Exercises: Formal theories

These exercises start to explore (informally, and in a very introductory way) a few aspects of the ideas of a formalized language and of a formalized theory.

Reading

1. *IGT2*, Ch. 4.

Exercises

1. Consider a first-order language $L$ for the theory of addition, whose logical apparatus comprises a suitable set of classical connectives, quantifiers, and identity. Its non-logical apparatus is to comprise the constants ‘0’ and ‘1’, the two-place function ‘+’, and the two-place relation ‘$<$’.
   (a) Define the terms of $L$.
   (b) Show that it is algorithmically decidable which expressions are $L$-terms.
   (c) Define the atomic wffs of $L$.
   (d) Show that it is algorithmically decidable which expressions are atomic $L$-wffs.
   (e) Define the wffs of $L$ [allowing wffs with free variables].
   (f) Show that it is algorithmically decidable which expressions are $L$-wffs.
   (g) Show that it is algorithmically decidable what is an $L$-wff which contains the variable ‘$x$’ free.
   (h) Show that it is algorithmically decidable which expressions are $L$-sentences (i.e. closed wffs without free variables).

2. Consider the following (uninterpreted) theory $H$. The alphabet of $H$’s language consists of the symbols $M, U, I$, and any finite string of symbols is a wff. The theory has one axiom: $MI$. $H$ also has five rules of inference ($\sigma$ indicates a string of symbols, possibly empty).
   1. Given a wff of the form $\sigma I$, you can infer the wff $\sigma IU$. (For example, from $MUI$ infer $MIUIU$.)
   2. Given a wff of the form $M\sigma$, infer $M\sigma\sigma$. (For example, from $MIUU$ infer $MIUIIIU$.)
   3. Given a wff which includes the string $UI$, infer the wff that results from replacing that string with $IU$. (For example, from $MIUIU$ infer $MIUIU$.)
   4. Given a wff which includes the string $UU$, infer the wff that results from deleting that string. (For example, from $MIUUU$ infer $MIU$.)
   5. Given a wff which includes the string $III$, infer the wff that results from replacing that string with a $U$. (For example, from $MIUIUU$ infer $MIUU$.)
   (a) Does $H$ count as an effectively axiomatized (uninterpreted) formal theory?
(b) Prove every H-theorem starts with the symbol M, and contains no other occurrence of M.

Don’t get bogged down on these next three little brain-teasers, but have a go before moving on:

(c) Can you derive MIU as a theorem?
(d) Can you derive MUUIU?
(e) Can you derive MU?

Now three more questions about H:

(f) Show that, for each rule, if it is applied to a wff whose number of contained ‘I’-s is not a multiple of 3, the result is a wff whose number of ‘I’-s is also not a multiple of 3.

(g) Can you derive MIUIUIII?
(h) Now revisit question (e) again.

3. [This is really for philosophers: mathematicians will have seen this all before, in one guise or another.]

Consider the formal first-order theory G whose non-logical vocabulary comprises just a two-place function expression ‘·’ and a constant ‘e’. We’ll in fact write the function ‘infix’ like ordinary multiplication, so we put e.g. ‘(x · y)’ rather than ·(x, y). We will also allow the dropping of outer brackets. The axioms of G are:

1. ∀x∀y x · (y · z) = (x · y) · z
2. ∀x x · e = x
3. ∀x∃y x · y = e

(a) Is G an effectively formalized theory?
(b) Prove ∀x∃y x · y = e.
(c) Prove ∀x e · x = x.
(d) Prove that the ‘unit’ e is unique: in other words, if e and e’ both satisfy (2) and (3), then e = e’.
(e) Prove that ‘inverses’ are unique: i.e., ∀x∀y∀y’((x · y = e ∧ x · y’ = e) → y = y’).
(f) Prove G is consistent by giving three interestingly different interpretations for the language of G on which G’s axioms are all true.
(g) Is ∀x∀y x · y = y · x a G-theorem?
(h) Is G negation complete?

4. A reality check. Suppose T is an effectively axiomatized formal theory. Can T be

(a) Inconsistent and negation-complete?
(b) Consistent, negation-incomplete and decidable?
(c) Inconsistent and undecidable?
(d) Consistent and undecidable?
(e) Consistent, negation-complete and undecidable?