

Gödel's Theorems

Exercises: Formal Languages, Formal Theories

Read: *GWT 1: IGT* §§3.1–3.4. If you've not come across the idea of an algorithm before (and so the idea of a property's being effectively decidable is a new one to you), then for more explanation it's hard to beat the §1.1 of the Hartley Rogers's classic *Theory of Recursive Functions and Effective Computability* (McGraw-Hill, 1967).

1. What is an algorithm? Show that there are algorithms for
 - (a) deciding whether a natural number is prime;
 - (b) finding the highest common factor of two natural numbers;
 - (c) deciding whether a given wff of your favourite system of the propositional calculus is a tautology;
 - (d) deciding whether a traditional syllogism (i.e. argument with two premisses and a conclusion, all of *A, E, I* or *O*, form) is valid.

Is there an algorithm for deciding whether an arbitrary real number is greater than one?

2. Carefully define the language of your favourite system of the propositional calculus. Find an algorithm for determining of an arbitrary string of symbols from its alphabet whether or not the string is a wff.
3. A language of basic arithmetic is a first-order language with identity containing a constant to denote zero, and symbols for the one-place successor function, and the two-place addition and multiplication functions. Let L be a particular such language, which allows wffs with free variables, and which uses '0' to denote zero, a postfix prime '′' for successor (so '0′' denotes the successor of zero), and infix '+' and '×' with brackets for addition and multiplication (as in '(x + y)' and '(x × y)'), and has no other non-logical vocabulary.
 - (a) Define the terms of L .
 - (b) Show that it is algorithmically decidable what is an L -term.
 - (c) Define the atomic wffs of L .
 - (d) Show that it is algorithmically decidable what is an atomic L -wff.
 - (e) Define the wffs of L .
 - (f) Show that it is algorithmically decidable what is an L -wff.
 - (g) Show that it is algorithmically decidable what is an L -wff which contains the variable 'x' free.
 - (h) Show that it is algorithmically decidable what is an L -wff which contains *only* the variable 'x' free.
 - (i) Show that it is algorithmically decidable what is an L -sentence (i.e. closed wff without free variable).

4. [Due to Douglas Hofstadter] Consider the (uninterpreted) theory H . Its wffs are any finite string of the symbols M, I, U. It has a sole axiom MI. Its rules of inference are as follows:
- R1. From a wff of the form φI (where φ is any wff), you can infer φIU . (For example, from MMI you can get MMIU.)
 - R2. From a wff of the form $M\varphi$ you may infer $M\varphi\varphi$. (From MIU you can get MIUIU, from MUM you get MUMUM and from MU you can get MUU, etc.)
 - R3. From a wff that contains the string III, you can infer the wff you get by replacing the III with U . (So, from UMIIIMU you can get UMUMU and from MIII you get MIU or MUI. From MIII you can make MU.)
 - R4. From a wff that contains the string UU, you can infer the wff with that string deleted. (So from MUUUM you can get MUM, and from MUUIII you can get MIII.)
- (a) Show that this toy theory H is indeed an (interpreted) formalized theory according to our definition.
 - (b) Can you derive MUIU?
 - (c) Can you derive MUIIU?
 - (d) Can you derive MUII?
 - (e) Can you derive MU?
5. Still concerning Hofstadter's toy theory H ,
- (a) Show that, for each rule, if it is applied to a wff of the beginning with M, the result is a wff beginning with M.
 - (b) Infer that U isn't a theorem of H .
 - (c) Show that, for each rule, if it is applied to a wff whose number of contained Is is *not* a multiple of 3, the result is a wff whose number of Is is also not a multiple of 3.
 - (d) Infer that MUIIIII is not a theorem.
 - (e) Now revisit the Question 4e.