Gödel's Theorems

Exercises: Enumerability and Countability

These exercises go much further than you need for understanding either Gödel's Theorems or *Gödel's Theorems*. But if you haven't come across the ideas of enumerabiity, etc., before then it will be worth pausing over them, as you really ought to know about them.

Read: *IGT* Ch. 2. For parallel reading, see Boolos, Burgess and Jeffrey, *Computability and Logic* (4th or 5th Edn), Ch. 1, which has further useful exercises on enumerability.

1. Define what it is for a set to be enumerable. Show that the following sets are enumerable:
   (a) The set of even numbers
   (b) The set of all unordered pairs of numbers
   (c) The set of rational numbers
   (d) The set of wffs of your favourite system of the propositional calculus
   (e) Any subset of an enumerable set
   (f) The union of any two enumerable sets

2. Two sets Δ and Γ are said to be *equinumerous* iff there is a one-one correspondence between them, i.e. there is some bijection $f : \Delta \to \Gamma$. Show that equinumerosity is an equivalence relation. In other words, writing '$\Delta \approx \Gamma$' for 'Δ is equinumerous to Γ', show that
   (a) $\Delta \approx \Delta$,
   (b) if $\Delta \approx \Gamma$ then $\Gamma \approx \Delta$,
   (c) if $\Delta \approx \Gamma$ and $\Gamma \approx \Theta$, then $\Delta \approx \Theta$.

3. Show that
   (a) a finite set of natural numbers can not be equinumerous with one of its proper subsets (i.e. with some some subset strictly contained in it);
   (b) an infinite set of natural numbers *can* be equinumerous with one of its proper subsets;
   (c) the set of natural numbers is equinumerous with the set of ordered pairs of natural numbers;
   (d) the set of natural numbers is equinumerous with the set of positive rational numbers;
   (e) the set of natural numbers is equinumerous with the set of ordered triples of natural numbers;
   (f) the set of natural numbers is equinumerous with the set containing all ordered pairs of natural numbers, triples of natural numbers, quadruples, quintuples, . . . , and n-tuples (for any finite n).

4. A set is *finite* iff it is either empty or equinumerous with the set $\{1, 2, \ldots, n\}$ for some n. A set is *countably infinite* iff it is equinumerous with the natural numbers. A set is *countable* iff it is either finite or countably infinite. Show that

   (a) all infinite sets of natural numbers are countable;

   (b) a set is countable iff it is equinumerous with a subset of the natural numbers. (Remember, as a limiting case, a set is a subset of itself!)

   What is the difference in the definitions of countability and enumerability (in the sense of *IGT*, p. 13)? Show that

   (c) if Δ is countable, it is enumerable;

   (d) if Δ is enumerable, it is countable.

5. The powerset $\mathcal{P}\Delta$ of Δ is the set of all subsets of Δ. Using Theorem 2.1 of *IGT* and the ensuing discussion, show that

   (a) the set of natural numbers is not equinumerous with the set of positive real numbers – in symbols, $\mathbb{N} \not\approx \mathbb{R}$;

   (b) the powerset of the natural numbers is equinumerous with the set of positive real numbers – in symbols, $\mathcal{P}\mathbb{N} \approx \mathbb{R}$;

   (c) for any Δ, $\Delta \prec \mathcal{P}\Delta$.

   (d) if Δ is countably infinite, then the set of *finite* subsets of Δ is countably infinite.

   (e) if Δ is countably infinite, then $\mathcal{P}\Delta$ is uncountably infinite.

6. An easy reality check: are countably infinite sets all equinumerous? are uncountably infinite sets all equinumerous?

7. Consider first this way of enumerating ordered pairs of the form $\langle m, n\rangle$:

$$
\begin{array}{ccccccc}
\langle 0,0\rangle & \rightarrow & \langle 0,1\rangle & & \langle 0,2\rangle & & \langle 0,3\rangle \ldots \\
 & \swarrow & & \swarrow & & \swarrow & \\
\langle 1,0\rangle & & \langle 1,1\rangle & & \langle 1,2\rangle & & \langle 1,3\rangle \ldots \\
 & \swarrow & & \swarrow & & & \\
\langle 2,0\rangle & & \langle 2,1\rangle & & \langle 2,2\rangle & & \langle 2,3\rangle \ldots \\
 & \swarrow & & & & & \\
\langle 3,0\rangle & & \langle 3,1\rangle & & \langle 3,2\rangle & & \langle 3,3\rangle \ldots \\
\vdots & & & & & &
\end{array}
$$

   Give a formula $f(m, n)$ which gives the position of the pair $\langle m, n\rangle$ in the list. (Pointless exercise for mathos: Now do the same for the different zig-zag ordering given on p. 16 of *IGT*.)

8. A trickier question for mathmos! We'll say that Σ is a *nested* family of sets if you can somehow order the sets in $\Sigma = \{\ldots, \Delta_\alpha, \ldots \Delta_\beta, \ldots\}$ so that if $\alpha < \beta$ then $\Delta_\alpha \subset \Delta_\beta$.

   Suppose then that the members of the nested family Σ are all subsets of some *countable* set Δ (i.e. for every α, $\Delta_\alpha \subseteq \Delta$ where Δ is countable). Must Σ be countable too?

   [For discussion of this, see Béla Bollobás, *The Art of Mathematics*, pp. 61–62.]