

A Use  $QL^=$  trees to show the following inferences are valid:

1. Jack (m) is Fingers (n). Fingers is never caught (F). Whoever is never caught (F) escapes justice (G). So Jack escapes justice.

(1)	$m = n$		
(2)	$F_n$		
(3)	$\forall x(Fx \supset Gx)$		
(4)	$\neg G_m$	Negated conclusion	
(5)	$(F_n \supset G_n)$	$\checkmark$	
$\swarrow$			
(6)	$\neg F_n$	$G_n$	
(7)	*	$G_m$	By LL, from 1, 6
		*	

where LL is, of course, Leibniz's Law. Check that you understand why there is just the single check mark here!

2. There is a wise philosopher. There is a philosopher who isn't wise. So there are at least two philosophers.

We could choose the domain of quantification to be philosophers. Then the translation would be simply:

$$\exists xFx, \exists x\neg Fx \therefore \exists x\exists y\neg x = y$$

If we'd chosen our usual domain of humans, we'd need to carve out the philosophers by some predicate, as in:

$$\exists x(Gx \wedge Fx), \exists x(Gx \wedge \neg Fx) \therefore \exists x\exists y((Gx \wedge Gy) \wedge \neg x = y)$$

Here's the tree for the first, simpler translation:

(1)	$\exists xFx$		
(2)	$\exists x\neg Fx$		
(3)	$\neg\exists x\exists y\neg x = y$	Negated conclusion	

We follow the usual rule-of-thumb, deal with negated quantifiers first, then instantiate the existentials (each with a new name of course) – thereby now checking each of (1) to (3):

(4)	$\forall x\neg\exists y\neg x = y$	
(5)	$F_a$	
(6)	$\neg F_b$	

Now instantiate the universal quantification of (4)

(7)	$\neg\exists y\neg a = y$	$\checkmark$	
(8)	$\forall y\neg a = y$	From 7	
(9)	$\neg a = b$	$\checkmark$	From 8
(10)	$a = b$		
(11)	$F_b$	From 5, 10 by LL	
	*		

The tree for the more complex translation involves basically the same moves, but with a little more detail added ...:

(1)	$\exists x(Gx \wedge Fx)$	$\checkmark$	
(2)	$\exists x(Gx \wedge \neg Fx)$	$\checkmark$	
(3)	$\neg\exists x\exists y((Gx \wedge Gy) \wedge \neg x = y)$	$\checkmark$	Negated conclusion
(4)	$\forall x\neg\exists y((Gx \wedge Gy) \wedge \neg x = y)$		
(5)	$(G_a \wedge F_a)$	$\checkmark$	
(6)	$(G_b \wedge \neg F_b)$	$\checkmark$	
(7)	$G_a$		



- (6)  $\neg \exists y (\neg a = y \wedge Rya)$  From 5
- (7)  $\forall y \neg (\neg a = y \wedge Rya)$  From 6
- (8)  $\exists y Rya$  From 1
- (9)  $\neg Raa$  From 4

This again leaves us with two universals and an existential quantification as unchecked propositions. So again we instantiate the existential first to give ...

- (10)  $Rba$  From 8

We have a new name to play with and the obvious next move is to instantiate (7) with it to get

- (11)  $\neg (\neg a = b \wedge Rba)$  From 7
- |      |                   |            |                   |
|------|-------------------|------------|-------------------|
|      | $\neg \neg a = b$ | $\neg Rba$ |                   |
| (12) | $a = b$           | *          | From 12           |
| (13) | $Raa$             |            | From 10, 13 by LL |
| (14) | *                 |            |                   |

- 5. *No one who isn't Bryn loves Angharad. At least one person loves Angharad. So Bryn loves Angharad.*

We can just use 'Fx' for 'x loves Angharad' (as the internal structure of that predicate isn't relevant to the argument):

- (1)  $\forall x (\neg x = b \supset \neg Fx)$
  - (2)  $\exists x Fx$  ✓
  - (3)  $\neg Fb$  Negated conclusion
  - (4)  $Fa$  From 2
  - (5)  $(\neg a = b \supset \neg Fa)$  From 1
- |     |                   |           |  |
|-----|-------------------|-----------|--|
|     | $\neg \neg a = b$ | $\neg Fa$ |  |
| (6) | $a = b$           | *         |  |
| (7) | $Fa$              |           |  |
| (8) | *                 |           |  |

- 6. *Exactly one person admires Frank. All and only those who admire Frank love him. Hence exactly one person loves Frank.*

Use 'F' for 'admires Frank', and 'G' for 'loves Frank'. 'All and only Fs are Gs' can be translated as ' $\forall x (Fx \equiv Gx)$ ' or equivalently  $\forall x ((Fx \supset Gx) \wedge (Gx \supset Fx))$ .

- (1)  $\exists x (Fx \wedge \forall y (Fy \supset y = x))$
- (2)  $\forall x (Fx \equiv Gx)$
- (3)  $\neg \exists x (Gx \wedge \forall y (Gy \supset y = x))$  ✓ Negated conclusion
- (4)  $\forall x \neg (Gx \wedge \forall y (Gy \supset y = x))$  From 3

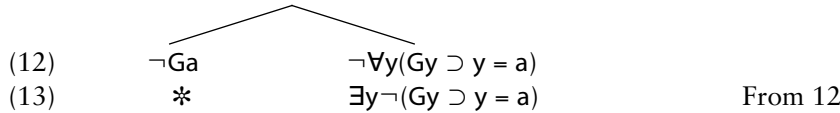
Once again, this leaves us with two universals and an existential quantification as unchecked propositions. So again we instantiate the existential first to give ...

- (5)  $(Fa \wedge \forall y (Fy \supset y = a))$  From 1
- (6)  $(Fa \equiv Ga)$  From 2
- (7)  $\neg (Ga \wedge \forall y (Gy \supset y = a))$  From 4

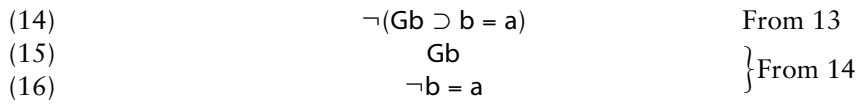
Next, on the principle 'apply non-branching connective rules first', we must next check off (5) then (6):

- (8)  $Fa$
  - (9)  $\forall y (Fy \supset y = a)$
- |      |      |           |          |
|------|------|-----------|----------|
|      | $Fa$ | $\neg Fa$ | } From 6 |
| (10) | $Ga$ | $\neg Ga$ |          |
| (11) |      | *         |          |

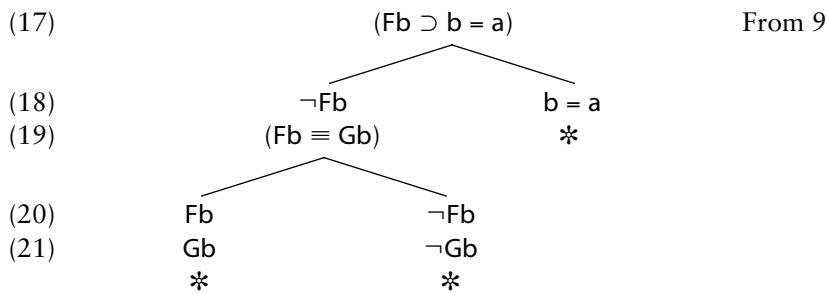
Now we continue the left-hand branch by unpacking (7):



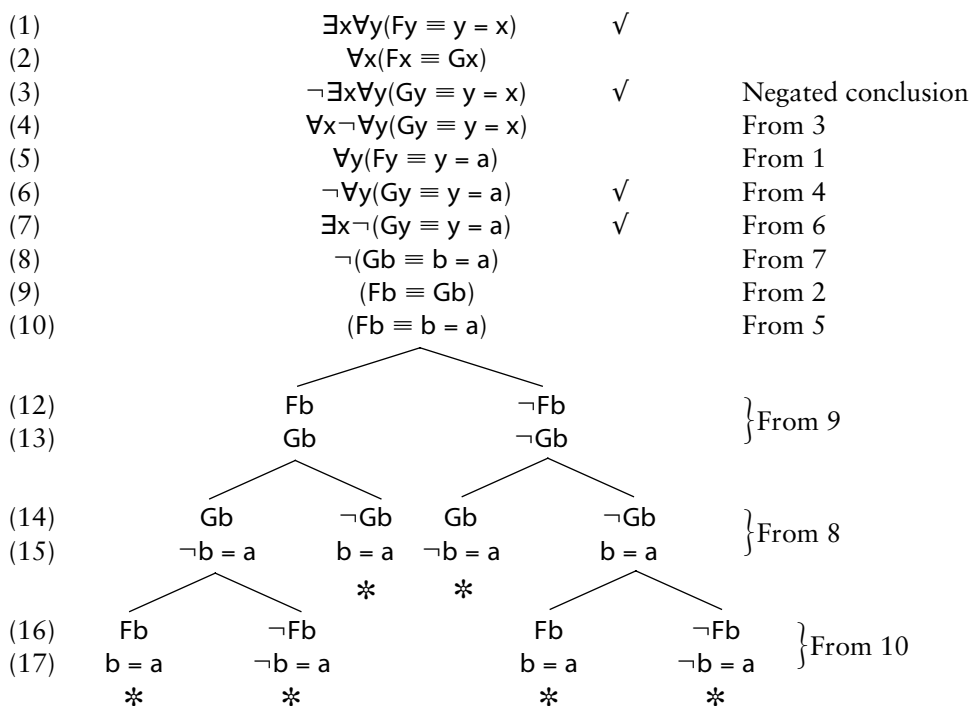
Next, to make use of (13) we instantiate the quantifier with a new name



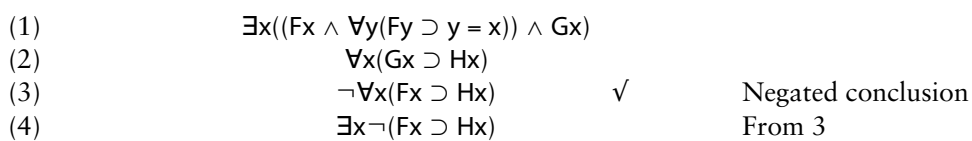
We have two universal quantifications, at (9) and (2) to instantiate with the new name 'b'.



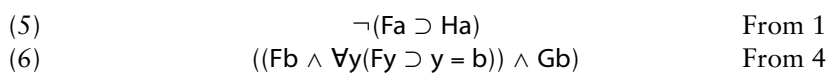
Here, without much commentary, is another tree for an alternative translation of the original argument:



7. *The present King of France is bald. Bald men are sexy. Hence whoever is a present King of France is sexy.*



We now have two existential quantifications to instantiate, with different names of course:



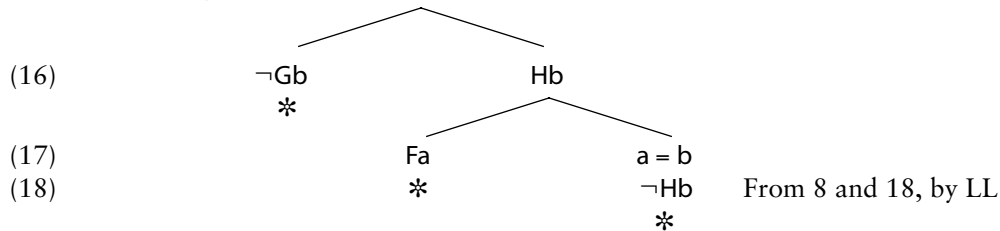
We can now disassemble (5) and (6) and check them off to get ...

- (7)  $Fa$
- (8)  $\neg Ha$
- (9)  $(Fb \wedge \forall y(Fy \supset y = b))$   $\checkmark$
- (10)  $Gb$
- (11)  $Fb$
- (12)  $\forall y(Fy \supset y = b)$

We have two universal quantifications to use, at lines (2) and (12): the obvious instantiations to make are ...

- (13)  $(Gb \supset Hb)$
- (14)  $(Fa \supset a = b)$

And the rest is straightforward:



Or, starting from an alternative translation for *The present King of France is bald* ...

- (1)  $\exists x \forall y ((Fy \equiv y = x) \wedge Gx)$   $\checkmark$
  - (2)  $\forall x (Gx \supset Hx)$
  - (3)  $\neg \forall x (Fx \supset Hx)$   $\checkmark$  Negated conclusion
  - (4)  $\exists x \neg (Fx \supset Hx)$   $\checkmark$  From 3
  - (5)  $\neg (Fa \supset Ha)$   $\checkmark$  From 1
  - (6)  $\forall y ((Fy \equiv y = b) \wedge Gb)$  From 4
  - (7)  $((Fa \equiv a = b) \wedge Gb)$   $\checkmark$  From 6
  - (8)  $(Gb \supset Hb)$   $\checkmark$  From 2
  - (9)  $(Fa \equiv a = b)$   $\}$  From 7
  - (10)  $Gb$
  - (11)  $Fa$   $\}$  From 5
  - (12)  $\neg Ha$
- |      |                          |                          |                   |
|------|--------------------------|--------------------------|-------------------|
|      | $\swarrow$<br>$\searrow$ |                          |                   |
| (13) | $\neg Gb$                | $Hb$                     |                   |
|      | *                        |                          |                   |
|      |                          | $\swarrow$<br>$\searrow$ |                   |
| (14) | $Fa$                     | $\neg Fa$                | $\}$ From 9       |
| (15) | $a = b$                  | $\neg a = b$             |                   |
| (16) | $\neg Hb$                | *                        | From 12, 15 by LL |

8. *Angharad loves only Bryn and Caradoc, who are different people. So Angharad loves exactly two people.*

Let's use 'Fx' for 'Angharad loves x', then we have

- (1)  $\{((Fb \wedge Fc) \wedge \neg b = c) \wedge \forall x(Fx \supset (x = b \vee x = c))\}$
  - (2)  $\neg \exists y \exists z \{((Fy \wedge Fz) \wedge \neg y = z) \wedge \forall x(Fx \supset (x = y \vee x = z))\}$   $\checkmark$  Negated conclusion
  - (3)  $\forall y \neg \exists z \{((Fy \wedge Fz) \wedge \neg y = z) \wedge \forall x(Fx \supset (x = y \vee x = z))\}$  From 2
  - (4)  $\neg \exists z \{((Fb \wedge Fz) \wedge \neg b = z) \wedge \forall x(Fx \supset (x = b \vee x = z))\}$   $\checkmark$  From 3
  - (5)  $\forall z \neg \{((Fb \wedge Fz) \wedge \neg b = z) \wedge \forall x(Fx \supset (x = b \vee x = z))\}$  From 4
  - (6)  $\neg \{((Fb \wedge Fc) \wedge \neg b = c) \wedge \forall x(Fx \supset (x = b \vee x = c))\}$  From 5
- \*

This closes particularly quickly because of a cunning choice of variables in our initial translation. Suppose we had started with the translations

- (1)  $\{((Fb \wedge Fc) \wedge \neg b = c) \wedge \forall x(Fx \supset (x = b \vee x = c))\}$
- (2)  $\neg \exists x \exists y \{((Fx \wedge Fy) \wedge \neg x = y) \wedge \forall z(Fz \supset (z = x \vee z = y))\}$  ✓ Negated conclusion
- (3)  $\forall x \neg \exists y \{((Fx \wedge Fy) \wedge \neg x = y) \wedge \forall z(Fz \supset (z = x \vee z = y))\}$  From 2
- (4)  $\neg \exists y \{((Fb \wedge Fy) \wedge \neg b = y) \wedge \forall z(Fz \supset (z = b \vee z = y))\}$  ✓ From 3
- (5)  $\forall y \neg \{((Fb \wedge Fy) \wedge \neg b = y) \wedge \forall z(Fz \supset (z = b \vee z = y))\}$  From 4
- (6)  $\neg \{((Fb \wedge Fc) \wedge \neg b = c) \wedge \forall z(Fz \supset (z = b \vee z = c))\}$  From 5

And now we can't immediately close the tree as (1) and (6) aren't literal contradictions. So we now check off (1) and (6) and unpack them to get

- (7)  $((Fb \wedge Fc) \wedge \neg b = c)$  From 1
  - (8)  $\forall x(Fx \supset (x = b \vee x = c))$  From 1
- 
- (9)  $\neg((Fb \wedge Fc) \wedge \neg b = c)$  From 6
  - (10)  $\neg \forall z(Fz \supset (z = b \vee z = c))$  ✓ From 6
  - (11)  $\exists z \neg(Fz \supset (z = b \vee z = c))$  ✓ From 9
  - (12)  $\neg(Fa \supset (a = b \vee a = c))$  From 10
  - (13)  $(Fa \supset (a = b \vee a = c))$  From 8

9. *Juliet kisses exactly two philosophers. Hence there is more than one philosopher.*

Let's use 'Fx' for 'Juliet kisses x' and take the domain to be philosophers. Then we get the simple translation

- (1)  $\exists x \exists y \{((Fx \wedge Fy) \wedge \neg x = y) \wedge \forall z(Fz \supset (z = x \vee z = y))\}$
- (2)  $\neg \exists x \exists y \neg x = y$  Negated conclusion

Instantiating (1) and then instantiating the result gives us

- (3)  $\exists y \{((Fa \wedge Fy) \wedge \neg a = y) \wedge \forall z(Fz \supset (z = a \vee z = y))\}$
- (4)  $\{((Fa \wedge Fb) \wedge \neg a = b) \wedge \forall z(Fz \supset (z = a \vee z = b))\}$

Unpacking the conjunctions in (4) gives us

- (5)  $((Fa \wedge Fb) \wedge \neg a = b)$
- (6)  $\forall z(Fz \supset (z = a \vee z = b))$
- (7)  $(Fa \wedge Fb)$
- (8)  $\neg a = b$

Now we can quickly expose a contradiction between (8) and (2): for from (2) we get ...

- (9)  $\forall x \neg \exists y \neg x = y$
- (10)  $\neg \exists y \neg a = y$
- (11)  $\forall y \neg \neg a = y$
- (12)  $\neg \neg a = b$

Note we couldn't have used 'Kjx' for 'Juliet kisses x' and still taken the domain to be philosophers. For any name must name something in the domain, and we aren't given that Juliet is a philosopher! So we'd have to take a wider domain and then explicitly pick out the philosophers by a predicate (say 'G'), as in ...

- (1)  $\exists x \exists y \{((Ljx \wedge Gx) \wedge [Ljy \wedge Gy]) \wedge \neg x = y) \wedge \forall z([Ljz \wedge Gz] \supset (z = x \vee z = y))\}$

and everything gets more longwinded.

**B** Show that the following wffs are q-logical truths

1.  $\forall x \forall y (x = y \supset (Fx \supset Fy))$
2.  $\forall y \forall z (y = z \supset (\forall x (Lxy \wedge Fy) \supset \forall x (Lxz \wedge Fz)))$  [final closing bracket added!]

Thinking about the structure of those proofs, conclude that each way of filling out the following schema from §33.1 yields a q-logical truth:

$$(LS) \quad \forall v \forall w (v = w \supset (C(\dots v \dots v \dots) \supset C(\dots w \dots w \dots)))$$

We establish something is a q-logical truth using trees by assuming its negation and proving a contradiction, thus ...

- |      |  |                 |
|------|--|-----------------|
| (1)  | $\neg \forall x \forall y (x = y \supset (Fx \supset Fy))$ | ✓               |
| (2)  | $\exists x \neg \forall y (x = y \supset (Fx \supset Fy))$ | ✓               |
| (3)  | $\neg \forall y (a = y \supset (Fa \supset Fy))$           | ✓               |
| (4)  | $\exists y \neg (a = y \supset (Fa \supset Fy))$           | ✓               |
| (5)  | $\neg (a = b \supset (Fa \supset Fb))$                     | ✓               |
| (6)  | $a = b$  |                 |
| (7)  | $\neg (Fa \supset Fb)$                                     | ✓               |
| (8)  | $Fa$   |                 |
| (9)  | $\neg Fb$  |                 |
| (10) | $Fb$   | From 6, 8 by LL |
|      | *  |                 |

- |      |  |                 |
|------|--|-----------------|
| (1)  | $\neg \forall y \forall z (y = z \supset (\forall x (Lxy \wedge Fy) \supset \forall x (Lxz \wedge Fz)))$ | ✓               |
| (2)  | $\exists y \neg \forall z (y = z \supset (\forall x (Lxy \wedge Fy) \supset \forall x (Lxz \wedge Fz)))$ | ✓               |
| (3)  | $\neg \forall z (a = z \supset (\forall x (Lxa \wedge Fa) \supset \forall x (Lxz \wedge Fz)))$           | ✓               |
| (4)  | $\exists z \neg (a = z \supset (\forall x (Lxa \wedge Fa) \supset \forall x (Lxz \wedge Fz)))$           | ✓               |
| (5)  | $\neg (a = b \supset (\forall x (Lxa \wedge Fa) \supset \forall x (Lxb \wedge Fb)))$                     | ✓               |
| (6)  | $a = b$  |                 |
| (7)  | $\neg (\forall x (Lxa \wedge Fa) \supset \forall x (Lxb \wedge Fb))$                                     | ✓               |
| (8)  | $\forall x (Lxa \wedge Fa)$  |                 |
| (9)  | $\neg \forall x (Lxb \wedge Fb)$   |                 |
| (10) | $\forall x (Lxb \wedge Fb)$  | From 6, 8 by LL |
|      | *  |                 |

The common pattern here can be exposed like this ...

- |      |  |                 |
|------|--|-----------------|
| (1)  | $\neg \forall v \forall w (v = w \supset (C(\dots v \dots v \dots) \supset C(\dots w \dots w \dots)))$ | ✓               |
| (2)  | $\exists v \neg \forall w (v = w \supset (C(\dots v \dots v \dots) \supset C(\dots w \dots w \dots)))$ | ✓               |
| (3)  | $\neg \forall w (a = w \supset (C(\dots a \dots a \dots) \supset C(\dots w \dots w \dots)))$           | ✓               |
| (4)  | $\exists w \neg (a = w \supset (C(\dots a \dots a \dots) \supset C(\dots w \dots w \dots)))$           | ✓               |
| (5)  | $(a = b \supset (C(\dots a \dots a \dots) \supset C(\dots b \dots b \dots)))$                          | ✓               |
| (6)  | $a = b$  |                 |
| (7)  | $\neg (C(\dots a \dots a \dots) \supset C(\dots a \dots a \dots))$                                     | ✓               |
| (8)  | $C(\dots a \dots a \dots)$   |                 |
| (9)  | $\neg C(\dots a \dots a \dots)$  |                 |
| (10) | $C(\dots a \dots a \dots)$   | From 6, 8 by LL |
|      | *  |                 |

assuming ‘a’ and ‘b’ are new names, not occurring in the context C (if they do, we can use different names of course – and so in every case show an instance of (LS) is indeed a q-logical truth.



(9)	$\neg(Fb \supset Gb)$	$\checkmark$	
(10)	$Fb$		} From 9
(11)	$\neg Gb$		
(12)	$(Fb \supset b = a)$	$\checkmark$	From 8
$\swarrow$			
(13)	$\neg Fb$	$b = a$	From 12
(14)	*	$\neg Ga$	From 11, 13 by LL
		*	

4.  $\exists x \forall y ((Fy \equiv y = x) \wedge Gx)$  *q-entails*  $\exists x ((Fx \wedge \forall y (Fy \supset y = x)) \wedge Gx)$ .

(1)	$\exists x \forall y ((Fy \equiv y = x) \wedge Gx)$	$\checkmark$	
(2)	$\neg \exists x ((Fx \wedge \forall y (Fy \supset y = x)) \wedge Gx)$	$\checkmark$	Negation of conclusion
(3)	$\forall x \neg ((Fx \wedge \forall y (Fy \supset y = x)) \wedge Gx)$		
(4)	$\forall y ((Fy \equiv y = a) \wedge Ga)$		From 1
(5)	$\neg ((Fa \wedge \forall y (Fy \supset y = a)) \wedge Ga)$	$\checkmark$	From 3
(6)	$((Fa \equiv y = a) \wedge Ga)$	$\checkmark$	
(7)	$(Fa \equiv y = a)$		} From 6
(8)	$Ga$		
$\swarrow$			
(13)	$\neg (Fa \wedge \forall y (Fy \supset y = a))$	$\checkmark$	From 12
		$\neg Ga$	*
$\swarrow$			
(14)	$\neg Fa$	$\neg \forall y (Fy \supset y = a)$	$\checkmark$
(15)	*	$\exists y \neg (Fy \supset y = a)$	$\checkmark$
(16)		$\neg (Fb \supset b = a)$	$\checkmark$
(17)		$Fb$	} From 16
(18)		$\neg b = a$	

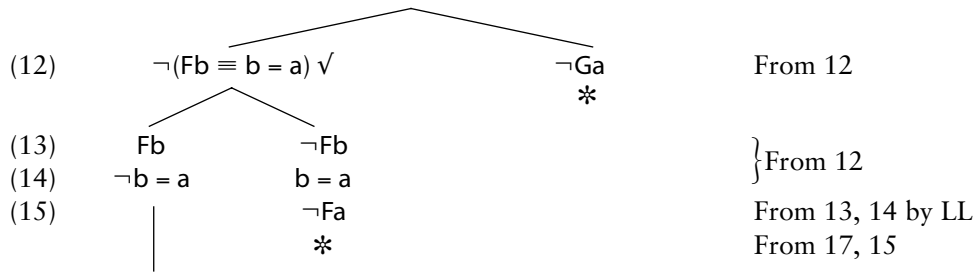
So far, that's pretty much all automatic. To finish off, we need more 'b'-involving info so we have to use one of our universally quantified wffs, and the obvious one to go for is (4):

(19)	$((Fb \equiv b = a) \wedge Ga)$	$\checkmark$	
(20)	$(Fb \equiv b = a)$		From 4
(21)	$Ga$		
$\swarrow$			
(22)	$Fb$	$\neg Fb$	} From 20
(23)	$b = a$	$\neg b = a$	
	*	*	

And finally ...

5.  $\exists x ((Fx \wedge \forall y (Fy \supset y = x)) \wedge Gx)$  *q-entails*  $\exists x \forall y ((Fy \equiv y = x) \wedge Gx)$ .

(1)	$\exists x ((Fx \wedge \forall y (Fy \supset y = x)) \wedge Gx)$	$\checkmark$	
(2)	$\neg \exists x \forall y ((Fy \equiv y = x) \wedge Gx)$	$\checkmark$	Negation of conclusion
(3)	$\forall x \neg \forall y ((Fy \equiv y = x) \wedge Gx)$		
(4)	$((Fa \wedge \forall y (Fy \supset y = a)) \wedge Ga)$	$\checkmark$	From 1
(5)	$(Fa \wedge \forall y (Fy \supset y = a))$	$\checkmark$	} From 4
(6)	$Ga$		
(7)	$Fa$		} From 5
(8)	$\forall y (Fy \supset y = a)$		
(9)	$\neg \forall y ((Fy \equiv y = a) \wedge Ga)$	$\checkmark$	From 3
(10)	$\exists y \neg ((Fy \equiv y = a) \wedge Ga)$	$\checkmark$	From 9
(11)	$\neg ((Fb \equiv b = a) \wedge Ga)$		From 10



To finish the left-most branch we need more more 'b'-involving info, so we have to use one of our universally quantified wffs, and this time nthe obvious one to go for is (8):

