

A Use the tree test to determine which of the following arguments (from Exercises 13A) are tautologically valid:

1. $\neg(P \vee Q) \therefore \neg P$

$$\begin{array}{l} \neg(P \vee Q) \quad \checkmark \\ \neg\neg P \\ \neg P \\ \neg Q \\ * \end{array}$$

The tree closes, so the inference is tautologically valid. Note you should have ticked off the one complex wff you have unpacked. NB line numbers aren't necessary!

2. $(P \wedge Q) \therefore (P \vee \neg Q)$

$$\begin{array}{l} (P \wedge Q) \quad \checkmark \\ \neg(P \vee \neg Q) \quad \checkmark \\ P \\ Q \\ \neg P \\ \neg\neg Q \\ * \end{array}$$

The tree closes, so the inference is tautologically valid. Of course, to expose the contradiction, we really only need to extract 'P' from the first line and '¬P' from the second. However, note that the unpacking rules on p.159 insist that you add both conjuncts when unpacking a conjunction etc. (That's to ensure that when you 'tick' off a wff you really have extracted all its implications.)

3. $(P \vee \neg Q), \neg P \therefore \neg\neg Q$

The tree starts as follows – NB the triple helping of negation signs in the negated conclusion!

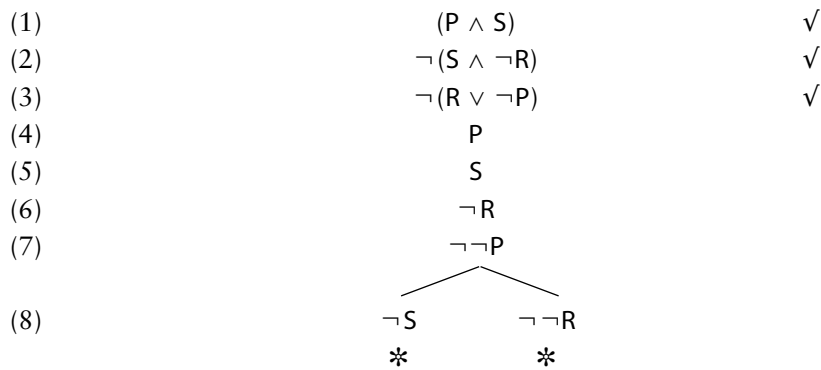
$$\begin{array}{l} (P \vee \neg Q) \\ \neg P \\ \neg\neg\neg Q \end{array}$$

Applying the non-branching rule to the last wff first, the completed tree looks like this:

$$\begin{array}{l} (P \vee \neg Q) \quad \checkmark \\ \neg P \\ \neg\neg\neg Q \quad \checkmark \\ \neg Q \\ \swarrow \quad \searrow \\ P \quad \neg Q \\ * \end{array}$$

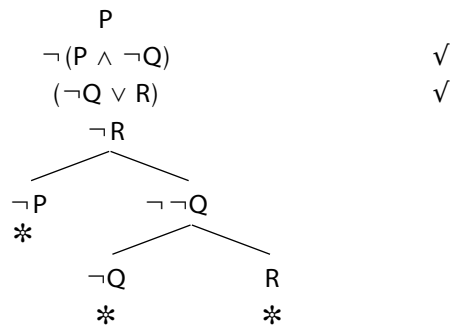
The tree is finished (we have ticked off every complex wff), and doesn't close. So the inference is invalid.

4. $(P \wedge S), \neg(S \wedge \neg R) \therefore (R \vee \neg P)$



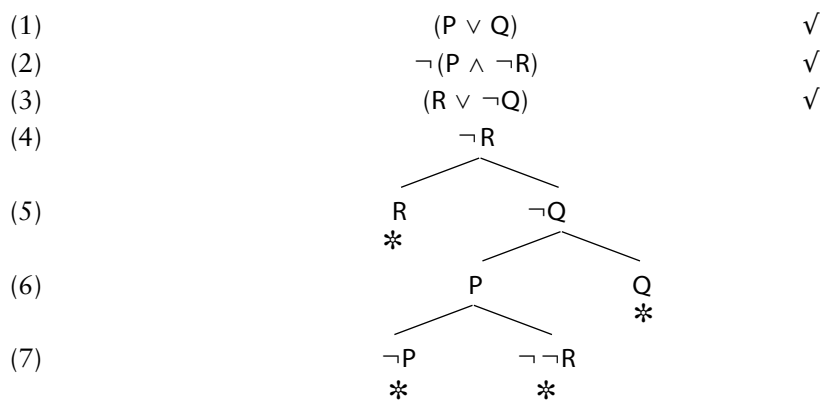
The inference is tautologically valid. Here we have first applied the non-branching rules to (1) and (3), before applying the branching rule to (2). We could have then unpacked (7) too – but we don't need to do so in order to get the tree to close.

5. $P, \neg(P \wedge \neg Q), (\neg Q \vee R) \therefore R$



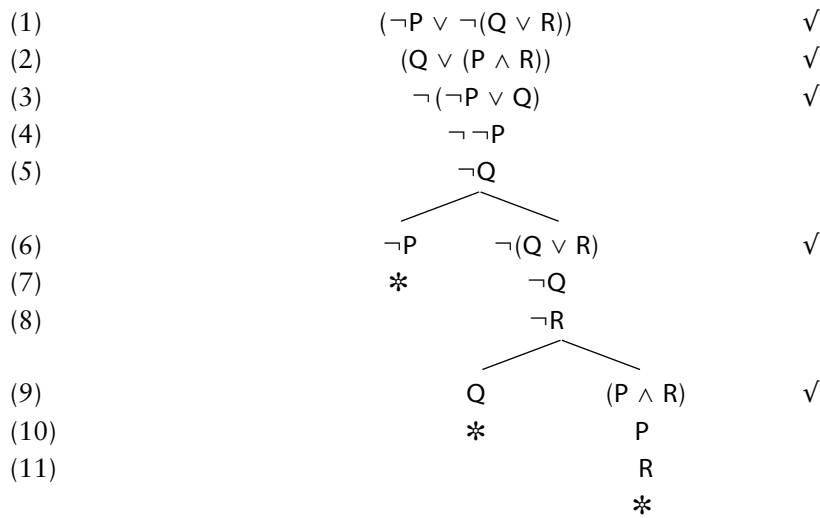
The inference is tautologically valid.

6. $(P \vee Q), \neg(P \wedge \neg R), (R \vee \neg Q) \therefore R$



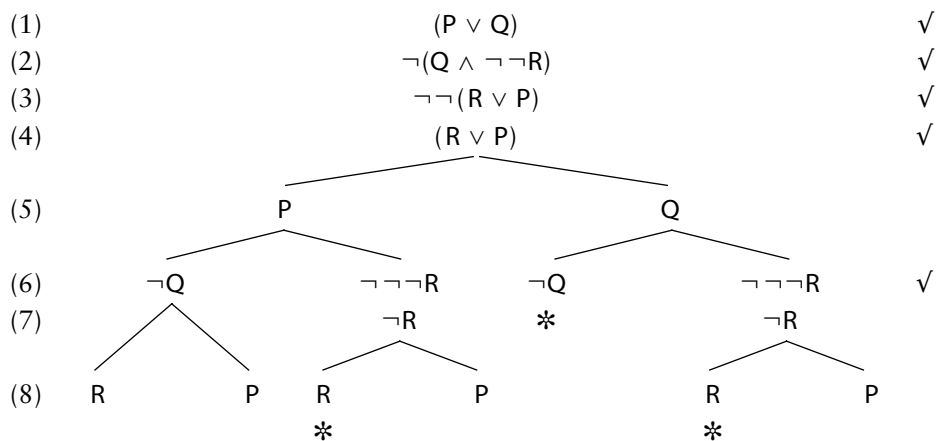
The inference is tautologically valid. We've chosen to unpack the three premisses in the order (3)–(1)–(2): why? Well, if we'd unpacked (1) first, no branch would immediately close, and the resulting tree would sprawl a bit.

7. $(\neg P \vee \neg(Q \vee R)), (Q \vee (P \wedge R)) \therefore (\neg P \vee Q)$



Apply the ‘straight’ rule to (3) first; similarly at (5) where we have a choice of unpacking (2) or (5), again apply the ‘straight’ rule first. Tree closes, so argument is valid.

8. $(P \vee Q), \neg(Q \wedge \neg\neg R) \therefore \neg(R \vee P)$

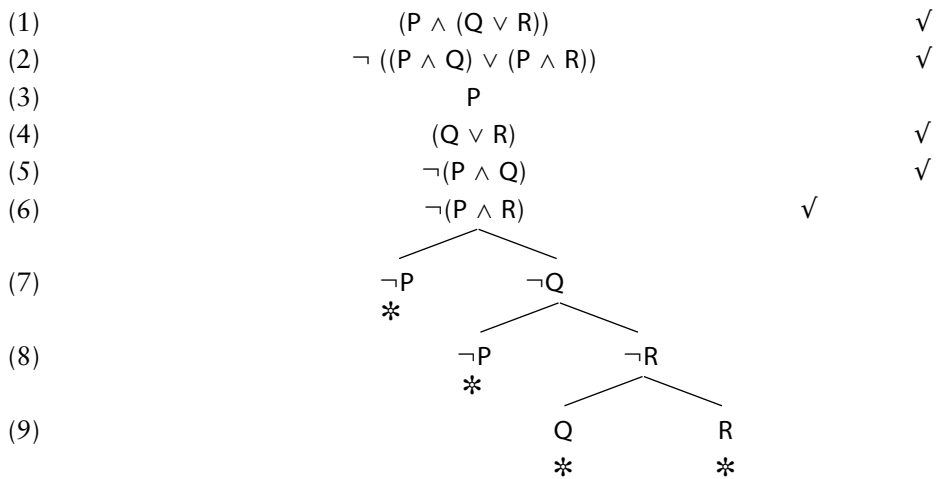


That leaves four open branches. These correspond (from the left) to the (non-exclusive!) valuations

- $P \Rightarrow T, Q \Rightarrow F, R \Rightarrow T$
- $P \Rightarrow T, Q \Rightarrow F$, and don't care about R
- $P \Rightarrow T, R \Rightarrow F$, and don't care about Q
- $P \Rightarrow T, Q \Rightarrow T, R \Rightarrow F$

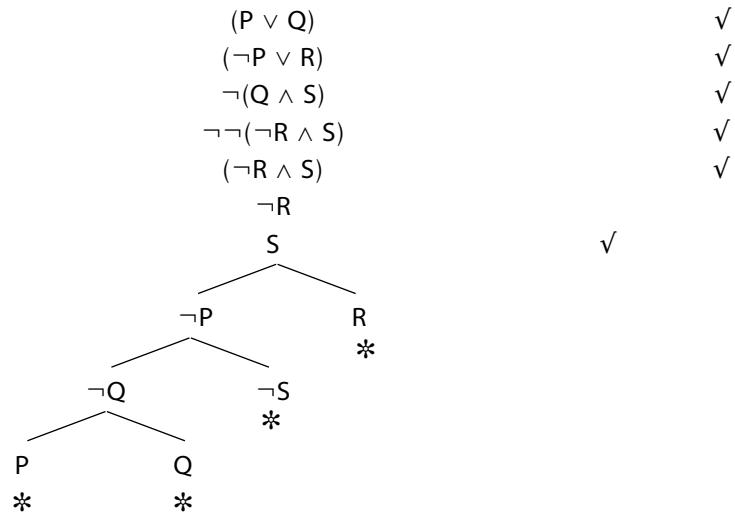
which, as is easily checked, all make the premisses true and conclusion false, so the inference is indeed invalid.

9. $(P \wedge (Q \vee R)) \therefore ((P \wedge Q) \vee (P \wedge R))$



Valid again!

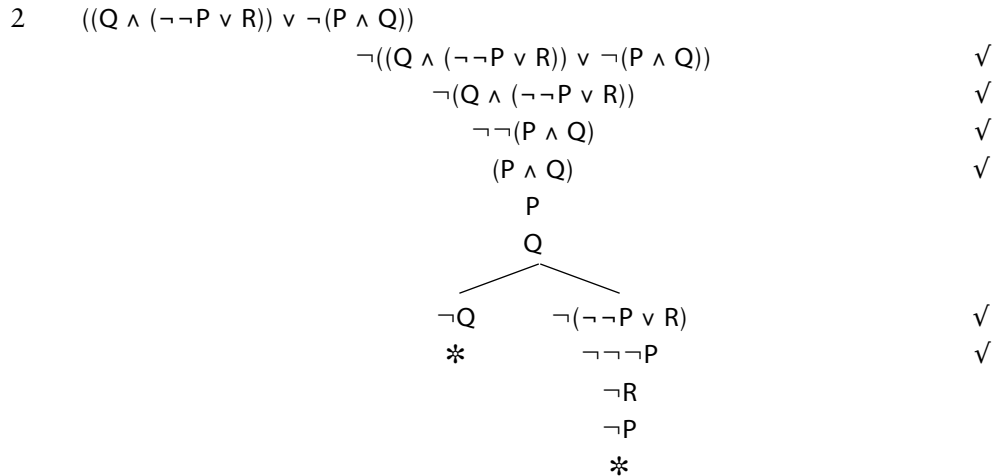
10. $(P \vee Q), (\neg P \vee R), \neg(Q \wedge S) \therefore \neg(\neg R \wedge S)$



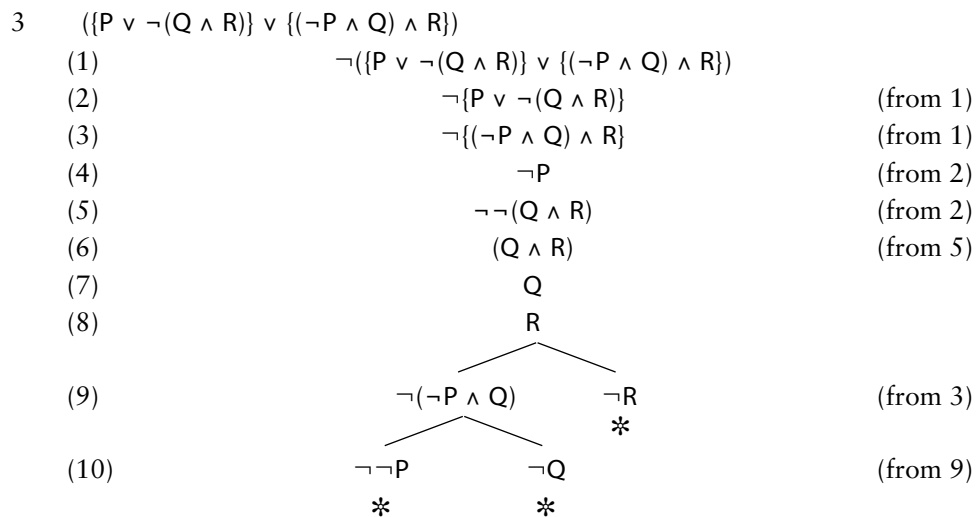
Valid – and just compare the brevity of that with the following cut-down, minimal truth table!!

P	Q	R	S	(P ∨ Q)	(¬P ∨ R)	¬(Q ∧ S)	¬(¬R ∧ S)
T	T	T	T				T
T	T	T	F				T
T	T	F	T	T	F		F
T	T	F	F				T
T	F	T	T				T
T	F	T	F				T
T	F	F	T	T	F		F
T	F	F	F				T
F	T	T	T				T
F	T	T	F				T
F	T	F	T	T	T	F	F
F	T	F	F				T
F	F	T	T				T
F	F	T	F				T
F	F	F	T	F			F
F	F	F	F				T
				3	2	4	1

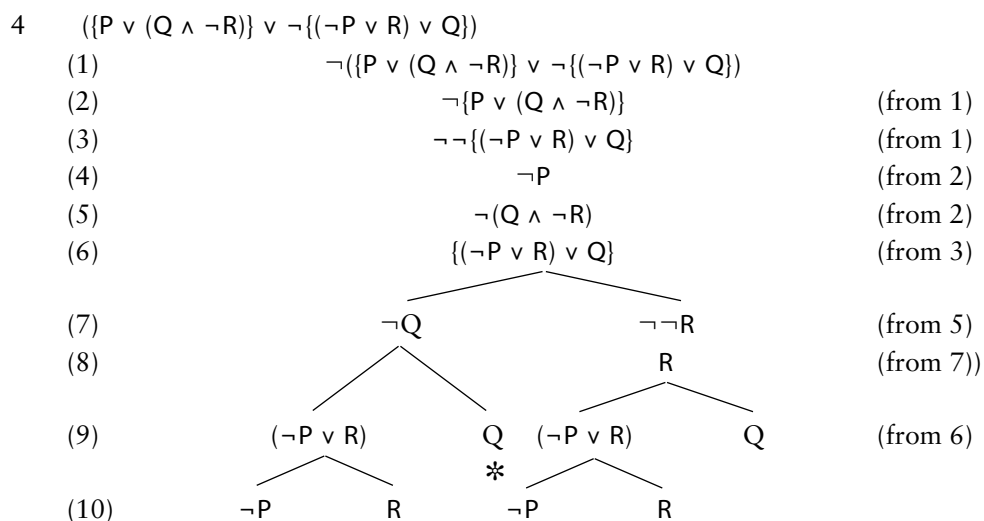
B Which of the following are tautologies?



We negated the wff to be tested for being a tautology, and use that wff to start the tree. We then apply the ‘straight’ rules first, and keep going ...

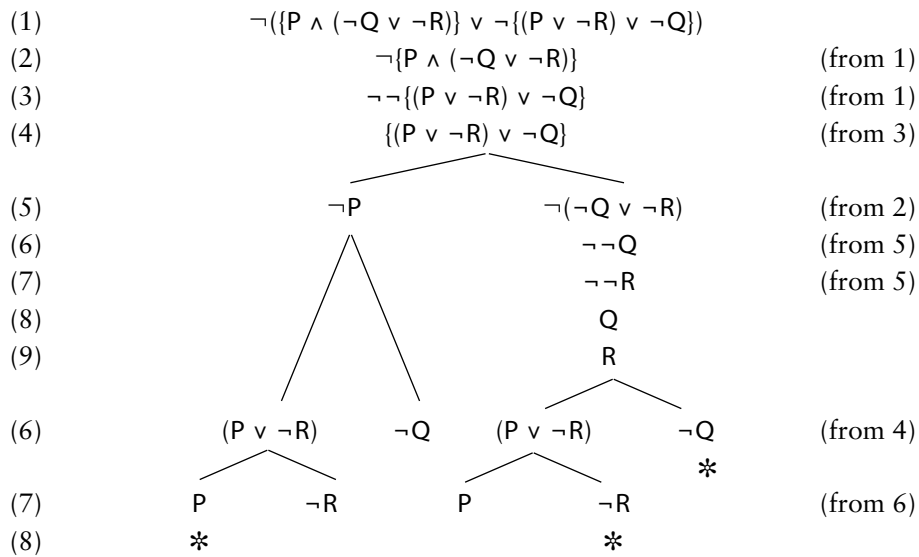


A tautology again.



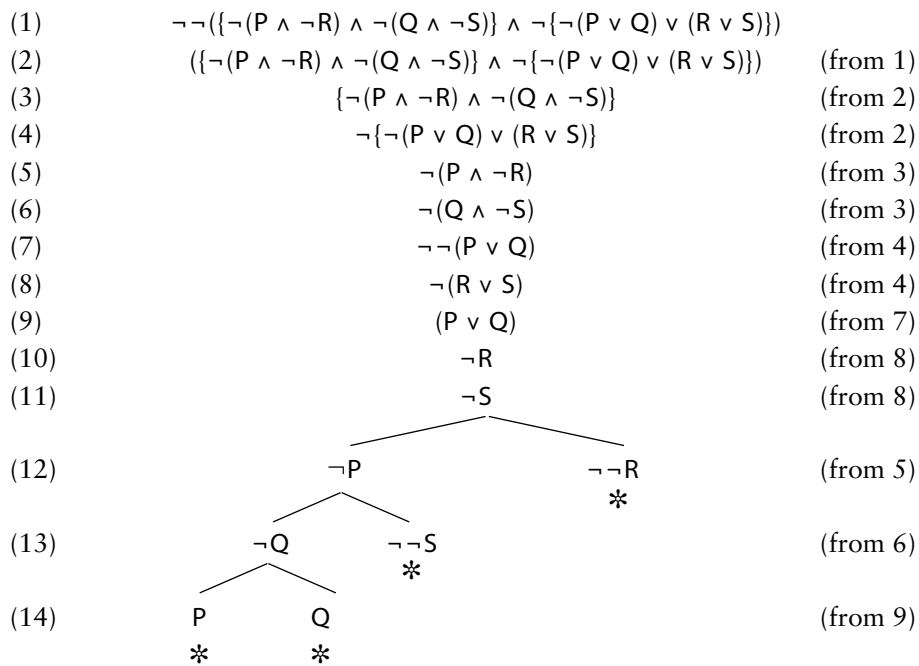
Lots of open branches – not a tautology!

5 $((P \wedge (\neg Q \vee \neg R)) \vee \neg((P \vee \neg R) \vee \neg Q))$



Lots of open branches – not a tautology!

6 $\neg(\neg(P \wedge \neg R) \wedge \neg(Q \wedge \neg S)) \wedge \neg(\neg(P \vee Q) \vee (R \vee S))$

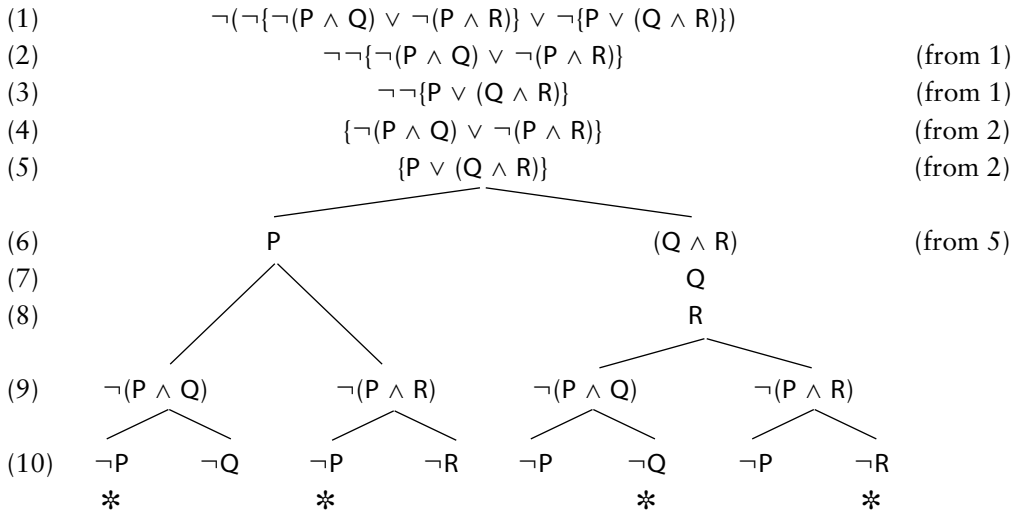


Tautology!

4. $\vDash (\neg(\neg(P \wedge Q) \vee \neg(P \wedge R)) \vee \neg(P \vee (Q \wedge R)))$

Is this wff a tautology? It is helpful to begin by bracket the wff more transparently. Rewrite it as

$$(\neg\{\neg(P \wedge Q) \vee \neg(P \wedge R)\} \vee \neg\{P \vee (Q \wedge R)\})$$



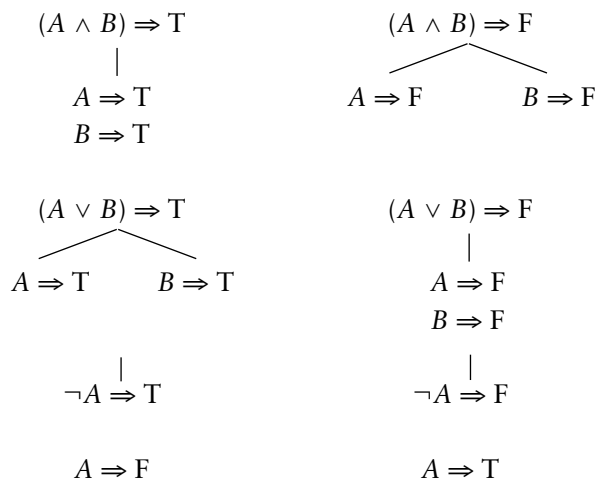
So tautological entailment claim is false.

5. $((P \wedge Q) \vee (R \wedge P)), \neg(\neg S \wedge P), \neg(Q \wedge R) \vDash (Q \wedge S)$

Another boring sprawling tree! I fear that the third premiss should have been $\neg(\neg Q \wedge R)$, then the tree would have closed.

D How would the rules and principles of tree-building need to be re-written if we had stuck to using *signed* trees, where wffs are explicitly assigned T or F?

We need ‘unpacking’ rules for conjunctions, disjunctions, and negations assigned ‘T’, and for conjunctions, disjunctions and negations assigned ‘F’ (and these will give us rules to cope with every non-atomic wff):



And the tree building rules are as before, except that we start the tree with the premisses assigned true and conclusion false; and close a branch if contains a pair $A \Rightarrow T, A \Rightarrow F$.