’s classical analysis of pure number theory is presented in a more modern terminology, followed by an explanation and proof of the famous result of Feferman and Schütte on the limits of predicativity.’ The first half of the book is probably manageable if (but only if) you already have done some of the other reading. But then the going indeed gets pretty tough.

H. Schwichtenberg and S. Wainer, *Proofs and Computations* (Association of Symbolic Logic/CUP 2012) ‘studies fundamental interactions between proof-theory and computability’. The first four chapters, at any rate, will be of wide interest, giving another take on some basic material and should be manageable given enough background – though sadly, I found the book to be not particularly well written and it sometimes makes heavier weather of its material than seems really necessary. Worth the effort though.

### 6.2 Beyond the model-theoretic basics

If you want to explore model theory beyond the introductory material in §4.1, why not start with a quick warm-up, with some reminders of headlines and some useful pointers to the road ahead:


Now, we noted in §4.1 that the wide-ranging texts by Hedman and Hinman eventually cover a substantial amount of model theory. But you will do better with three stand-alone treatments of the area which really choose themselves. Both in order of first publication and of eventual difficulty we have:
2. C. Chang and H. J. Keisler, *Model Theory* (originally North Holland 1973: the third edition has been inexpensively republished by Dover Books in 2012). This is the Old Testament, the first systematic text on model theory. Over 550 pages long, it proceeds at an engagingly leisurely pace. It is particularly lucid and is extremely nicely constructed with different chapters on different methods of model-building. A fine achievement that remains a good route in to the serious study of model theory.


4. David Marker, *Model Theory: An Introduction* (Springer 2002). Very highly regarded; another book which has also quickly become something of a standard text. Eventually another notch or two up again in difficulty and mathematical sophistication, and with later chapters probably going far over the horizon for all but the most enthusiastic readers of this Guide – it isn’t published in the series ‘Graduate Texts in Mathematics’ for nothing!

If you are serious about getting to grips with model theory, then there’s a lot to be said for reading all three in order! Or perhaps better, you could read the first three chapters of Chang and Keisler, and then pause to read


You could then return to Ch. 4 of C&K to look at (some of) their treatment of the ultra-product construction, before perhaps putting the rest of their book on hold and turning to Hodges and (eventually) Marker.
Postscript  I should perhaps mention an old fifty-page survey essay:

H. J. Keisler, ‘Fundamentals of Model Theory’, in J. Barwise, editor, *Handbook of Mathematical Logic*, pp. 47–103 (North-Holland, 1977). This is surely going to be too terse for most readers to read with full understanding at the outset, which is why I didn’t highlight it before. However, you could helpfully read it at an early stage to get a half-understanding of where you want to be going, and then re-read it at a later stage to check your understanding of some of the fundamentals (along, perhaps, with the next two essays in the *Handbook*).

And, to mention another essay-length piece, it is illuminating to read something about the history of model theory: there’s a good, and characteristically lucid, unpublished piece by a now-familiar author here:

W. Hodges, ‘Model Theory’.

Then, in rough order of difficulty, let me mention a number of other book-length options:

Philipp Rothmaler’s *Introduction to Model Theory* (Taylor and Francis 2000) is, overall, comparable in level of difficulty with, say, the first half of Hodges rather than with Manzano. As the blurb puts it: ‘This text introduces the model theory of first-order logic, avoiding syntactical issues not too relevant to model theory. In this spirit, the compactness theorem is proved via the algebraically useful ultraproduct technique (rather than via the completeness theorem of first-order logic). This leads fairly quickly to algebraic applications, …’. Now, the opening chapters are indeed very clear: but oddly the introduction of the crucial ultraproduct construction in Ch. 4 is done very briskly (compared, say, with Bell and Slomson). And thereafter it seems to me that there is some unevenness in the accessibility of the book. But others have recommended this text, so worth looking out.

There’s a short book by Kees Doets *Basic Model Theory* (CSLI 1996), which highlights so-called Ehrenfeucht games. This is enjoyable and instructive.

Bruno Poizat’s *A Course in Model Theory* (English edition, Springer 2000) starts from scratch and the early chapters give an interesting and helpful account of the model-theoretic basics, and the later chapters form a rather comprehensive introduction to stability theory. This often-recommended
book is written in a rather distinctive style, with rather more expansive class-room commentary than usual: so an unusually engaging read.

Chs. 2 and 3 of Alexander Prestel and Charles N. Delzell’s *Mathematical Logic and Model Theory: A Brief Introduction* (Springer 1986, 2011) are brisk but clear, and can be recommended if you wanting a speedy review of model theoretic basics. The key feature of the book, however, is the sophisticated final chapter on applications to algebra, which might appeal to mathematicians with special interests in that area. For a very little more on this book, see my Book Note.

As will quickly become clear as you explore model theory, we quickly get tangled up with algebraic questions. Now, as well as going (so to speak) in the direction from logic to algebra, we can make connections the other way about, starting from algebra. For something on this approach, see the following short, relatively accessible, and illuminating book:


One thing you will have noticed is that the absolutely key compactness theorem (for example) can be proved in a variety of ways – indirectly via the completeness proof, via a more direct Henkin construction, via ultraproducts, etc. How do these proofs inter-relate? Do they generalize in different ways? Do they differ in explanatory power? For a quite excellent essay on this – on the borders of mathematics and philosophy (and illustrating that there is indeed interesting work to be done on those borders), see


Finally, let me also briefly allude to the sub-area of Finite Model Theory which arises from consideration of problems in the theory of computation (where, of course, we are interested in *finite* structures – e.g. finite databases and finite computations over them). What happens, then, to model theory if we restrict our attention to finite models? Trakhtenbrot’s theorem, for example, tells that the class of sentences true in any finite model is not recursively enumerable. So there is no deductive theory for capturing such finitely valid sentences. As the Wikipedia entry on the theorem wonders, isn’t it counter-intuitive that the property of being valid over *all* structures should be ‘easier’ (nicely capturable in a sound and complete deductive system) when compared with the restricted
finite case? It turns out, then, that the study of finite models is surprisingly rich and interesting (at least for enthusiasts!). So why not dip into one or other of


Either is a good standard text to explore the area with, though I prefer Libkin’s.

### 6.3 Computability

In §4.2 we took a first look at the related topics of computability, Gödelian incompleteness, and theories of arithmetic. In this and the next two main sections, we return to these topics, taking them separately (though this division is necessarily slightly artificial).

#### 6.3.1 Computable functions

First, a recent and very admirable introductory text that was a candidate for being mentioned in our introductory readings in Chapter 4 instead of Epstein and Carnielli: but this book is just a step more challenging/sophisticated/abstract so probably belongs here:


   Enderton writes here with attractive zip and lightness of touch. The first chapter is on the informal Computability Concept. There are then chapters on general recursive functions and on register machines (showing that the register-computable functions are exactly the recursive ones), and a chapter on recursive enumerability. Chapter 5 makes ‘Connections to Logic’ (including proving Tarski’s
theorem on the undefinability of arithmetical truth and a semantic incompleteness theorem). The final two chapters push on to say something about ‘Degrees of Unsolvability’ and ‘Polynomial-time Computability’. This is all very nicely done, and in under 150 pages too.

Enderton’s book could appeal to, and be very manageable by, philosophers. However, if you are more mathematically minded and/or you have already coped well with the basic reading on computability mentioned in §4.2 you might well prefer to jump up half a notch in difficulty and go straight to the stand-out text:

2. Nigel Cutland, *Computability: An Introduction to Recursive Function Theory* (CUP 1980). This is a rightly much-reprinted classic and is beautifully lucid and well-organized. This *does* have a bit more of the look-and-feel of a traditional maths text book of its time (so with fewer of Enderton’s classroom asides). However, if you got through most of e.g. Boolos and Jeffrey without too much difficulty, you ought certainly to be able to tackle this as the next step. Very warmly recommended.

And of more recent general books covering similar ground, I particularly like

3. S. Barry Cooper, *Computability Theory* (Chapman & Hall/CRC 2004, 2nd edn. forthcoming 2015). This is a particularly nicely done modern textbook. Read at least Part I of the book (about the same level of sophistication as Cutland, but with some extra topics), and then you can press on as far as your curiosity takes you, and get to excitement like the Friedberg-Muchnik theorem.

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**Postscript** The inherited literature on computability is huge. But, being *very* selective, let me mention three classics from different generations:

Rósza Péter, *Recursive Functions* (originally published 1950: English translation Academic Press 1967). This is by one of those logicians who was
‘there at the beginning’. It has that old-school slow-and-steady un-flashy lucidity that makes it still a considerable pleasure to read. It remains very worth looking at.

Hartley Rogers, Jr., *Theory of Recursive Functions and Effective Computability* (McGraw-Hill 1967) is a heavy-weight state-of-the-art-then classic, written at the end of the glory days of the initial development of the logical theory of computation. It quite speedily gets advanced. But the opening chapters are still excellent reading and are action-packed. At least take it out of the library, read a few chapters, and admire!

Piergiorgio Odifreddi, *Classical Recursion Theory*, Vol. 1 (North Holland, 1989) is well-written and discursive, with numerous interesting asides. It’s over 650 pages long, so it goes further and deeper than other books on the main list above (and then there is Vol. 2). But it certainly starts off quite gently paced and very accessibly and can be warmly recommended.

A number of books we’ve already mentioned say something about the fascinating historical development of the idea of computability: as we noted before, Richard Epstein offers a very helpful 28 page timeline on ‘Computability and Undecidability’ at the end of the 2nd edn. of Epstein/Carnielli (see §4.2). Cooper’s short first chapter on ‘Hilbert and the Origins of Computability Theory’ also gives some of the headlines. Odifreddi too has many historical details. But here are two more good essays on the history:


6.3.2 Computational complexity

Computer scientists are – surprise, surprise! – interested in the theory of feasible computation, and it is certainly interesting to know at least a little about the topic of computational complexity.
1. Shawn Hedman *A First Course in Logic* (OUP 2004): Ch. 7 on ‘Computability and complexity’ has a nice review of basic computability theory before some lucid sections introducing notions of computational complexity.

2. Michael Sipser, *Introduction to the Theory of Computation* (Thomson, 2nd edn. 2006) is a standard and very well regarded text on computation aimed at computer scientists. It aims to be very accessible and to take its time giving clear explanations of key concepts and proof ideas. I think this is very successful as a general introduction and I could well have mentioned the book before. But I’m highlighting the book in this subsection because its last third is on computational complexity.


4. Ashley Montanaro, *Computational Complexity*. Excellent 2012 lecture notes that include a useful quick guide to further reading.

5. You could also look at the opening chapters of the pretty encyclopaedic Sanjeev Arora and Boaz Barak *Computational Complexity: A Modern Approach* (CUP, 2009). The authors say ‘Requiring essentially no background apart from mathematical maturity, the book can be used as a reference for self-study for anyone interested in complexity, including physicists, mathematicians, and other scientists, as well as a textbook for a variety of courses and seminars.’ And it at least starts very readably. A late draft of the book can be freely downloaded.

6.4 Gödelian incompleteness again

If you have looked at my book and/or Boolos and Jeffrey you should now be in a position to appreciate the terse elegance of
1. Raymond Smullyan, *Gödel’s Incompleteness Theorems*, Oxford Logic Guides 19 (Clarendon Press, 1992). This is delightfully short – under 140 pages – proving some beautiful, slightly abstract, versions of the incompleteness theorems. This is a modern classic which anyone with a taste for mathematical elegance will find rewarding.

2. Equally short and equally elegant is Melvin Fitting’s, *Incompleteness in the Land of Sets* (College Publications, 2007). This approaches things from a slightly different angle, relying on the fact that there is a simple correspondence between natural numbers and ‘hereditarily finite sets’ (i.e. sets which have a finite number of members which in turn have a finite number of members which in turn … where all downward membership chains bottom out with the empty set).

In terms of difficulty, these two lovely brief books could easily have appeared among our introductory readings in Chapter 4. I have put them here because (as I see it) the simpler, more abstract, stories they tell can probably only be fully appreciated if you’ve first met the basics of computability theory and the incompleteness theorems in a more conventional treatment.

You ought also at some stage read an even briefer, and still officially introductory, treatment of the incompleteness theorems,


After these, where should you go if you want to know more about matters more or less directly to do with the incompleteness theorems?

4. Raymond Smullyan’s *Diagonalization and Self-Reference*, Oxford Logic Guides 27 (Clarendon Press 1994) is an investigation-in-depth around and about the idea of diagonalization that figures so prominently in proofs of limitative results like the unsolvability of the halting problem, the arithmetical undefinability of arithmetical truth, and the incompleteness of arithmetic. Read at least Part I.
5. Torkel Franzén, *Inexhaustibility: A Non-exhaustive Treatment* (Association for Symbolic Logic/A. K. Peters, 2004). The first two-thirds of the book gives another take on logic, arithmetic, computability and incompleteness. The last third notes that Gödel’s incompleteness results have a positive consequence: ‘any system of axioms for mathematics that we recognize as correct can be properly extended by adding as a new axiom a formal statement expressing that the original system is consistent. This suggests that our mathematical knowledge is inexhaustible, an essentially philosophical topic to which this book is devoted.’ Not always easy (you will need to know something about ordinals before you read this), but very illuminating.


Going in a rather different direction, you will recall from my *IGT2* or other reading on the second incompleteness theorem that we introduced the so-called derivability conditions on $\Box \varphi$ (where this is an abbreviation for – or at any rate, is closely tied to – $\text{Prov}(\overline{\varphi})$, which expresses the claim that the wff $\varphi$, whose Gödel number is $\overline{\varphi}$, is provable in some given theory). The ‘$\Box$’ here functions rather like a modal operator: so what is its modal logic? This is investigated in a wonderful modern classic


### 6.5 Theories of arithmetic

The readings in §4.2 will have introduced you to the canonical first-order theory of arithmetic, first-order Peano Arithmetic, as well as to some subsystems of PA (in particular, Robinson Arithmetic) and second-order extensions. And you will already know that first-order PA has non-standard
models (in fact, it even has uncountably many non-isomorphic models which can be built out of natural numbers!).

So what to read next on arithmetic? There is perhaps something of a gap in the literature here and a need for a mid-level book on theories of arithmetic and their models. But taking a step up in difficulty, we get to

1. Richard Kaye’s *Models of Peano Arithmetic* (Oxford Logic Guides, OUP, 1991) which tells us a great deal about non-standard models of PA. This will reveal more about what PA can and can’t prove, and will introduce you to some non-Gödelian examples of incompleteness. This does get pretty challenging in places, and it is probably best if you’ve already worked through some model theory at a more-than-very-basic level. Still, this is a terrific book, and deservedly a modern classic.

(There’s also another volume in the Oxford Logic Guides series which can be thought of as a sequel to Kaye’s for real enthusiasts with more background in model theory, namely Roman Kossak and James Schmerl, *The Structure of Models of Peano Arithmetic*, OUP, 2006.)

Next, going in a rather different direction, with e.g. a lot about arithmetics weaker than full PA, here’s another modern classic:

2. Petr Hájek and Pavel Pudlák, *Metamathematics of First-Order Arithmetic*** (Springer 1993). Now freely available from projecteuclid.org. This is pretty encyclopaedic, but the long first three chapters, say, actually do remain surprisingly accessible for such a work. This is, eventually, a must-read if you have a serious interest in theories of arithmetic and incompleteness.

And what about going beyond first-order PA? We know that full second-order PA (where the second-order quantifiers are constrained to run over all possible sets of numbers) is unaxiomatizable, because the underlying second-order logic is unaxiomatiable. But there are axiomatizable subsystems of second order arithmetic. These are wonderfully investigated in another encyclopaedic modern classic:
3. Stephen Simpson, *Subsystems of Second-Order Logic* (Springer 1999; 2nd edn CUP 2009). The focus of this book is the project of ‘reverse mathematics’ (as it has become known): that is to say, the project of identifying the weakest theories of numbers-and-sets-of-numbers that are required for proving various characteristic theorems of classical mathematics.

We know that we can reconstruct classical analysis in pure set theory, and rather more neatly in set theory with natural numbers as unanalysed ‘urelemente’. But just *how much* set theory is needed to do the job? The answer is: stunningly little. The project of exploring what’s needed is introduced very clearly and accessibly in the first chapter, which is a must-read for anyone interested in the foundations of mathematics. This introduction is freely available at the book’s website.

6.6 *Serious set theory*

§4.3 gave suggestions for readings on the elements of set theory. These will have introduced you to the standard set theory ZFC, and the iterative hierarchy it seeks to describe. They also explained how we can construct the real number system in set theoretic terms (so giving you a sense of what might be involved in saying that set theory can be used as a ‘foundation’ for another mathematical theory). You will have in addition learnt something about the role of the axiom of choice, and about the arithmetic of infinite cardinal and ordinal numbers. If you looked at the books by Fraenkel/Bar-Hillel/Levy or by Potter, you will also have noted that standard ZFC is not the only set theory on the market. So where next?

We’ll take things in three stages, firstly focusing again on our canonical theory, ZFC (though this soon reaches into seriously hard mathematics). Then we’ll backtrack from those pointers towards the frontiers to consider the Axiom of Choice (as this is of particular conceptual interest). Then we will say something about non-standard set theories (again, the possi-
bility of different accounts, with different degrees of departure from the canonical theory, is of considerable conceptual interest and you don’t need a huge mathematical background to understand some of the options).

### 6.6.1 ZFC, with all the bells and whistles

One option is immediately to go for broke and dive in to the modern bible, which is impressive not just for its size:

1. Thomas Jech, *Set Theory*, The Third Millennium Edition, Revised and Expanded (Springer, 2003). The book is in three parts: the first, Jech says, every student should know; the second part every budding set-theorist should master; and the third consists of various results reflecting ‘the state of the art of set theory at the turn of the new millennium’. Start at page 1 and keep going to page 705 (or until you feel glutted with set theory, whichever comes first).

   For this is indeed a masterly achievement by a great expositor. And if you’ve happily read e.g. the introductory books by Enderton and then Moschovakis mentioned the Guide, then you should be able to cope pretty well with Part I of the book while it pushes on the story a little with some material on small large cardinals and other topics. Part II of the book starts by telling you about independence proofs. The Axiom of Choice is consistent with ZF and the Continuum Hypothesis is consistent with ZFC, as proved by Gödel using the idea of ‘constructible’ sets. And the Axiom of Choice is independent of ZF, and the Continuum Hypothesis is independent with ZFC, as proved by Cohen using the much more tricky idea of ‘forcing’. The rest of Part II tells you more about large cardinals, and about descriptive set theory. Part III is indeed for enthusiasts.

   Now, Jech’s book is wonderful, but let’s face it, the sheer size makes it a trifle daunting. It goes quite a bit further than many will need, and to get there it does speed along a bit faster than some will feel comfortable with. So what other options are there for taking things more slowly?
Well, you could well profit from starting with some preliminary historical orientation. If you looked at the old book by Fraenkel/Bar-Hillel/Levy which was recommended in the Guide, then you will already know something of the early days. Alternatively, there is a nice very short introductory overview

2. José Ferreirós, ‘The early development of set theory’, *The Stanford Encycl. of Philosophy*. (Ferreirós has also written a terrific book *Labyrinth of Thought: A History of Set Theory and its Role in Modern Mathematics* (Birkhäuser 1999), which at some stage in the future you might well want to read.)

You could also browse through the substantial article

3. Akhiro Kanamori, ‘The Mathematical Development of Set Theory from Cantor to Cohen’, *The Bulletin of Symbolic Logic* (1996) pp. 1-71, a revised version of which is downloadable here. (You will very probably need to skip chunks of this at a first pass: but even a partial grasp will help give you a good sense of the lie of the land.)

Then to start filling in details, a much admired older book (not up-to-date of course, but still a fine first treatment of its topic) is

4. Frank R. Drake, *Set Theory: An Introduction to Large Cardinals* (North-Holland, 1974), which – at a gentler pace? – overlaps with Part I of Jech’s bible, but also will tell you about Gödel’s Constructible Universe and some more about large cardinals.

But the crucial next step – that perhaps marks the point where set theory gets challenging – is to get your head around Cohen’s idea of forcing used in independence proofs. However, there is not getting away from it, this is tough. In the admirable

5. Timothy Y. Chow, ‘A beginner’s guide to forcing’,

Chow writes
All mathematicians are familiar with the concept of an open research problem. I propose the less familiar concept of an open exposition problem. Solving an open exposition problem means explaining a mathematical subject in a way that renders it totally perspicuous. Every step should be motivated and clear; ideally, students should feel that they could have arrived at the results themselves. The proofs should be ‘natural’ . . . [i.e., lack] any ad hoc constructions or brilliances. I believe that it is an open exposition problem to explain forcing.

In short: if you find that expositions of forcing tend to be hard going, then join the club.

Now, there is a new book which aims to fill the expositional gap, clearly and briefly:


A cursory glance suggests that this is very promising, but at the moment I can’t give a firm recommendation since I haven’t read it!

So back to a very widely used and much reprinted textbook, which nicely complements Drake’s book and which has (inter alia) a pretty good presentation of forcing:

7. Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs* (North-Holland, 1980). Again, if you have read (some of) the introductory set theory books mentioned in the Guide, you should actually find much of this classic text now pretty accessible, and can speed through some of the earlier chapters, slowing down later, until you get to the penultimate chapter on forcing which you’ll need to take slowly and carefully. (Kunen has lately published another, totally rewritten, version of this book as *Set Theory* (College Publications, 2011). This later book is quite significantly longer, covering an amount of more difficult material that has come to prominence since 1980. Not just because of the additional material, my current sense is that the earlier book may remain the slightly more approachable
read. But you’ll probably want to tackle the later version at some point.)

Kunen’s classic text takes a ‘straight down the middle’ approach, starting with what is basically Cohen’s original treatment of forcing, though he does relate this to some variant approaches to forcing. Here are two of them:

8. Raymond Smullyan and Melvin Fitting, *Set Theory and the Continuum Problem* (OUP 1996, Dover Publications 2010). This medium-sized book is divided into three parts. Part I is a nice introduction to axiomatic set theory. The shorter Part II concerns matters round and about Gödel’s consistency proofs via the idea of constructible sets. Part III gives a different take on forcing (a variant of the approach taken in Fitting’s earlier *Intuitionistic Logic, Model Theory, and Forcing*, North Holland, 1969). This is beautifully done, as you might expect from two writers with a quite enviable knack for wonderfully clear explanations and an eye for elegance.

9. Keith Devlin, *The Joy of Sets* (Springer 1979, 2nd edn. 1993) Ch. 6 introduces the idea of Boolean-Valued Models and their use in independence proofs. The basic idea is fairly easily grasped, but details get hairy. For more on this theme, see John L. Bell’s classic *Set Theory: Boolean-Valued Models and Independence Proofs* (Oxford Logic Guides, OUP, 3rd edn. 2005). The relation between this approach and other approaches to forcing is discussed e.g. in Chow’s paper and the last chapter of Smullyan and Fitting.

And after those? It’s back to Jech’s bible and/or the more recent

10. Lorenz J. Halbeisen, *Combinatorial Set Theory, With a Gentle Introduction to Forcing* (Springer 2011, with a late draft freely downloadable from the author’s website). This is particularly attractively written for a set theory book, and has been widely recommended.

And then – oh heavens! – there is another blockbuster awaiting you:
6.6.2 The Axiom of Choice

But leave the Higher Infinite and get back down to earth! In fact, to return to the beginning, we might wonder: is ZFC the ‘right’ set theory?

Start by thinking about the Axiom of Choice in particular. It is comforting to know from Gödel that AC is consistent with ZF (so adding it doesn’t lead to contradiction). But we also know from Cohen’s forcing argument that AC is independent with ZF (so accepting ZF doesn’t commit you to accepting AC too). So why buy AC? Is it an optional extra?

Some of the readings already mentioned will have touched on the question of AC’s status and role. But for an overview/revision of some basics, see


And for a short book also explaining some of the consequences of AC (and some of the results that you need AC to prove), see


That already probably tells you more than enough about the impact of AC: but there’s also a famous book by H. Rubin and J.E. Rubin, Equivalents of the Axiom of Choice (North-Holland 1963; 2nd edn. 1985) which gives over two hundred equivalents of AC! Then next there is the nice short classic

Hypothesis). In particular, there is a nice presentation of the so-called Fraenkel-Mostowski method of using ‘permutation models’. Then later parts of the book tell us something about what mathematics without choice, and about alternative axioms that are inconsistent with choice.

And for a more recent short book, taking you into new territories (e.g. making links with category theory), enthusiasts might enjoy


6.6.3 Other set theories?

From earlier reading you should have picked up the idea that, although ZFC is the canonical modern set theory, there are other theories on the market. I mention just a selection here (I’m not suggesting you follow up all these – the point is to stress that set theory is not quite the monolithic edifice that some introductory presentations might suggest).

**NBG** You will have come across mention of this already (e.g. even in the early pages of Enderton’s set theory book). And in fact – in many of the respects that matter – it isn’t really an ‘alternative’ set theory. So let’s get it out of the way first. We know that the universe of sets in ZFC is not itself a set. But we might think that this universe is a *sort* of big collection. Should we explicitly recognize, then, two sorts of collection, sets and (as they are called in the trade) proper classes which are too big to be sets? NBG (named for von Neumann, Bernays, Gödel: some say VBG) is one such theory of collections. So NBG in some sense recognizes proper classes, objects having members but that cannot be members of other entities. NBG’s principle of class comprehension is predicative; i.e. quantified variables in the defining formula can’t range over proper classes but range only over sets, and we get a conservative extension of ZFC (nothing in the language of sets can be proved in NBG which can’t already be proved in ZFC).

2. Elliott Mendelson, *Introduction to Mathematical Logic* (CRC, 4th edition 1997), Ch.4. is a classic and influential textbook presentation of set-theory via NBG.

**SP** This again is by way of reminder. Recall, earlier in the Guide, we very warmly recommended Michael Potter’s book which we just mentioned again. This presents a version of an axiomatization of set theory due to Dana Scott (hence ‘Scott-Potter set theory’). This axiomatization is consciously guided by the conception of the set theoretic universe as built up in levels (the conception that, supposedly, also warrants the axioms of ZF). What Potter’s book aims to reveal is that we can get a rich hierarchy of sets, more than enough for mathematical purposes, without committing ourselves to *all* of ZFC (whose extreme richness comes from the full Axiom of Replacement). If you haven’t read Potter’s book before, now is the time to look at it.

**ZFA** (i.e. ZF − AF + AFA) Here again is the now-familiar hierarchical conception of the set universe: We start with some non-sets (maybe zero of them in the case of pure set theory). We collect them into sets (as many different ways as we can). Now we collect what we’ve already formed into sets (as many as we can). Keep on going, as far as we can. On this ‘bottom-up’ picture, the Axiom of Foundation is compelling (any downward chain linked by set-membership will bottom out, and won’t go round in a circle). But now here’s another alternative conception of the set universe. Think of a set as a gadget that points you at some some things, its members. And those members, if sets, point to their members. And so on and so forth. On this ‘top-down’ picture, the Axiom of Foundation is not so compelling. As we follow the pointers, can’t we for example come back to where we started? It is well known that in much of the usual development of ZFC the Axiom of Foundation AF does little work. So what about considering a theory of sets which drops AF and instead has
an Anti-Foundation Axiom (AFA), which allows self-membered sets? To explore this idea,


3. Peter Aczel’s, Non-well-founded sets, (CSLI Lecture Notes 1988) is a very readable short classic book.

4. Luca Incurvati, ‘The graph conception of set’ Journal of Philosophical Logic (published online Dec 2012), very illuminatingly explores the motivation for such set theories.

NF Now for a much more radical departure from ZF. Standard set theory lacks a universal set because, together with other standard assumptions, the idea that there is a set of all sets leads to contradiction. But by tinkering with those other assumptions, there are coherent theories with universal sets. For very readable presentations concentrating on Quine’s NF (‘New Foundations’), and explaining motivations as well as technical details, see


ETCS Famously, Zermelo constructed his theory of sets by gathering together some principles of set-theoretic reasoning that seemed actually
to be used by working mathematicians (engaged in e.g. the rigorization of analysis or the development of point set topology), hoping to get a theory strong enough for mathematical use while weak enough to avoid paradox. But does he overshoot? We’ve already noted that SP is a weaker theory which may suffice. For a more radical approach, see


2. F. William Lawvere and Robert Rosebrugh, *Sets for Mathematicians* (CUP 2003) gives a very accessible presentation which in principle doesn’t require that you have already done any category theory.

But perhaps to fully appreciate what’s going on, you will have to go on to dabble in some more category theory.

**IZF, CZF** ZF/ZFC has a classical logic: what if we change the logic to intuitionistic logic? what if we have more general constructivist scruples? The place to start exploring is


Then for one interesting possibility, look at the version of constructive ZF in


**IST** Leibniz and Newton invented infinitesimal calculus in the 1660s: a century and a half later we learnt how to rigorize the calculus without invoking infinitely small quantities. Still, the idea of infinitesimals retains a certain intuitive appeal, and in the 1960s, Abraham Robinson created a theory of hyperreal numbers: this yields a rigorous formal treatment of
infinitesimal calculus (you will have seen this mentioned in e.g. Enderton’s *Mathematical Introduction to Logic*, §2.8, or van Dalen’s *Logic and Structure*, p. 123). Later, a simpler and arguably more natural approach, based on so-called Internal Set Theory, was invented by Edward Nelson. As put it, ‘IST is an extension of Zermelo-Fraenkel set theory in that alongside the basic binary membership relation, it introduces a new unary predicate ‘standard’ which can be applied to elements of the mathematical universe together with some axioms for reasoning with this new predicate.’ Starting in this way we can recover features of Robinson’s theory in a simpler framework.


**Yet more?** Well yes, we can keep on going. Take a look, for example, at *SEAR*. But we must call a halt! For a brisk overview, putting many of these various set theories into some sort of order, and mentioning yet further alternatives, see


If that’s a bit *too* brisk, then (if you can get access to it) there’s what can be thought of as a bigger, better, version here:

Chapter 7

What else?

Mathematical logicians and philosophers interested in the philosophy of maths will want to know about yet more areas that fall outside the traditional math logic curriculum. For example:

- With roots going right back to *Principia Mathematica*, there’s the topic of *modern type theories*.

- Relatedly, we should explore *the lambda calculus*.

- Again relatedly, of central computer-science interest, there’s the business of *logic programming* and *formal proof verification*.

- Even in elementary model theory we relax the notion of a language to allow e.g. for uncountably many names: what if we further relax and allow for e.g. sentences which are infinite conjunctions? Pursuing such questions leads us to consider *infinitary logics*.

- Going in the opposite direction, as it were, an intuitionist worries about applying classical reasoning to infinite domains, opening up the whole topic of constructive reasoning and *constructive mathematics*.

- If set theory traditionally counts as part of mathematical logic, because of its generality, breadth and foundational interest, what about *category theory* (on which there is now a supplementary webpage)?
These topics go beyond the remit of this basic Guide. But perhaps, over time, I will produce supplements to cover some of them.

Meanwhile, those who feel that – even with the Big Books mentioned in the Book Notes – they haven’t got enough to read (or think that the mainstream diet so far is a bit restricted) can always consult the modestly-sized articles in the old-but-still-useful *Handbook of Mathematical Logic* edited by Jon Barwise, or the much longer, wide-ranging, articles in the seventeen(!) volume *Handbook of Philosophical Logic* edited by the indefatigable Dov Gabbay together with F. Guenthner (the latter is somewhat mistitled: a lot of the articles are straight mathematics which might be of interest to technically minded philosophers but could equally be of interest to others too). Or you can try some of the articles in the fascinating nine-volumes-and-counting of the *Handbook of the History of Logic* (edited by Dov Gabbay together with John Woods, with contents listed here), and/or dip into the five volumes of the *Handbook of Logic in Computer Science* (edited by S. Abramsky, Dov Gabbay – again! – and T. S. E. Maibaum).

But enough already!
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