

2 Eliminative Structuralism and Nominalism

§§8–11 **Introducing eliminative structuralism** The first four sections of this chapter cover some rather familiar territory, and I don't have much to add by way of commentary. So I'll mostly confine myself to a quick overview summary.

§§8, 9: Parsons says that he himself thinks that “something close to the structuralist view is true”. But structuralist in what sense?

It is often said, perhaps in a Bourbarchiste spirit, that modern mathematics is the study of structures. But *that* claim taken by itself leaves it wide open what picture we should adopt of the ontology of mathematical objects and the structures they sit in. The Bourbarchiste mathematician can – indeed typically does – go on to take structures to be sets (comprising domains with distinguished elements and equipped with relations and/or functions which are themselves treated as more sets). So for her, the structures are set-theoretic objects. But then that view can still be filled in with whatever story about the ontology of which sets takes your philosophical fancy. Put it this way: Bourbarkie leaves the ontology open.

Parsons stresses, then, that he is concerned with varieties of philosophical structuralism(s) which *do* involve an ontological manifesto – views along the lines suggested by “the objects of mathematics are positions in structures, [and] have no identity or features outside of a structure” (to quote from Michael Resnik's well-known 1981 *Noûs* paper). Though, Parsons notes, that can't be the *whole* story about quasi-concrete mathematical objects if such there be (“because the representation relation is something additional to intrastructural relations”)

But if objects are “positions in structures”, what are structures? If we take the Bourbarchiste line of treating all structures as *sets*, and then tried to be a global structuralists in something like Resnik's sense – i.e. try to go on to explain the nature of sets in structuralist terms – it looks as though some kind of circularity threatens. However, let's not worry about that for the moment. Let's begin – at any rate – by backing off from a global Bourbarchisme that would have it that all mathematics is about sets, and let's start by looking at more local structuralisms.

§10: In particular, then, let's begin be looking at Dedekind's treatment of the natural numbers (read in a way that is not true to the historical Dedekind, but we won't fuss about that now, particularly as Dedekind's own view, which we'll return to touch on in discussing §18, is rather murky).

Dedekind defines conditions for a domain N , with distinguished element 0 , and a one-one mapping $S : N \rightarrow N - \{0\}$, to be ‘simply infinite’. Abbreviate those (categorical) conditions $\Omega(N, 0, S)$. With some effort, an ordinary statement of arithmetic can be correlated with a version $A(N, 0, S)$ whose primitives are again just $N, 0, S$ (that takes effort, because we'll need to define arithmetical functions other than the successor function using tricks about which more in just a moment). Then on one reading of Dedekind – the *eliminative* reading – the suggestion is that *an ordinary statement of arithmetic can be treated as elliptical for the corresponding claim*

For any $N, 0, S$, if $\Omega(N, 0, S)$ then $A(N, 0, S)$.

This is ‘eliminative’ in that a statement apparently about one kind of thing, numbers, is paraphrased away as a disguised generalization about other kinds of things, domains, elements of domains, and mappings. And it might be said to be structuralist in the something like the Resnik sense Parsons is interested in exploring, since it's just what is in common to all simply infinite systems – not the objects occupying positions but

the relations between the positions themselves – which matters for arithmetical truth. (Though note, this isn't quite a structuralism in Resnik's sense; for it isn't that numbers are being treated as 'thin' objects, mere positions in structures: reference to numbers is in this story being paraphrased right away.)

This eliminative structuralism, as Parsons calls it, neatly sidesteps the 'multiple reduction' problems for more straightforward attempts to reduce arithmetic by identifying the numbers with a particular ω -sequence – namely there's no reason to choose one candidate rather than another. But now note that if there are *no* simply infinite systems, so instances of $\Omega(N, 0, S)$ are always false, then any and every ordinary arithmetical statement (so both $2 + 2 = 4$ and $2 + 2 = 5$) comes out as vacuously true. So on this structuralist view, there has to be an infinite system somewhere in our universe if we are to get the right truth-values for arithmetic statements.

Of course, that's not straightforwardly a problem if we *do* already buy into a suitable background universe of sets where there are infinite systems aplenty (though the threat of vacuity will evidently become more urgent if and when we try to repeat the trick and give a similar eliminative structuralist account of larger mathematical universes).

But, as Parsons notes, there's not just an issue about getting the truth-values right, there's an issue about getting the ontological commitments right. For Dedekind's conditions $\Omega(N, 0, S)$ – spelt out his kind of way – will involve quantification over sets. And likewise indeed there will be more quantifications over sets in many of the typical instances of $A(N, 0, S)$, as when we have to deploy explicit definitions of e.g. addition and multiplication in terms of successor. We might ask: do we really want a structuralist account of a particular familiar kind of mathematical object, numbers, to tell us that really we've been generalizing about some *other* rather less familiar kind of object, sets?

I'm not sure, however, quite how telling this second thought is. For are we to take eliminative structuralism 'hermeneutically', as a story about what we've been meaning all along, or in a more 'revolutionary' spirit, as supposedly offering us the best chance to preserve the truth of proxies for common-or-garden arithmetical claims while keeping our ontology respectable? Parsons doesn't explore the distinction, and it looks as though it should be rather significant here. (My sense is that Parsons is aiming for a hermeneutic structuralist account. But working mathematicians tend to be happy enough with revolutionary stories. After all, when at the beginning of an analysis course, real numbers are treated as Dedekind cuts on the rationals, is the story offered as an account of what the naive mathematics of the reals was talking about all along? Surely not.)

Another point Parsons makes at this stage seems more telling. Whether or not we are finally happy with the eliminative structuralist story about numbers, it does illustrate a general possibility: an account of numbers can be *structuralist* in flavour without referring to *structures* (and hence without treating structures as objects). His eliminative structuralism not only eliminates the numbers but also the structures in which they sit (there's a vivid contrast, then, with e.g. Stewart Shapiro's 'ante rem' structuralism which argues for a rich ontology of structures).

§11: Now, Parsons suggests, we can perhaps sidestep *some* of the issues about our first version of eliminative structuralism about numbers by trading in an explicitly set-theoretic presentation for a version couched in second-order logical terms, so we quantify over properties and functions rather than sets, and use a full second-order logic. We then get a new second-order definition of being simply infinite, $\Omega'(N, 0, S)$, a new correlate of an ordinary arithmetical claim, $A'(N, 0, S)$, and correspondingly a new suggestion that the ordinary statement can be treated as elliptical for

For any $N, 0, S$, if $\Omega'(N, 0, S)$ then $A'(N, 0, S)$,

where now ‘any N ’ and ‘any S ’ are treated as second-order. If we are relaxed enough about second-order quantification (though that’s quite a big ‘if’), we might find this a bit easier to swallow than the previous version of structuralism that traded more explicitly in sets; and if (another big ‘if’) we think that second order logic is indeed pure logic, this kind of ‘if-thenism’ about arithmetic could be advertised as a form of logicism.

Moreover – if we are going second-order to avoid committing ourselves already to sets or other rich mathematical structures – then this kind of ‘if-thenism’ about arithmetic is of course more threatened by the possibility of vacuity (prescinding from sets, are we sure there are any simply infinite systems?). What to do?

Well, one option is to read the conditional as stronger-than-material (so $\Omega'(N, 0, S)$ being false doesn’t make ‘if $\Omega'(N, 0, S)$ then $A'(N, 0, S)$ ’ true): e.g. we might discern an implicit modal operator governing the conditional. But that opens up a whole new set of problems. For a start, what kind of modality is involved here?

Some of these issues are to be pursued in the following chapter. But we might wonder first if we can perhaps deflate issues about modality by giving a very modest possibility-as-consistency reading. Perhaps “we interpret the theories in an if-thenist way, but deal with the problem of possibility by appealing to consistency, nominalistically interpreted.” Well, this suggestion is to be pursued critically in the following section §12 (but I should say that I found it, and the rest of this chapter, something of a jumble of concerns).

§12 Nominalism Parsons understands ‘nominalism’ Harvard-style – no surprise there, then! – to mean the rejection of abstract entities and the eschewing of (ineliminable) modality. What hope, then, for giving a response to the potential-vacuity problem for eliminative structuralism about arithmetic (say) which meets nominalist constraints? We can’t, by temporary hypothesis, go ineliminably modal: so what to do?

Well, as the physical world actually is (or so we might well now believe), there are in fact enough physical things – e.g. space time points – and suitable physical orderings on them to give us physically realized simply infinite structures. But Parsons is unhappy with this way of bluntly rejecting the vacuity worry, and for familiar reasons: “[S]hould it be taken as a presupposition of elementary mathematics that the real world instantiates a mathematical conception of the infinite? This would have the consequence that mathematics is hostage to the future possible development of physics.”

But (although I have no particular nominalist sympathies myself), I’m not sure how worried the hard-core nominalist should be about giving such hostages to fortune if he treats arithmetic, say, in the eliminative structuralist way. For as things are, given how we believe the world actually to be, such a nominalist can believe that there *are* physical instances of simply infinite systems, and can hence continue to speak with the vulgar and treat arithmetical claims (as he construes them) as non-vacuously true or false.

However, suppose the worst happens, so we change our minds and come to believe the world is in fact ultimately grainy and finite in all respects: it’s not that the ‘school-room’ arithmetic of feasibly computable numbers is going to get undermined. At most, it is the idealizing rounding out of school-room arithmetic, which insists on an infinitude of numbers, which is in trouble. And if indeed it should emerge that this rounding out, construed the eliminative-structuralist way, makes any proposition of idealized arithmetic vacuously *true* without discrimination, that doesn’t make the mathematician’s game of seeing what follows from what inside idealized arithmetic vacuous or trivial. Nor does it stop idealized arithmetic being useful in giving us quick ways of establishing claims about ‘real arithmetic’, the arithmetic of the feasibly computable. It just means that the idealized game doesn’t track the truth, any more than do other bits of more infinitary mathematics (according to the nominalist).

Parsons says “a great deal of the historically given mathematics would have to be jettisoned in this case [i.e. on the eliminative structuralist reading, if e.g. there are no simply infinite systems]”. But talk of ‘jettisoning’ seems to cover a slide here. For ‘jettisoning’ idealized arithmetic in the sense of no longer thinking of it as literally true does not imply ‘jettisoning’ it in the sense of simply throwing arithmetic into the trash-can – as any fictionalist, looking over his shoulder at Hilbert’s programme, will be quick to insist.

What about the other line that offered to the nominalist at the end of §11? – i.e. sidestep the vacuity problem by going modal but in a nominalistically tolerable way (remember: “interpret the theories in an if-thenist way, but deal with the problem of possibility by appealing to consistency, nominalistically interpreted”). Well, again Parsons sees trouble, this time arising from the fact that there might be physical limitations in how big a proof-token could be, and so a theory could count as (nominalistically) consistent – because no proof of an inconsistency could be tokened – even if we can show that there is a process which, if only there were world enough and time, would produce an inconsistency.

At the end of this section, Parsons revisits the question of how to frame an eliminative structuralism for arithmetic. He looked at a move from a set-theoretic formulation to a more ‘logical’, second-order formulation. But now he asks: could we in fact go first-order, in a way more congenial no doubt to those of nominalist inclinations? The trouble is, of course, that we won’t get categoricity (whatever we build into the axioms), so the eliminative structuralist who goes first-order runs up against the intuition that the natural numbers have a unique structure. But how secure, we might wonder, is that intuition? Parsons raises the question only to shelve it until Ch. 8. So we’ll have to return to that later.

§13 Nominalism and second-order logic Having raised the possibility of giving a second-order version of Dedekind eliminative structuralism, Parsons now pauses to think some more about the interpretation of second order logic, in particular interpretations (unlike Frege’s own) that could appeal to a nominalist. So this long section falls into two parts. First, Parsons offers some remarks on the Fieldian project of using mereology to do the work of second-order logic. The key thought is this. For mereology to do all the work Field wants, it needs an (impredicative) comprehension principle: “Given a predicate of individuals that is true of at least one individual, there is a sum of just the individuals of which the predicate is true, and moreover, the admissible predicates will be closed under quantification over all individuals, including those very sums.” (Cf. the principle ‘Cs’ in Field’s paper ‘On Conservativeness and Incompleteness’.) But what entitles Field to such a strong comprehension principle? Well, Parsons notes that it’s not clear that Field can offer any direct a priori argument (but then, I wonder, would he want to?). The justification will be that “the comprehension principle is a hypothesis justified by its consequences in systematizing the geometrical basis of physics”. But then “Field’s view, on this reading, puts him in a position in which we have found other formulations of nominalism: making the justification of mathematics turn on some hypothesis about the physical world, which is more vulnerable to refutation than the mathematics.”

But again, just how troubled should be someone of nominalist inclinations be? Suppose that we decide that our physical theory of the world doesn’t require such a strong comprehension principle: we can get away with recognizing a less wide-ranging plurality of regions. That’s not at all implausible, actually, given that nearly all the mathematics required for physics can be reconstructed in a weak second-order arithmetic like ACA_0 with only predicative comprehension (a point that has become much better known since

Field’s original proposals in *Science without Numbers*). Then, in Fieldian spirit, we would just demote the full mathematical apparatus of the classical reals from its alleged status as being justified as a required tool for getting more nominalistically acceptable consequences out of our best physics. It turns out that it is no longer so justified. In that sense, for the Fieldian-style nominalist, the “justification” of a bit of mathematics is indeed wrapped up with our hypotheses about the physical world, and Parsons’s complaint will seem question-begging.

The second part of this section considers Boolos’s attempt to make second-order logic ontologically tame by giving a plural reading to the second-order quantifiers. The thought under scrutiny is that plural quantification is ontologically innocent because, in plurally quantifying over F s, we are just committing ourselves to the F s (not to sets or to Fregean concepts). Parsons’s discussion – or am I missing something here? – initially advances now rather familiar sorts of worries about this claim of innocence. But Parsons does make one point towards the end of the section that I find very congenial (i.e. I’ve argued similarly myself!).

Consider (say) the range of second-order arithmetics that Simpson discusses in his *SOSO*. As we advance through theories with stronger and stronger comprehension principles, then – on a standard platonist construal – we are countenancing more and more sets of numbers. If we reconstrue the second-order quantifiers plural-wise, then, as we go from theory to theory, we are countenancing more and more . . . well, more what? It is tempting to say ‘pluralities’. And indeed it is convenient to give an informal gloss of the plural reading using talk of pluralities. But – if this isn’t to smuggle back reference to pluralities-as-single-entities, i.e. sets – this convenient way of talking needs to be eliminable.¹² So how do we eliminate it here? We might, I suppose, trade in talk of countenancing more and more pluralities for talk of allowing more and more different *ways* we can take numbers together: but quantifying over ways of taking numbers together seems tantamount to re-instating Fregean concepts as the values of the second-order variables – which is fine by me, but then the supposed ontological gain of interpreting the second-order quantifiers via plurals is lost.

The question then is this: if we accept the pluralist’s contention that we can treat second-order numerical quantifiers as ontologically committing just us to numbers, period, then how are we to think of the surely varying commitments we take on with varying strengths of comprehension principle? As Parsons nicely puts it, “If there is no enlargement of ontological commitment as one passes to less restricted versions of the comprehension schema, then perhaps that speaks against the importance of the notion.”

§14 Structuralism and application We’re considering – as a first sample version of a structuralist view of numbers – the schematic idea that an ordinary arithmetical statement is elliptical for something general along the lines of

For any $N, 0, S$, if $\Omega(N, 0, S)$ then $A(N, 0, S)$,

where $\Omega(N, 0, S)$ lays down the conditions for a domain N (equipped with a distinguished element 0 , and a mapping $S : N \rightarrow N - \{0\}$) to be ‘simply infinite’, and $A(N, 0, S)$ is appropriately correlated with the ordinary statement.

Does this eliminative structuralist view have a problem accounting for the application of numbers as cardinals? What about structuralisms more generally? Recall Frege’s remark: “It is applicability alone that raises arithmetic from the rank of a game to that of a science. Applicability therefore belongs to it of necessity.” And Frege further takes

¹²See here Linnebo’s nice article on plural logic in the Stanford Encyclopedia.

it that an account of numbers should start from their use in counting (so a structuralist understanding that explains the nature of arithmetical truths prior to explaining their application is going wrong). But, Parsons argues, our structuralist in fact can resist that further thought.

I'm not sure I fully have the measure of Parsons thinking here.¹³ But I take it that the idea is something like this. Counting some objects involves putting them into one-one correspondence with an initial segment of some paradigm simply infinite system (of numerals, say). That involves setting up some external relations between some members of the relevant simply infinite system, over and above the internal relations which constitute their being a such a system. But now, via the Dedekind categoricity theorem, we see that these external relations will engender a one-one correspondence with an isomorphic initial segment of any simply infinite system. So, in counting, there is an implicit generalization over simply infinite systems in the offing – which is what, according to the eliminative structuralist, talk of numbers amounts to. Hence, as Frege wanted, even on the structuralist view, we *do* after all have an essential connection between numbers and their application in counting.

That, I think, does rather neatly deal with the supposed general problem. Now, Dummett has raised a more specific problem – roughly, defining a simply infinite system doesn't tell us whether its initial element is to be treated as 0 or 1. But Parsons (rightly in my view) doesn't find this worry a telling one for the structuralist. He can regard it as just a matter of pragmatic convention whether, in applications, we start counting at 0 or 1, depending on how much we care about having a number for empty collections.

A last remark. Parsons notes that as well as the natural number 3, we have the integer 3, the rational 3, the real number 3 and the complex number 3 (not to mention more exotic constructions). And the structuralist can then say that the use of '3' each time signifies not the same entity but the same structural role, a point congenial to his general account of the significance of number words. The suggestion is that our multiple use of '3' here favours the structuralist construal. However, I don't see that the multiple use of '3' counts in fact against the Fregean view that numbers are "specific objects" (as Dummett would insist). The Fregean can just say that there is here a range of different terms ('natural number 3', 'rational number 3', 'real number 3', etc.) which have different objects as their referents: as some of my local friendly mathematicians are wont to say, we should keep our objects strongly typed. Though the occurrence of '3' in each of those terms is of course not just a pun: for the denoted objects, intuitively, have sufficiently analogous roles in the respective families. Or to put it more fancily, the naturals have a canonical embedding into rationals and reals, etc. But note that an embedding is a *mapping* not a literal placing of one structure inside another, and '3' – used as a natural number term – denotes a different object from '3/1' i.e. '3' used as a rational.

¹³I think that Parsons wants to consider structuralisms generally in this section: he does say the points he wants to make can be "discussed with a version of eliminative reading of Dedekind as our paradigm" (p. 73) though he also immediately slips back into talking of numbers as objects (e.g. at pp. 74–75).