Answers to Exercises 11

A *Give truth-tables for the following wffs (i.e. calculate the value of the wff for every assignment of values to the atoms: you can, however, use short-cuts).*

1. ¬(P ∧ ¬P)
2. (P ∧ ¬(P ∧ Q))
3. ((R ∨ Q) ∨ ¬P)
4. (¬(P ∧ ¬Q) ∧ ¬¬R)
5. ((P ∧ Q) ∨ (¬P ∨ ¬Q))
6. ¬((P ∧ ¬Q) ∨ (¬R ∨ ¬(P ∨ Q)))
7. (((P ∨ ¬Q) ∧ (Q ∨ R)) ∨ ¬¬(Q ∨ ¬R))
8. (¬(¬P ∨ ¬(Q ∧ ¬R)) ∨ ¬¬(Q ∨ ¬P))
9. (¬((R ∨ ¬Q) ∧ ¬S) ∧ (¬(¬P ∧ Q) ∧ S))

B *Give equivalent wffs in DNF for each of A.1 to 7. (Read off the truth-table and use the construction in §11.7.)*

1 ¬(P ∧ ¬P)

P	¬	(P ∧ ¬P)
T	T	F
F	T	F

DNF: (P ∨ ¬P)

2 (P ∧ ¬(P ∧ Q))

P	Q	(P	∧	¬	(P ∧ Q))
T	T	T	F	F	T
T	F	T	T	T	F
F	T	F	F		
F	F	F	F		
		1	2,5	4	3

(here the numbers '2,5' indicate that part of the column can be filled in at stage 2, and the rest at stage 5 of the working)

DNF: (P ∧ ¬Q) — a single 'basic conjunction'.

3 ((R ∨ Q) ∨ ¬P)

P	Q	R	((R ∨ Q)	∨	¬P)
T	T	T	T	T	F
T	T	F	T	T	F
T	F	T	T	T	F
T	F	F	F	F	F
F	T	T		T	T
F	T	F		T	T
F	F	T		T	T
F	F	F		T	T
			3	2,4	1

The DNF will disjoin basic conjunctions for 7 lines, so being careless about brackets will have …

(P ∧ Q ∧ R) ∨ (P ∧ Q ∧ ¬R) ∨ (P ∧ ¬Q ∧ R) ∨ (¬P ∧ Q ∧ R) ∨ (¬P ∧ Q ∧ ¬R) ∨ (¬P ∧ ¬Q ∧ R) ∨ (¬P ∧ ¬Q ∧ ¬R)

Now correct the bracketing of each basic conjunction, e.g.

(P ∧ (Q ∧ R)) ∨ (P ∧ (Q ∧ ¬R)) ∨ (P ∧ (¬Q ∧ R)) ∨ (¬P ∧ (Q ∧ R)) ∨ (¬P ∧ (Q ∧ ¬R)) ∨ (¬P ∧ (¬Q ∧ R)) ∨ (¬P ∧ (¬Q ∧ ¬R))

And now group disjunctions properly (I've used different style brackets to make it clearer ..)

((P ∧ (Q ∧ R)) ∨ [(P ∧ (Q ∧ ¬R)) ∨ {(P ∧ (¬Q ∧ R)) ∨ ((¬P ∧ (Q ∧ R)) ∨ [(¬P ∧ (Q ∧ ¬R)) ∨ {(¬P ∧ (¬Q ∧ R)) ∨ (¬P ∧ (¬Q ∧ ¬R))}])}])

Obviously that way of bracketing isn't the only legal one. We could have grouped from the left, or adopted some other more jumbled policy.

4 (¬(P ∧ ¬Q) ∧ ¬¬R)

P	Q	R	(¬	(P ∧ ¬Q)	∧	¬¬R)
T	T	T	T	F	T	T
T	T	F			F	F
T	F	T	F	T	F	T
T	F	F			F	F
F	T	T	T	F	T	T
F	T	F			F	F
F	F	T	T	F	T	T
F	F	F			F	F
			4	3	2,5	1

DNF: informally, with some bracketing suppressed:

((P ∧ Q ∧ R) ∨ (¬P ∧ Q ∧ R) ∨ (¬P ∧ ¬Q ∧ R))

More officially:

((P ∧ (Q ∧ R)) ∨ ((¬P ∧ (Q ∧ R)) ∨ (¬P ∧ (¬Q ∧ R))))

5 ((P ∧ Q) ∨ (¬P ∨ ¬Q))

P	Q	((P ∧ Q)	∨	(¬P ∨ ¬Q))
T	T	T	T	F
T	F		T	T
F	T		T	T
F	F		T	T
		3	2,4	1

DNF: informally, with some bracketing suppressed:

((P ∧ Q) ∨ (¬P ∧ Q) ∨ (P ∧ ¬Q) ∨ (¬P ∧ ¬Q))

Bracketed would become:

((P ∧ Q) ∨ ((¬P ∧ Q) ∨ ((P ∧ ¬Q) ∨ (¬P ∧ ¬Q))))

6 ¬((P ∧ ¬Q) ∨ (¬R ∨ ¬(P ∨ Q)))

P	Q	R	¬	((P ∧ ¬Q)	∨	(¬R	∨	¬(P ∨ Q)))
T	T	T	T	F	F	F	F	F
T	T	F	F		T	T	T	
T	F	T	F	T	T	F	F	F
T	F	F	F		T	T	T	
F	T	T	T	F	F	F	F	F
F	T	F	F		T	T	T	
F	F	T	F		T	F	T	T
F	F	F	F		T	T	T	
			6	4	5	1	3	2

DNF: informally, with some bracketing suppressed:

((P ∧ Q ∧ R) ∨ (¬P ∧ Q ∧ R))

Bracketed would become:

((P ∧ (Q ∧ R)) ∨ (¬P ∧ (Q ∧ R)))

7 (((P ∨ ¬Q) ∧ (Q ∨ R)) ∨ ¬¬(Q ∨ ¬R))

P	Q	R	(((P ∨	¬Q) ∧	(Q ∨ R))	∨	¬¬(Q ∨ ¬R))
T	T	T				T	T
T	T	F				T	T
T	F	T	T	T	T	T	F
T	F	F				T	T
F	T	T				T	T
F	T	F				T	T
F	F	T	T	T	T	T	F
F	F	F				T	T
			3	5	4	2,6	1

DNF: informally, with some bracketing suppressed:

((P ∧ Q ∧ R) ∨ (P ∧ Q ∧ ¬R)) ∨ (P ∧ ¬Q ∧ R) ∨ (P ∧ ¬Q ∧ ¬R) ∨ (¬P ∧ Q ∧ R)
∨ (¬P ∧ Q ∧ ¬R) ∨ (¬P ∧ ¬Q ∧ R) ∨ (¬P ∧ ¬Q ∧ ¬R))

The formal version with all the brackets in place becomes horrendous again!!!!

8 (¬(¬P ∨ ¬(Q ∧ ¬R)) ∨ ¬¬(Q ∨ ¬P))

P	Q	R	(¬	(¬P	∨	¬(Q ∧ ¬R))	∨	¬¬(Q ∨ ¬P))
T	T	T					T	T
T	T	F					T	T
T	F	T	F	F	T	T	F	F
T	F	F	F	F	T	T	F	F
F	T	T					T	T
F	T	F					T	T
F	F	T					T	T
F	F	F					T	T
			6	3	5	4	2/7	1

9 (¬((R ∨ ¬ Q) ∧ ¬ S) ∧ (¬(¬ P ∧ Q) ∧ S))

P	Q	R	S	(¬	((R ∨ ¬ Q)	∧	¬ S)	∧	(¬	(¬ P ∧ Q)	∧	S))
T	T	T	T	T		F	F	T	T	F	T	T
T	T	T	F	F	T	T	T	F				
T	T	F	T	T		F	F	T	T	F	T	T
T	T	F	F	T	F	F	T	F			F	F
T	F	T	T	T		F	F	T	T	F	T	T
T	F	T	F	F	T	T	T	F				
T	F	F	T	T		F	F	T	T	F	T	T
T	F	F	F	F	T	T	T	F				
F	T	T	T	T		F	F	F	F	T	F	T
F	T	T	F	F	T	T	T	F				
F	T	F	T	T		F	F	F	F	T	F	T
F	T	F	F	T	F	F	T	F			F	F
F	F	T	T	T		F	F	T	T	F	T	T
F	F	T	F	F	T	T	T	F				
F	F	F	T	T		F	F	T	T	F	T	T
F	F	F	F	F	T	T	T	F				
				5	3	2,4	1	6,11	9	8	10	7

C *Show that '↓' and '↑' are the only two-place connectives which are expressively adequate taken by themselves.*

Let's introduce '×' as a two-place connective which is a candidate for being expressively adequate. Suppose $(A \times B)$ is T whenever A, B are both T. Then, however complex a wff we build up using '×', if the atoms in the wff are T then the complex wff will be T. So, in particular, we could never use '×' to construct a wff which is equivalent to $\neg A$ which *flips* values.

Suppose $(A \times B)$ is F whenever A, B are both F. Then, however complex a wff we build up using '×' , if the atoms in the wff are F then the complex wff will be F. So, in particular, we could again never use '×' to construct a wff which is equivalent to $\neg A$.

That shows that the only possible truth-tables for our candidate connective are of the form

A	B	$(A \times B)$
T	T	F
T	F	?
F	T	?
F	F	T

But plainly

A	B	$(A \times B)$
T	T	F
T	F	F
F	T	T
F	F	T

A	B	$(A \times B)$
T	T	F
T	F	T
F	T	F
F	F	T

which are equivalent to $\neg A$ and $\neg B$ respectively can't be used to define e.g. $(A \wedge B)$ whose value depends on *both* A and B. So that leaves

A	B	$(A \downarrow B)$
T	T	F
T	F	F
F	T	F
F	F	T

A	B	$(A \uparrow B)$
T	T	F
T	F	T
F	T	T
F	F	T

as the only possible (and we now know, successful) candidates for being two-place connectives which we could use to define both negation and conjunction (and hence all truth-functions).

D *Show that* '((P ∧ Q) ∧ R)' *and* '(P ∧ (Q ∧ R))' *have the same truth-table. Show, more generally, that pairs of wffs of the forms* $((A \wedge B) \wedge C)$ *and* $(A \wedge (B \wedge C))$ *will have the same truth-table. Generalize to show that how you bracket an unmixed conjunction,* $A \wedge B \wedge C \wedge D \wedge \ldots$*, doesn't affect truth-values. Show similarly that it doesn't matter how you bracket an unmixed disjunction.*

It is a simple exercise to check that '((P ∧ Q) ∧ R)' and '(P ∧ (Q ∧ R))' have the same truth-table. Take any wff $((A \wedge B) \wedge C)$. Whatever the values of the embedded propositional atoms, they fix the truth-values of A, B and C, and whatever those truth-values, the same truth-table calcuations establishes what we'll call *the equivalence*, namely that the values of $((A \wedge B) \wedge C)$ and $(A \wedge (B \wedge C))$ are the same.

Now consider a four-ply conjunction,

(1) (((P ∧ Q) ∧ R) ∧ S);

applying the equivalence, that must have the same value as

(2) ((P ∧ (Q ∧ R)) ∧ S).

And applying the equivalence to (1) again, this time with '(P ∧ Q)' in place of 'A', it must also have the same truth value as

(3) $((P \land Q) \land (R \land S))$

Applying the equivalence to (3) this has the same truth-value as

(4) $(P \land (Q \land (R \land S)))$

and finally, applying the equivalent (right to left, with 'Q' in place of '*A*', etc.) this is equivalent to

(5) $(P \land ((Q \land R) \land S))$

Thus by applying and re-applying the equivalence, each of the five ways of bracketing up $P \land Q \land R \land S$ has the same truth-value. And the same sort of argument evidently applies however complex the conjunct and however many conjunts there are.

Here's another argument for the same conclusion. Take any long conjunction, however bracketed up. For example

$$(A \land (((B \land C) \land D) \land (E \land F)))$$

And suppose an ultimate conjunct, say C is false. Then that makes false the conjunction it is an immediate conjunct of, i.e. $(B \land C)$. And that makes false the conjunction *it* is an immediate conjunct of, i.e. $((B \land C) \land D)$. And the falsity of *that* makes false the conjunction *it* is an immediate conjunct of, i.e. $(((B \land C) \land D) \land (E \land F))$. Which makes false the conjunction it is an immediate conjunct of, which is the whole wff. Falsity, in short percolates up to the whole wff. So an unmixed conjunction, however bracketed, is false if *any* of the conjuncts is false. And likewise an unmixed conjunction, however bracketed, is true in the other case where *all* of the conjuncts are true. So bracketing doesn't matter.

The case of unmixed disjunctions is exactly similar.

E *Show that for any A, B, $\neg(A \land B)$ is equivalent to $(\neg A \lor \neg B)$, and $\neg(A \lor B)$ is equivalent to $(\neg A \land \neg B)$. Along with the equivalences of $\neg(\neg A \land \neg B)$ with $(A \lor B)$, and $\neg(\neg A \lor \neg B)$ with $(A \land B)$, these are called De Morgan's Laws. Use De Morgan's Laws to show that every* **PL** *wff is equivalent to one where the only negation signs attach to atoms.*

Checking that De Morgan's Laws hold is trivial.

A general result: Suppose a complicated **PL** wff K involves the subformula C; and let L result from replacing C in K by the subformula D. Then if C is tautologically equivalent to D, then K is equivalent to L. (That is to say: substituting equivalent subformulae produces equivalents.) Why so? Because by construction the value of L depends on the value of D in just the way that K depends on C (and otherwise the wffs are the same). So if D and C must have the same value on any valuation (being equivalents), so must K and L.

Applied to the present case. If K involves the subformula C where C is of the form $\neg(A \land B)$ or $\neg(A \lor B)$, use one De Morgan's Laws to find an equivalent D with the initial negation sign driven inside the brackets, and substitute D for C to get an equivalent wff L. Now keep repeating this process, at each step driving a negation sign further and further inside brackets, until all negation signs are inside any brackets, giving strings of negation signs attached to atoms. The result will be equivalent to our orignal given wff. Delete all pairs of adjacent negation signs, which we can of course do yielding another equivalent wff. And we are left with an equivalent wff where the only negation signs attach immediately to atoms.

F *Instead of denoting the truth-value 'T' and 'F', use '1' and '0'; and instead of writing the connectives '$\land$', '$\lor$', write '×' and '+'. What rule of binary arithmetic corresponds to our rule for distributing the 'T' and 'F's to a sequence of atoms when constructing a truth-table? Explore the parallels between the truth-functional logic of '$\land$' and '$\lor$' and binary arithmetic. What is the 'arithmetic' correlate of negation?*

Note that the sequence of assignments e.g. to four variables, TTTT, TTTF, TTFT, TTFF , … corresponds to 1111, 1110, 1101, 1100, … i.e. the sixteen numbers from 15 to 0 in binary form in descending order. The rest of this question just invited exploration.