

Gödel's Theorems

Exercises: Enumerability and Countability

These exercises go much further than you need for understanding either Gödel's Theorems or *Gödel's Theorems*. But if you haven't come across the ideas of enumerability, etc., before then it will be worth pausing over them, as you really ought to know about them.

Read: *IGT* Ch. 2. For parallel reading, see Boolos, Burgess and Jeffrey, *Computability and Logic* (4th or 5th Edn), Ch. 1, which has further useful exercises on enumerability.

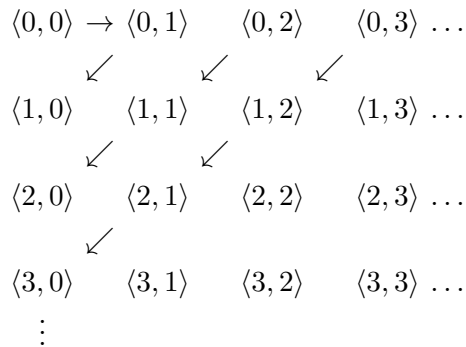
1. Define what it is for a set to be enumerable. Show that the following sets are enumerable:
 - (a) The set of even numbers
 - (b) The set of all unordered pairs of numbers
 - (c) The set of rational numbers
 - (d) The set of wffs of your favourite system of the propositional calculus
 - (e) Any subset of an enumerable set
 - (f) The union of any two enumerable sets
2. Two sets Δ and Γ are said to be *equinumerous* iff there is a one-one correspondence between them, i.e. there is some bijection $f : \Delta \rightarrow \Gamma$. Show that equinumerosity is an equivalence relation. In other words, writing ' $\Delta \approx \Gamma$ ' for ' Δ is equinumerous to Γ ', show that
 - (a) $\Delta \approx \Delta$,
 - (b) if $\Delta \approx \Gamma$ then $\Gamma \approx \Delta$,
 - (c) if $\Delta \approx \Gamma$ and $\Gamma \approx \Theta$, then $\Delta \approx \Theta$.
3. Show that
 - (a) a finite set of natural numbers can not be equinumerous with one of its proper subsets (i.e. with some subset strictly contained in it);
 - (b) an infinite set of natural numbers *can* be equinumerous with one of its proper subsets;
 - (c) the set of natural numbers is equinumerous with the set of ordered pairs of natural numbers;
 - (d) the set of natural numbers is equinumerous with the set of positive rational numbers;
 - (e) the set of natural numbers is equinumerous with the set of ordered triples of natural numbers;
 - (f) the set of natural numbers is equinumerous with the set containing all ordered pairs of natural numbers, triples of natural numbers, quadruples, quintuples, ..., and n -tuples (for any finite n).

4. A set is *finite* iff it is either empty or equinumerous with the set $\{1, 2, \dots, n\}$ for some n . A set is *countably infinite* iff it is equinumerous with the natural numbers. A set is *countable* iff it is either finite or countably infinite. Show that

- (a) all infinite sets of natural numbers are countable;
- (b) a set is countable iff it is equinumerous with a subset of the natural numbers. (Remember, as a limiting case, a set is a subset of itself!)

What is the difference in the definitions of countability and enumerability (in the sense of *IGT*, p. 13)? Show that

- (c) if Δ is countable, it is enumerable;
 - (d) if Δ is enumerable, it is countable.
5. The powerset $\mathcal{P}\Delta$ of Δ is the set of all subsets of Δ . Using Theorem 2.1 of *IGT* and the ensuing discussion, show that
- (a) the set of natural numbers is not equinumerous with the set of positive real numbers – in symbols, $\mathbb{N} \not\approx \mathbb{R}$;
 - (b) the powerset of the natural numbers is equinumerous with the set of positive real numbers – in symbols, $\mathcal{P}\mathbb{N} \approx \mathbb{R}$;
 - (c) for any Δ , $\Delta \prec \mathcal{P}\Delta$.
 - (d) if Δ is countably infinite, then the set of *finite* subsets of Δ is countably infinite.
 - (e) if Δ is countably infinite, then $\mathcal{P}\Delta$ is uncountably infinite.
6. An easy reality check: are countably infinite sets all equinumerous? are uncountably infinite sets all equinumerous?
7. Consider first this way of enumerating ordered pairs of the form $\langle m, n \rangle$:



Give a formula $f(m, n)$ which gives the position of the pair $\langle m, n \rangle$ in the list. (Pointless exercise for mathos: Now do the same for the different zig-zag ordering given on p. 16 of *IGT*.)

8. A trickier question for mathmos! We'll say that Σ is a *nested* family of sets if you can somehow order the sets in $\Sigma = \{\dots, \Delta_\alpha, \dots, \Delta_\beta, \dots\}$ so that if $\alpha < \beta$ then $\Delta_\alpha \subset \Delta_\beta$.

Suppose then that the members of the nested family Σ are all subsets of some *countable* set Δ (i.e. for every α , $\Delta_\alpha \subseteq \Delta$ where Δ is countable). Must Σ be countable too?

[For discussion of this, see Béla Bollobás, *The Art of Mathematics*, pp. 61–62.]