

Question

In addition to the properties of relations already defined, we say that a relation R is *Euclidean* just if, whenever a has R to b , and a has R to c , then b has R to c . R is *asymmetric* if, whenever a has R to b , then b does *not* have R to a . R is *irreflexive* if no object has R to itself. These properties, along with the properties of being *reflexive*, *symmetric* and *transitive*, can all be captured by QL wffs. For example, R 's being symmetric is captured by

$$\forall x \forall y (Rxy \supset Ryx)$$

Give wffs that similarly capture R 's having each of the other five properties. Then give both informal arguments and QL trees to show

1. If R is asymmetric, it is irreflexive.
2. If R is transitive and irreflexive, it is asymmetric.
3. If R is an equivalence relation, it is Euclidean.
4. If R is Euclidean and reflexive, it is an equivalence relation.

What about

5. If R is transitive and symmetric, it is reflexive?

Answer

Translations:

- R is symmetric: $\forall x \forall y (Rxy \supset Ryx)$
- R is asymmetric: $\forall x \forall y (Rxy \supset \neg Ryx)$
- R is reflexive: $\forall x Rxx$
- R is irreflexive: $\forall x \neg Rxx$
- R is transitive: $\forall x \forall y \forall z ((Rxy \wedge Ryz) \supset Rxz)$
- R is Euclidean: $\forall x \forall y \forall z ((Rxy \wedge Rxz) \supset Ryz)$

1. To show that *if R is asymmetric, it is irreflexive*: By definition, R is *asymmetric* if, for any a and b , whenever a has R to b , then b does *not* have R to a . Take the particular case where a is b . Then in particular, if a has R to a , then a does not have R to a – hence (by the principle that *if P then not- P implies not- P*) a does not have R to a , and R is irreflexive.

Tree version:

(1)	$\forall x \forall y (Rxy \supset \neg Ryx)$		Asymmetry assumption
(2)	$\neg \forall x \neg Rxx$	✓	Neg. conclusion, irreflexivity
(3)	$\exists x \neg \neg Rxx$	✓	From 2, by $\neg \forall$ rule
(4)	$\neg \neg Raa$	✓	Instantiate with new name
(5)	Raa	✓	Remove double negation
(6)	$\forall y (Ray \supset \neg Rya)$		From 1, instantiate $\forall x$
(7)	$(Raa \supset \neg Raa)$	✓	From 6, instantiate $\forall y$
<div style="display: flex; justify-content: space-around; align-items: center;"> <div style="text-align: center;"> $\neg Raa$ *</div> </div> <div style="display: flex; justify-content: space-between; width: 100%; border-top: 1px solid black; margin: 0 10px;"> <div style="width: 45%;"></div> <div style="width: 10%;"></div> <div style="width: 45%;"></div> </div> <div style="text-align: center;"> $\neg Raa$ *</div>			

2. To show that *if R is transitive and irreflexive, it is asymmetric*. If R is transitive, then if $a R b$ and $b R c$, then $a R c$. So in particular, if $a R b$ and $b R a$, then $a R a$. Hence, if not- $(a R a)$, then not- $(a R b$ and $b R a)$. In other words, if R is irreflexive, then if $a R b$ then not- $(b R a)$, i.e. R is asymmetric.

Tree version:

(1)	$\forall x\forall y\forall z((Rxy \wedge Ryz) \supset Rxz)$		Transitivity
(2)	$\forall x\neg Rxx$		Irreflexivity
(3)	$\neg\forall x\forall y(Rxy \supset \neg Ryx)$	✓	Neg. concl.: Assymmetry
(4)	$\exists x\neg\forall y(Rxy \supset \neg Ryx)$	✓	From 3, $y \neg\forall$ rule
(5)	$\neg\forall y(Ray \supset \neg Rya)$	✓	From 4, instantiating $\exists x$
(6)	$\exists y\neg(Ray \supset \neg Rya)$	✓	From 5, by $\neg\forall$ rule
(7)	$\neg(Rab \supset \neg Rba)$	✓	From 6, instantiating $\exists y$
(8)	Rab		Unpacking the neg. \supset
(9)	$\neg\neg Rba$	✓	
(10)	Rba		From 9

So far, so automatic! There is nothing else to do at steps (3) to (7) if we stick to the basic principle of not instantiating universals with new names. Now we do instantiate (1) and (2) in the obvious ways.

(11)	$\forall y\forall z((Ray \wedge Ryz) \supset Raz)$		From 1, instantiating $\forall x$
(12)	$\forall z((Rab \wedge Rbz) \supset Raz)$		From 12, instantiating $\forall y$
(13)	$((Rab \wedge Rba) \supset Raa)$	✓	From 13, instantiating $\forall z$
(14)	$\neg Raa$		From 2, instantiating $\forall x$

(15)	$\neg(Rab \wedge Rba)$	✓	Raa *
(16)	$\neg Rab$	*	$\neg Rba$ *

3. A relation is an equivalence relation if it is transitive, symmetric and reflexive. Suppose R is an equivalence relation; and suppose that $a R b$ and $a R c$. Since R is symmetric, it follows that $b R a$. And from $b R a$ and $a R c$, transitivity yields $b R c$. So we've shown that if $a R b$ and $a R c$ then $b R c$. So R is Euclidean.

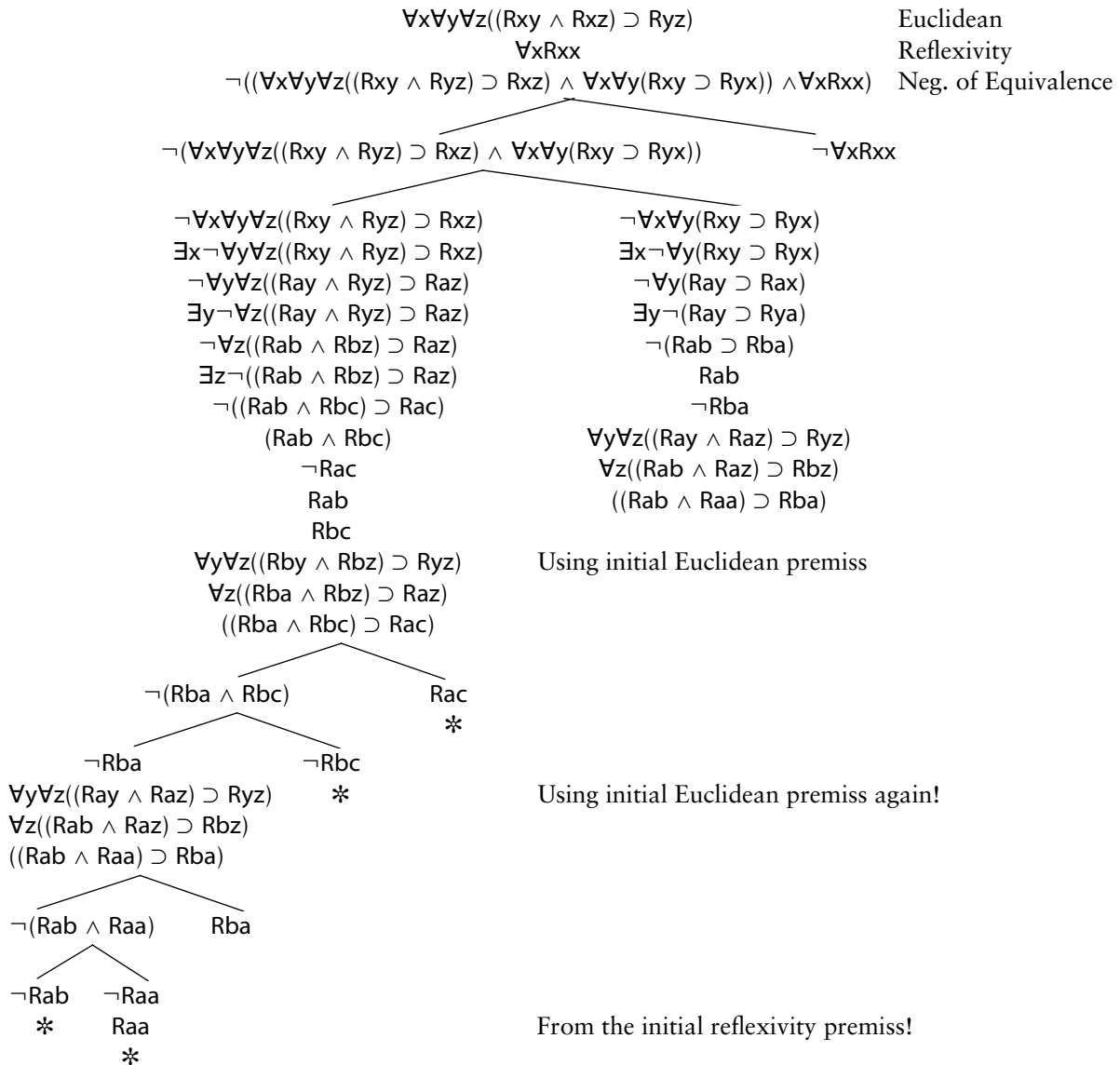
	$\forall x\forall y\forall z((Rxy \wedge Ryz) \supset Rxz)$		Transitivity
	$\forall x\forall y(Rxy \supset Ryx)$		Symmetry
	$\neg\forall x\forall y\forall z((Rxy \wedge Rxz) \supset Ryz)$		Neg. Euclidean
	$\exists x\neg\forall y\forall z((Rxy \wedge Rxz) \supset Ryz)$		By $\neg\forall$ rule
	$\neg\forall y\forall z((Ray \wedge Raz) \supset Ryz)$		Instantiating $\exists x$
	$\exists y\neg\forall z((Ray \wedge Raz) \supset Ryz)$		Etc.
	$\neg\forall z((Rab \wedge Raz) \supset Rbz)$		
	$\exists z\neg((Rab \wedge Raz) \supset Rbz)$		
	$\neg((Rab \wedge Rac) \supset Rbc)$		
	$(Rab \wedge Rac)$		Unpacking $\neg\supset$
	$\neg Rbc$		
	Rab		
	Rac		
	$\forall y\forall z((Rby \wedge Ryz) \supset Rbz)$		Instantiating $\forall x$
	$\forall z((Rba \wedge Raz) \supset Rbz)$		Instantiating $\forall y$
	$((Rba \wedge Rac) \supset Rbc)$		Instantiating $\forall z$

	$\neg(Rba \wedge Rac)$		Rbc *
	$\neg Rba$		$\neg Rac$ *
	$\forall y(Ray \supset Rya)$		Instantiating $\forall x$ in Symmetry
	$(Rab \supset Rba)$		Instantiating $\forall y$
	$\neg Rab$	*	Rba *

(Now insert the check marks in the proper places!)

4. We want to show: if R is Euclidean and reflexive, it is an equivalence relation. (1) Being Euclidean, if $a R b$ and $a R c$, then $b R c$; so in particular, if $a R b$, and $a R a$, then $b R a$ – so R is symmetric. (2) Now suppose R is Euclidean and reflexive and therefore symmetric. Then if $a R b$ and $b R c$, then (by symmetry) $b R a$ and $b R c$, so (by Euclidean property) $a R c$ – hence R is transitive. (3) In sum, if R is Euclidean and reflexive then R is reflexive, symmetric and transitive, so is an equivalence relation.

For the tree (again, left to you to insert check marks!):



The complication here is in the way we choose to instantiate the variables in the Euclidean premiss. Our choice the first time is designed to deliver quickly closing branches (so ‘Ryz’ needs to become ‘Rac’ to give a contradiction with ‘ $\neg Ra$ ’).

5. What about if R is transitive and symmetric, it is reflexive?

If R is transitive, then if $a R b$ and $b R c$, then $a R c$. So in particular, if $a R b$ and $b R a$, then $a R a$. But if R is symmetric, if $a R b$, then $a R b$ and $b R a$. So – putting those together – if $a R b$, then if $a R a$. That tells us that if R relates a to anything in the domain, then it relates a to itself. But maybe R relates nothing in the domain of discourse!

Take an extreme example: every a, b, c is such that IF a is the same god as b , and b is the same god as c , then a is the same god as c (transitivity), and IF a is the same god as b , then b is the same god as a (symmetry). But it doesn't follow that every a is the same god as itself (for that can only be true if a is a god, and there may be none).

Consider now the tree:

$\forall x \forall y \forall z ((Rxy \wedge Ryz) \supset Rxz)$	Transitivity
$\forall x \forall y (Rxy \supset Ryx)$	Symmetry
$\neg \forall x Rxx$	Neg. conclusion, Reflexivity
$\exists x \neg Rxx$	
$\neg Raa$	

Now, we only have the name 'a' in play. If we instantiate the universal quantifications using the only name we have in play every time, we'll get ...

$$\begin{aligned} & ((Raa \wedge Raa) \supset Raa) \\ & (Raa \supset Raa) \end{aligned}$$

Press on and the tree won't close!

On the other hand the tree for

$\forall x \forall y \forall z ((Rxy \wedge Ryz) \supset Rxz)$	Transitivity
$\forall x \forall y (Rxy \supset Ryx)$	Symmetry
$\forall x \exists y Rxy$	R relates everything to something
$\neg \forall x Rxx$	Neg. conclusion, Reflexivity

Does close:

$\exists x \neg Rxx$	By $\neg \forall$ rule
$\neg Raa$	
$\exists y Ray$	Instantiating new premiss
Rab	
$\forall y \forall z ((Ray \wedge Ryz) \supset Raz)$	
$\forall z ((Rab \wedge Rbz) \supset Raz)$	
$((Rab \wedge Rba) \supset Raa)$	
$\forall y (Ray \supset Rya)$	
$(Rab \supset Rba)$	

And the rest is easy!