Gödel Without Tears – 8

The First Incompleteness Theorem

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- How to construct a 'canonical' Gödel sentence
- If PA is sound, it is negation incomplete
- Generalizing that result to sound p.r. axiomatized theories whose language extends L_A
- ω-incompleteness, ω-inconsistency
- If PA is ω-consistent, it is negation incomplete
- Generalizing that result to ω-consistent p.r. axiomatized theories which extend Q
- The historical First Theorem

The pieces we need to prove the First Theorem are finally all in place. So in this episode we at long last learn how to construct 'Gödel sentences' and use them to prove that PA is incomplete. We also show how generalize the result to other theories.

i. We fix on some acceptable scheme for coding up wffs of PA's language L_A by using Gödel numbers ('g.n.' for short), and coding up sequences of wffs by super Gödel numbers. If φ is an expression, then we'll denote its Gödel number in our logician's English by '$\ulcorner\varphi\urcorner$'. We use '$\overline{\ulcorner\varphi\urcorner}$' as an abbreviation inside L_A for the standard numeral for $\ulcorner\varphi\urcorner$. (§26) Later, when we start generalizing Gödel's results to other theories, we'll use the same notation for Gödel numberings of other languages.

ii. The diagonalization of φ is $\exists\mathsf{y}(\mathsf{y} = \overline{\ulcorner\varphi\urcorner} \wedge \varphi)$. The diagonalization of $\varphi(\mathsf{y})$ is thus equivalent to $\varphi(\overline{\ulcorner\varphi\urcorner})$. (§27)

iii. *diag*(n) is the p.r. function which, when applied to a number n which is the g.n. of some wff φ, yields the g.n. of φ's diagonalization. (§27)

iv. *Prf*(m, n) is the relation which holds just if m is the super g.n. of a sequence of wffs that is a PA proof of a sentence with g.n. n (assume we've fixed on some definite version of PA). This relation is p.r. decidable. (§25.4)

v. Any p.r. function or relation can be *expressed* by a wff of PA's language L_A. In particular, we can choose a Σ_1 wff which '*canonically*' expresses a given p.r. relation by recapitulating its p.r. definition (or more strictly, by recapitulating the definition of the relation's characteristic function). (§20)

vi. Any p.r. function or relation can be *captured* in Q and hence in PA (and captured by a Σ_1 wff which canonically expresses it). (§22)

For what follows, it isn't necessary that you remember the *proofs* of the claims we've just summarized: but do check that you at least fully understand what the various claims *say*.

28 Constructing a Gödel sentence

In this section, we construct a Gödel sentence for PA in particular. But the mode of construction will evidently generalize – a point we return to in the next section. First, another definition:

Defn. 43. *The relation $Gdl(m, n)$ is defined to hold just when m is the super g.n. for a* PA *proof of the diagonalization of the wff with g.n. n.*

Theorem 32. *$Gdl(m, n)$ is p.r. decidable.*

Proof. Either we can informally note that we can mechanically check whether $Gdl(m, n)$ holds without open-ended searches.

Or we can note that $Gdl(m, n)$ holds, by definition, when $Prf(m, diag(n))$. The characteristic function of Gdl is therefore definable by composition from the characteristic function of Prf and the function $diag$, and hence is p.r., given facts (iii) and (iv) from the preamble. ☐

Since Gdl it can be expressed in L_A by a Σ_1 wff (by fact v), which in fact captures Gdl in PA (by fact vi). Of course there won't be a unique such Σ_1 wff. For a start, there will be more than one way of constructing a full definition of the (characteristic function) for the p.r. relation Gdl, so more than one way of tracking such a definition. But we'll adopt the following definition:

Defn. 44. $\mathsf{Gdl}(\mathsf{x}, \mathsf{y})$ *stands in for some Σ_1 wff which canonically expresses and captures Gdl.*

And we next follow Gödel in first constructing the corresponding wff

Defn. 45. $\mathsf{U}(\mathsf{y}) =_{\mathrm{def}} \forall \mathsf{x} \neg \mathsf{Gdl}(\mathsf{x}, \mathsf{y})$.

(For reasons that will become clear in just a moment, you can think of that U as standing for 'unprovable'.) And now we diagonalize U, to give

Defn. 46. $\mathsf{G} =_{\mathrm{def}} \exists \mathsf{y}(\mathsf{y} = \overline{\ulcorner \mathsf{U} \urcorner} \wedge \mathsf{U}(\mathsf{y}))$.

Trivially, G is equivalent to $\mathsf{U}(\overline{\ulcorner \mathsf{U} \urcorner})$. Or unpacking that a bit, G is equivalent to $\forall \mathsf{x} \neg \mathsf{Gdl}(\mathsf{x}, \overline{\ulcorner \mathsf{U} \urcorner})$.

G – meaning of course the L_A sentence you get when you unpack the abbreviations! – is our 'Gödel sentence' for PA. We might indeed call it a *canonical* Gödel sentence for three reasons: (a) it is defined in terms of a wff that we said canonically expresses/captures Gdl, and (b) because it is roughly the sort of sentence that Gödel himself constructed, so (c) it is the kind of sentence people standardly have in mind when they talk of '*the*' Gödel sentence for PA.

Note that G will in be horribly long when spelt out in unabbreviated L_A. But in another way, it is relatively simple. In the terminology of §13, we have the easy result that

Theorem 33. G *is* Π_1.

Proof. $\mathsf{Gdl}(\mathsf{x}, \mathsf{y})$ is Σ_1. So $\mathsf{Gdl}(\mathsf{x}, \overline{\ulcorner \mathsf{U} \urcorner})$ is Σ_1. So its negation $\neg \mathsf{Gdl}(\mathsf{x}, \overline{\ulcorner \mathsf{U} \urcorner})$ is Π_1. Hence $\forall \mathsf{x} \neg \mathsf{Gdl}(\mathsf{x}, \overline{\ulcorner \mathsf{U} \urcorner})$ is Π_1 too. Its logical equivalent G is therefore also Π_1. ☐

And now the key observation:

Theorem 34. G *is true if and only if it is unprovable in* PA.

Proof. Consider what it takes for G to be true (on the interpretation built into L_A of course), given that the formal predicate Gdl expresses the numerical relation Gdl.

G is true if and only if for all numbers m it isn't the case that $Gdl(m, \ulcorner \mathsf{U} \urcorner)$. That is to say, given the definition of Gdl, G is true if and only if there is no number m such that m is the code number for a PA proof of the diagonalization of the wff with g.n. $\ulcorner \mathsf{U} \urcorner$. But the wff with g.n. $\ulcorner \mathsf{U} \urcorner$ is of course U; and its diagonalization is G.

So, G is true if and only if there is no number m such that m is the code number for a PA proof of G. But if G is provable, some number would be the code number of a proof of it. Hence G is true if and only if it is unprovable in PA. ☐

29 The First Theorem – the semantic version

29.1 If PA is sound, it is incomplete

Suppose PA is a sound theory, i.e. it proves no falsehoods (because its axioms are true and its logic is truth-preserving). If G (which is true if and only if it is *not* provable) could be proved in PA, then PA *would* prove a false theorem, contradicting our supposition. Hence, G is not provable in PA.

But that shows that G *is* true. So $\neg$G must be false. Hence $\neg$G cannot be proved in PA either, supposing PA is sound. In Gödel's words, G is a 'formally undecidable' sentence of PA (see Defn. 6).

Which establishes

Theorem 35. *If* PA *is sound, then there is a true* Π_1 *sentence* G *such that* PA $\nvdash$ G *and* PA $\nvdash \neg$G*, so* PA *is negation incomplete.*

If we are happy with the semantic assumption that PA's axioms *are* true on interpretation and so PA *is* sound, the argument for incompleteness is as simple as that – or at least, it's that simple once we have constructed G.

29.2 Generalizing the proof

The proof evidently generalizes. Suppose T is any theory at all, that is put together so that we can mechanically check whether a purported T-proof is indeed a kosher proof without going off on an open-ended search. Then, assuming a sensible scheme for Gödel-number wffs of T, the relation $Prf_T(m,n)$ which holds when m numbers a proof of the wff with number n will be primitive recursive again. Let's say that a theory is *p.r. axiomatized* when it is indeed axiomatized so as to make Prf_T primitive recursive: then indeed any normal theory you dream up which is formally axiomatized is p.r. axiomatized.

Suppose now that T's language includes the language of basic arithmetic, L_A (see Defn. 8), so T can form standard numerals, and we can form the diagonalization of a T-wff. Then we can also define the relation $Gld_T(m,n)$ which holds when m numbers a T-proof of the diagonalization of the wff with number n. This too will be primitive recursive again.

Continuing to suppose that T's language includes the language of basic arithmetic, T will be able to express the p.r. relation Gld_T by a Σ_1 wff Gld_T. Then, just as we did for PA, we'll be able to construct the corresponding Π_1 wff G_T. And then exactly the same argument as before will show, more generally,

Theorem 36. *If T is a sound p.r. axiomatized theory whose language contains the language of basic arithmetic, then there will be a true Π_1 sentence G_T such that $T \nvdash \mathsf{G}_T$ and $T \nvdash \neg\mathsf{G}_T$, so T is negation incomplete.*

Which is our first, 'semantic', version of the general Incompleteness Theorem!

29.3 Comparisons

Compare Theorem 36 with our initially announced

Theorem 1. *If T is a sound formalized theory whose language contains the language of basic arithmetic, then there will be a true sentence G_T of basic arithmetic such that $T \nvdash \mathsf{G}_T$ and $T \nvdash \neg\mathsf{G}_T$, so T is negation incomplete.*

Our new theorem is stronger in one respect, weaker in another. But the gain is much more than the loss.

Our new theorem is stronger, because it tells us more about the character of the undecidable Gödel sentence – namely it has minimal quantifier complexity. The unprovable sentence G_T is a Π_1 sentence of arithmetic, i.e. is the universal quantification of a decidable condition. As far as quantifier complexity is concerned, it is on a par with Goldbach's conjecture that every number is such that, if even and greater than two, it is the sum of two primes (for note it is decidable whether a number is the sum of two primes). Indeed it is sometimes said that a Gödel sentence like G_T is *of Goldbach type.*

Our new theorem is weaker, however, as it only applies to p.r. axiomatized theories, not to formalized theories more generally. But that's not much loss. For what would a theory look like that was axiomatized but not p.r. axiomatized? It would be a matter, for example, of only being able to tell what's an axiom on the basis of an open-ended search: but that would require a *very* unnatural way of specifying the theorem's axioms in the first place. As I noted before, any normally presented axiomatized theory will be p.r. axiomatized. (Later, in Episode 12, we will say something about how to extend Gödel's theorem to cover the case of abnormally though still decidably axiomatized theories – but that really is a minor extension.)

29.4 Our Incompleteness Theorem is better called an *incompletability* theorem

Here, we just repeat the argument of §2.7: but the point is central enough to bear repetition. Suppose T is a sound p.r. axiomatized theory which can express claims of basic arithmetic. Then Theorem 36 we can find a true G_T such that $T \nvdash \mathsf{G}_T$ and $T \nvdash \neg\mathsf{G}_T$. That *doesn't* mean that G_T is 'absolutely unprovable' in any sense: it just means that G_T-is-unprovable-in-T.

Now, we might want to 'repair the gap' in T by adding G_T as a new axiom. So consider the theory $T' = T + \mathsf{G}_T$. Then (i) T' is still sound (for the old T-axioms are true, the added new axiom is true, and the logic is still truth-preserving). (ii) T is still a p.r. axiomatized theory, since adding an specified axiom to T doesn't commit us to any open-ended searches to determine what is an axiom of the augmented theory. (iii) We haven't changed the language. So our Incompleteness Theorem applies, and we can find a sentence G'_T such that $T' \nvdash \mathsf{G}_{T'}$ and $T' \nvdash \neg\mathsf{G}_{T'}$. And since T' is stronger than T, we have a fortiori, $T \nvdash \mathsf{G}_{T'}$ and $T \nvdash \neg\mathsf{G}_{T'}$. In other words, 'repairing the gap' in T by adding G_T as a new axiom leaves some other sentences that are undecidable in T *still* undecidable in the augmented theory.

And so it goes. Our theorem tells us that if we keep chucking more and more additional true axioms at T, our theory will still remain negation-incomplete, unless it either stops being sound or stops being p.r. axiomatized. In a good sense, T is *incompletable.*

Now do pause here: have a think, have a coffee! Are you absolutely clear about how G is constructed? Are you absolutely clear why it is true iff and only if unprovable? Do uou understand why it it must be formally undecidable assuming PA is sound? Do you understand how and why the result generalizes?

If you answer 'no' to any of those, re-read more carefully! If you answer 'yes' to all, excellent: on we go ...

30 ω-completeness, ω-consistency

Before we turn to the second version of the First Incompleteness Theorem – the version that downgrades the semantic assumption that we're dealing with a sound theory to the much weaker syntactic assumption that the theory is consistent (and a bit more) – we need to pause to define two key notions.

Techie note: in this section, take the quantifiers mentioned to be arithmetical ones – if necessary, therefore, replacing $\forall\mathsf{x}\varphi(\mathsf{x})$ by $\forall\mathsf{x}(\mathsf{Nx} \to \varphi(\mathsf{x}))$, where '$\mathsf{N}$' picks out the numbers from the domain of the theory's native quantifiers (see Defn. 8).

Defn. 47. *A theory T is ω-*incomplete *iff, for some open wff $\varphi(\mathsf{x})$, T can prove $\varphi(\overline{\mathsf{n}})$ for each natural number m, but T can't go on to prove $\forall\mathsf{x}\varphi(\mathsf{x})$.*

We saw in §10.3 that Q is ω-incomplete: that's because it can prove each instance of $\mathsf{0} + \overline{\mathsf{n}} = \overline{\mathsf{n}}$, but can't prove $\forall\mathsf{x}(\mathsf{0} + \mathsf{x} = \mathsf{x})$. We could repair ω-incompleteness if we could add the ω-rule (see §11.1), but that's an infinitary rule that is not available in a formalized theory given the usual finitary restrictions on the checkability of proofs. We instead added induction to Q hoping to repair as much incompleteness as we could: but, as we'll see, PA remains ω-incomplete (assuming it is consistent).

Defn. 48. *A theory T is ω-*inconsistent *iff, for some open wff $\varphi(\mathsf{x})$, T can prove each $\varphi(\overline{\mathsf{n}})$ and T can also prove $\neg\forall\mathsf{x}\varphi(\mathsf{x})$.*

Or, entirely equivalently, we could of course say that T is ω-inconsistent if, for some open wff $\varphi'(\mathsf{x})$, $T \vdash \exists\mathsf{x}\varphi'(\mathsf{x})$, yet for each number n we have $T \vdash \neg\varphi'(\overline{\mathsf{n}})$.

Note that ω-inconsistency, like ordinary inconsistency, is a syntactically defined property: it is characterized in terms of what wffs can be proved, not in terms of what they mean. Note too that, in a classical context, ω-consistency – defined of course as not being ω-inconsistent! – trivially implies plain consistency. That's because T's being ω-consistent is a matter of its *not* being able to prove a certain combination of wffs, which entails that T can't be inconsistent and prove *all* wffs.

Now compare and contrast. Suppose T can prove $\varphi(\overline{\mathsf{n}})$ for each m. T is ω-incomplete if it can't also prove something we'd like it to prove, namely $\forall\mathsf{x}\varphi(\mathsf{x})$. While T is ω-inconsistent if it can actually prove the *negation* of what we'd like it to prove, i.e. it can prove $\neg\forall\mathsf{x}\varphi(\mathsf{x})$.

So ω-incompleteness in a theory of arithmetic is a regrettable weakness; but ω-inconsistency is a Very Bad Thing (not as bad as outright inconsistency, maybe, but still bad enough). For evidently, a theory that can prove each of $\varphi(\overline{\mathsf{n}})$ and yet also prove $\neg\forall\mathsf{x}\varphi(\mathsf{x})$ is just not going to be an acceptable candidate for regimenting arithmetic.

That last observation can be made vivid if we temporarily bring semantic ideas back into play. Suppose the theory T is given a *arithmetically standard* interpretation, by which we here mean just an interpretation which takes numerical quantifiers as running over a domain comprising the natural numbers, and on which T's standard numerals denote the intended numbers (with the logical apparatus also being treated as normal, so that inferences in T are truth-preserving). And suppose further that on this interpretation, the axioms of T are all true. Then T's theorems will all be true too. So now imagine that, for some $\varphi(\mathsf{x})$, T does prove each of $\varphi(\mathsf{0})$, $\varphi(\overline{\mathsf{1}})$, $\varphi(\overline{\mathsf{2}})$, ... By hypothesis, these theorems will then be true on the given standard interpretation; so this means that every natural number must satisfy $\varphi(\mathsf{x})$; so $\forall\mathsf{x}\varphi(\mathsf{x})$ is true since the domain contains only natural numbers. Hence $\neg\forall\mathsf{x}\varphi(\mathsf{x})$ will have to be false on this standard interpretation. Therefore $\neg\forall\mathsf{x}\varphi(\mathsf{x})$ can't be a theorem, and T must be ω-consistent.

Hence, contraposing, we have

Theorem 37. *If T is ω-inconsistent then T's axioms can't all be true on an arithmetically standard interpretation.*

Given that we want formal arithmetics to have axioms which *are* all true on a standard interpretation, we must therefore want ω-consistent arithmetics. And given that we think e.g. PA *is* sound on its standard interpretation, we are committed to thinking that it *is* ω-consistent.

31 The First Theorem – the syntactic version

31.1 If PA is consistent, it can't prove G

So far, we have actually only made use of the weak result that PA's language can *express* the relation *Gdl*. But remember Defn. 44: our chosen Gdl doesn't just express *Gdl* but *captures* it. Using this fact about Gdl, we can again show that PA does not prove G, but this time *without* making the semantic assumption that PA is sound.

Theorem 38. *If PA is consistent, $\mathsf{PA} \nvdash \mathsf{G}$.*

Proof. Suppose G *is* provable in PA. If G has a proof, then there is some super g.n. m that codes its proof. But by definition, G is the diagonalization of the wff U. Hence, by definition, $Gdl(m, \ulcorner\mathsf{U}\urcorner)$.

Now we use the fact that Gdl captures the relation *Gdl*. That implies that, since $Gdl(m, \ulcorner\mathsf{U}\urcorner)$, we have (i) $\mathsf{PA} \vdash \mathsf{Gdl}(\overline{\mathsf{m}}, \overline{\ulcorner\mathsf{U}\urcorner})$.

But since G is logically equivalent to $\forall \mathsf{x} \neg \mathsf{Gdl}(\mathsf{x}, \overline{\ulcorner \mathsf{U} \urcorner})$, the assumption that G is provable comes to this: $\mathsf{PA} \vdash \forall \mathsf{x} \neg \mathsf{Gdl}(\mathsf{x}, \overline{\ulcorner \mathsf{U} \urcorner})$. The universal quantification here entails any instance. Hence (ii) $\mathsf{PA} \vdash \neg \mathsf{Gdl}(\overline{\mathsf{m}}, \overline{\ulcorner \mathsf{U} \urcorner})$.

So, combining (i) and (ii), the assumption that G is provable entails that PA is inconsistent. Hence, if PA is consistent, there can be no PA proof of G. ☐

31.2 If PA is consistent, it is ω-incomplete

Here's an immediate corollary of that last theorem:

Theorem 39. *If* PA *is consistent, it is* ω*-incomplete.*

Proof. Assume PA's consistency. Then we've shown that $\mathsf{PA} \nvdash \mathsf{G}$, i.e.,

1. $\mathsf{PA} \nvdash \forall \mathsf{x} \neg \mathsf{Gdl}(\mathsf{x}, \overline{\ulcorner \mathsf{U} \urcorner})$.

Since G is unprovable, that means that no number is the super g.n. of a proof of G. That is to say, no number numbers a proof of the diagonalization of U. That is to say, for any particular m, it *isn't* the case that $Gdl(m, \ulcorner \mathsf{U} \urcorner)$. Hence, again by the fact that Gdl captures Gdl, we have

2. For each m, $\mathsf{PA} \vdash \neg \mathsf{Gdl}(\overline{\mathsf{m}}, \overline{\ulcorner \mathsf{U} \urcorner})$.

Putting $\varphi(\mathsf{x}) =_{\mathrm{def}} \neg \mathsf{Gdl}(\mathsf{x}, \overline{\ulcorner \mathsf{U} \urcorner})$, the combination of (1) and (2) therefore shows that PA is ω-incomplete. ☐

31.3 If PA is ω-consistent, it can't prove $\neg$G

We'll now show that PA can't prove the negation of G, without assuming PA's soundness: we'll just make the syntactic assumption of ω-consistency.

Theorem 40. *If* PA *is* ω*-consistent,* $\mathsf{PA} \nvdash \neg \mathsf{G}$.

Proof. Suppose $\neg \mathsf{G}$ is provable in PA. That's equivalent to assuming

1. $\mathsf{PA} \vdash \exists \mathsf{x} \mathsf{Gdl}(\mathsf{x}, \overline{\ulcorner \mathsf{U} \urcorner})$.

Now suppose too that PA is ω-consistent. Then, as we remarked before, that implies that PA is consistent. So if $\neg \mathsf{G}$ is provable, G is *not* provable. Hence for any m, m cannot code for a proof of G. But G is (again!) the wff you get by diagonalizing U. Therefore, by the definition of Gdl, our assumptions imply that $Gdl(m, \ulcorner \mathsf{U} \urcorner)$ is false, for each m. So, by the requirement that Gdl captures Gdl, we have

2. $\mathsf{PA} \vdash \neg \mathsf{Gdl}(\overline{\mathsf{m}}, \ulcorner \mathsf{U} \urcorner)$ for each m.

But (1) and (2) together make PA ω-inconsistent after all, contrary to hypothesis. Hence, if PA is ω-consistent, $\neg \mathsf{G}$ is unprovable. ☐

31.4 Putting together syntactic Incompleteness Theorem for PA

Let's put all the ingredients together. Recall that G is a Π_1 sentence (i.e. of the same quantifier complexity as e.g. Goldbach's Conjecture). And we know from Theorem 34 that G is true if and only if it is unprovable. That observation put together with what we've shown so far this section entails

Theorem 41. *If* PA *is consistent, then there is a* Π_1 *sentence* G *such that* $\mathsf{PA} \nvdash \mathsf{G}$*, and if* PA *is* ω*-consistent* $\mathsf{PA} \nvdash \neg \mathsf{G}$*, so – assuming* ω*-consistency and hence consistency –* PA *is negation incomplete.*

31.5 Generalizing the proof

The proof for Theorem 41 evidently generalizes. Suppose T is a p.r. axiomatized theory which contains Q – so (perhaps after introducing some new vocabulary by definitions) the language of T extends the language of basic arithmetic, and T can prove Q's axioms. Then assuming a sensible scheme for Gödel-number wffs of T, the relation $Gdl_T(m, n)$ which holds when m numbers a T-proof of the diagonalization of the wff with number n will be primitive recursive again.

Since T can prove everything Q proves, T will be able to capture the p.r. relation Gld_T by a Σ_1 wff Gld_T. Just as did for PA, we'll be able to construct the corresponding Π_1 wff G_T. And, exactly the same arguments as before will then show, more generally,

Theorem 42. *If T is a consistent p.r. axiomatized theory which contains Q, then there will be a Π_1 sentence G_T such that $T \nvdash \mathsf{G}_T$, and if T is ω-consistent, $T \nvdash \neg\mathsf{G}_T$, so T is negation incomplete.*

When people refer to 'The First Incompleteness Theorem' (without qualification), they typically mean something like this second general result, deriving incompleteness from syntactic assumptions.

31.6 Comparisons

Compare Theorem 42 with our initially announced

Theorem 2. *For any consistent formalized theory T which contains a certain modest amount of arithmetic (and has a certain additional desirable property that any sensible formalized arithmetic will share), there is a sentence of basic arithmetic G_T such that $T \nvdash \mathsf{G}_T$ and $T \nvdash \neg\mathsf{G}_T$, so T is negation incomplete.*

Our new theorem fills out the old one in various respects, but it is weaker in another respect. But the gain is much more than the loss.

Our new theorem tells us more about the 'modest amount of arithmetic' that T is assumed to contain and it also spells out the 'additional desirable property' which we previously left mysterious (and we now know the condition is only applied in half the theorem). Further it tells us more about the undecidable Gödel sentence – namely it has minimal quantifier complexity, i.e. it is a Π_1 sentence of arithmetic. Our new theorem is weaker, however, as it only applies to p.r. axiomatized theories, not to formalized theories more generally. But we've already note that that's not much loss. Later we'll make up the shortfall.

32 The historical First Theorem

Theorem 42, or something like it, is what people usually mean when they speak without qualification of 'The First Incompleteness Theorem'. But since the stated theorem refers to Robinson Arithmetic Q (developed by Robinson in 1950!), and Gödel didn't originally know about that (in 1931), our version can't be quite what Gödel originally proved. But it is a near miss.

Looking again at our analysis of the syntactic argument for incompleteness, we see that we are interested in theories which extend Q *because we are interested in theories which can capture p.r. relations like Gdl.* It's being able to capture Gdl that is the crucial condition for a theory's being incomplete. So let's say

Defn. 49. *A theory T is* p.r. adequate *if it can capture all primitive recursive functions and relations.*

Then, instead of mentioning Q, let's instead explicitly write in the requirement of p.r. adequacy. So, by just the same arguments,

Theorem 43. *If T is a p.r. adequate, p.r. axiomatized theory whose language includes L_A, then there is Π_1 sentence φ such that, if T is consistent then $T \nvdash \varphi$, and if T is ω-consistent then $T \nvdash \neg\varphi$.*

And this is pretty much Gödel's own general version of the incompleteness result. I suppose that it has as much historical right as any to be called *Gödel's* First Theorem.[1]

For in his 1931 paper, Gödel first proves his Theorem VI, which with a bit of help from his Theorem VIII shows that the formal system P – which is his simplified version of the hierarchical type-theory of *Principia Mathematica* – has a formally undecidable Π_1 sentence (or sentence 'of Goldbach type', see §29.3). Then he immediately generalizes:

> In the proof of Theorem VI no properties of the system P were used besides the following:
>
> 1. The class of axioms and the rules of inference (that is, the relation 'immediate consequence') are [primitive] recursively definable (as soon as we replace the primitive signs in some way by the natural numbers).
> 2. Every [primitive] recursive relation is definable [i.e. is 'capturable'] in the system P.
>
> Therefore, in every formal system that satisfies the assumptions 1 and 2 and is ω-consistent, there are undecidable propositions of the form $(x)F(x)$ [i.e. $\forall x F(x)$], where F is a [primitive] recursively defined property of natural numbers, and likewise in every extension of such a system by a recursively definable ω-consistent class of axioms.

Which gives us our Theorem 43.

At this point, make sure you really understand at least what the core theorems in this episode *mean*. Then read *IGT*, Chs. 16 and 17. And then re-read those chapters! – for they are at the very heart of the book, and of this course.

Then when you feel reasonably confident of the techie details, have a look at Ch. 18 (perhaps skilling §18.3).

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[1] 'Hold on! If *that's* the First Theorem, we didn't need to do all the hard work showing that Q and PA are p.r. adequate, did we?' Well, yes and no. No, proving *this* original version of the Theorem of course doesn't depend on proving that any particular theory is p.r. adequate. But yes, showing that this Theorem has real bite, showing that it applies to familiar arithmetics, does depend on proving the adequacy theorem.