Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 41: $\mathsf{QL}^=$ proofs

(a) *Use* $\mathsf{QL}^=$ *derivations to show the following inferences are valid:*

Four *very* simple examples as warm-up exercises!

(1) *Mrs Jones isn't Kate. So Kate isn't Mrs Jones.*

Translation $\neg \mathsf{j} = \mathsf{k} \;\therefore\; \neg \mathsf{k} = \mathsf{j}$

Feel free to use shorthand such as $\mathsf{j} \neq \mathsf{k}$ for negated identities. But in these answers I'll mostly write in longhand, with the negations prefixed as usual.

(1)	$\neg \mathsf{j} = \mathsf{k}$	(Prem)
(2)	$\quad \mathsf{k} = \mathsf{j}$	(Supp)
(3)	$\quad \neg \mathsf{j} = \mathsf{j}$	(=E 2, 1)
(4)	$\quad \mathsf{j} = \mathsf{j}$	(=I)
(5)	$\quad \bot$	(Abs 4, 3)
(6)	$\neg \mathsf{k} = \mathsf{j}$	(RAA 2–5)

Here we treat the premiss as having the form $\alpha(\mathsf{k})$ when applying (=E) using $\mathsf{k} = \mathsf{j}$. But of course we could equally well treat the premiss as having the form $\alpha(\mathsf{j})$, apply (=E) using $\mathsf{k} = \mathsf{j}$ to derive $\neg \mathsf{k} = \mathsf{k}$, and so get a different contradiction. Our 'two-sided' version of (=E) allows either: see the comment on exercise (3) below.

(2) *No one who isn't Bryn loves Angharad. At least one person loves Angharad. So Bryn loves Angharad.*

Translation $\forall \mathsf{x}(\neg \mathsf{x} = \mathsf{n} \to \neg \mathsf{Lx}), \exists \mathsf{xLx} \;\therefore\; \mathsf{Ln}$

Note, the internal structure of 'loves Angharard' does no work, so we might as well translate that as an unstructured unary predicate. Since we are by default using $\mathsf{a}, \mathsf{b}, \mathsf{c}$ for dummy names, it wouldn't be the best idea to use b to denote Bryn – though it wouldn't be a wicked mistake either. After all, we are in charge when it comes to notational conventions! And remember, 'at least one' is just the simple existential quantifier. For a proof, we'll need to use the second premiss, presumably by instantiating with a view to a proof by (∃E). So let's try assuming La and aiming for the desired conclusion Ln (which will follow if we can get $\mathsf{a} = \mathsf{n}$).

(1)	$\forall \mathsf{x}(\neg \mathsf{x} = \mathsf{n} \to \neg \mathsf{Lx})$	(Prem)
(2)	$\exists \mathsf{xLx}$	(Prem)
(3)	$\quad \mathsf{La}$	(Supp)
(4)	$\quad \neg\, \mathsf{a} = \mathsf{n} \to \neg \mathsf{La}$	(∀E 1)
(5)	$\quad\quad \neg\, \mathsf{a} = \mathsf{n}$	(Supp)
(6)	$\quad\quad \neg \mathsf{La}$	(MP 5, 4)
(7)	$\quad\quad \bot$	(Abs 3, 6)
(8)	$\quad \neg\neg\, \mathsf{a} = \mathsf{n}$	(RAA 3–7)
(9)	$\quad \mathsf{a} = \mathsf{n}$	(DN 8)
(10)	$\quad \mathsf{Ln}$	(=E 9, 3)
(11)	Ln	(∃E 2 3–10)

(3) *If Clark Kent isn't Superman, then Clark isn't even himself. Superman can fly. So Clark can fly.*

Avoiding the use of c which we officially reserve for dummy names, let's use k to denote Clark Kent. Then the we can render this argument

$(\neg k = s \to \neg k = k),\ Fs\ \therefore\ Fk$

And the proof is trivial.

(1)	$(\neg k = s \to \neg k = k)$	(Prem)
(2)	Fs	(Prem)
(3)	$\quad\mid\ \neg k = s$	(Supp)
(4)	$\quad\mid\ \neg k = k$	(MP 3, 1)
(5)	$\quad\mid\ k = k$	(=I)
(6)	$\quad\mid\ \bot$	(Abs 5, 4)
(7)	$\neg\neg k = s$	(RAA 3–6)
(8)	$k = s$	(DN 7)
(9)	Fk	(=E 8, 2)

The example is a silly one. But it is quite difficult to find more interesting proofs where we need to use (=E). Why so?

If we'd stuck to a 'one-sided' version of Leibniz's Law in the form *from* $\alpha(\tau_1)$ *and* $\tau_1 = \tau_2$ *infer* $\alpha(\tau_2)$, we would have needed to invoke (=E) every time we wanted to warrant an inference of the form *from* $\alpha(\tau_1)$ *and* $\tau_2 = \tau_1$ *infer* $\alpha(\tau_2)$ which has the terms of the identity the other way around (why?). But we have formalized Leibniz's Law – i.e. (=E) – as a two-part rule, covering both forms of inference, with the terms of the identity appearing in either order. And now note that, as soon as we have *any* identity involving the term τ_1, e.g. $\tau_1 = \tau_2$, we can iterate it, then apply our (=E) to infer the self-identity $\tau_1 = \tau_1$! So the special self-identity rule (=I) which allows us to infer $\tau_1 = \tau_1$ from no assumptions is pretty rarely needed!

(4) *The goods were stolen by someone. Whoever stole the goods knew the safe combination. Only Jack knew the safe combination. Hence Jack stole the goods.*

Translation $\exists x Sx,\ \forall x(Sx \to Kx),\ (Kj \land \forall x(Kx \to x = j))\ \therefore\ Sj$

The first premiss is just another way of saying that someone stole the goods. For the third premiss, note that 'only Jack knew' implies that Jack *did* know, hence the first conjunct of the translation, even though it isn't used in the proof as we will see. The 'only' part could be translated either by $\forall x(Kx \to x = j)$ or the contraposed $\forall x(\neg x = j \to \neg Kx)$.

For a proof, we'll need to use the first premiss, presumably by instantiating it with a view to a proof by (∃E). So let's try that:

(1)	$\exists x Sx$	(Prem)
(2)	$\forall x(Sx \to Kx)$	(Prem)
(3)	$(Kj \land \forall x(Kx \to x = j))$	(Prem)
(4)	$\quad\mid\ Sa$	(Supp)
(5)	$\quad\mid\ (Sa \to Ka)$	(∀E 2)
(6)	$\quad\mid\ Ka$	(MP 4, 5)
(7)	$\quad\mid\ \forall x(Kx \to x = j)$	(∧E 3)
(8)	$\quad\mid\ (Ka \to a = j)$	(∀E 7)
(9)	$\quad\mid\ a = j$	(MP 6, 8)
(10)	$\quad\mid\ Sj$	(=E 9, 4)
(11)	Sj	(∃E 1 4–10)

(5) *Take two people (perhaps the same): if the first is taller than the second, the second is not taller than the first. Therefore, if Kurt is taller than Gerhard, they are different people.*

Translation: the first premiss is naturally rendered as a double universal quantification, giving us the rendition

$\forall x \forall y(Txy \to \neg Tyx) \therefore (Tkg \to \neg k = g)$

(1)	$\forall x \forall y(Txy \to \neg Tyx)$	(Prem)
(2)	$\quad Tkg$	(Supp)
(3)	$\quad \forall y(Tky \to \neg Tyk)$	(∀E 2)
(4)	$\quad (Tkg \to \neg Tgk)$	(∀E 3)
(5)	$\quad \neg Tgk$	(MP 2, 4)
(6)	$\quad\quad k = g$	(Supp)
(7)	$\quad\quad Tkk$	(=E 6, 2)
(8)	$\quad\quad Tgk$	(=E 6, 7)
(9)	$\quad\quad \bot$	(Abs)
(10)	$\quad \neg k = g$	(RAA 6–9)
(11)	$(Tkg \to \neg k = g)$	(RAA 2–10)

The only thing to comment on in this proof is that it takes *two* applications of Leibniz's Law (=E) to use $k = g$ to get from Tkg to Tgk: why?

(6) *There is a wise philosopher. There is a philosopher who isn't wise. So there are at least two philosophers.*

Translation: $\exists x(Px \land Wx), \exists x(Px \land \neg Wx) \therefore \exists x \exists y((Px \land Py) \land \neg x = y)$

The proof is a bit long but the proof idea is again very simple. It is obvious that the proof is going to need to *start* by instantiating the two existential quantifications (with two different dummy names of course), with a view to two applications of (∃E). So the overall shape of the proof will be:

$\exists x(Px \land Wx)$	(Prem)
$\exists x(Px \land \neg Wx)$	(Prem)
$\quad (Pa \land Wa)$	(Supp instantiating 1)
$\quad\quad (Pb \land \neg Wb)$	(Supp instantiating 2))
$\quad\quad \vdots$	
$\quad\quad \exists x \exists y((Px \land Py) \land \neg x = y)$	
$\quad \exists x \exists y((Px \land Py) \land \neg x = y)$	(∃E using line 2 and inner subproof)
$\exists x \exists y((Px \land Py) \land \neg x = y)$	(∃E using line 1 and outer subproof)

So how are we going to prove that double quantified existential wff? Presumably by twice existentially quantifying a wff like $((Pa \land Pb) \land \neg a = b)$. What else would be sensible? And how are we going to derive *that*?

Well, we already have both conjuncts of $(Pa \land Pb)$ in play. And we'll prove the negation of $a = b$ by reductio – suppose it true, and then use Leibniz's Law to get a contradiction from Wa and $\neg Wb$. So with all the details filled in we get the following derivation

(1)	$\exists x(Px \land Wx)$	(Prem)
(2)	$\exists x(Px \land \neg Wx)$	(Prem)
(3)	$\quad (Pa \land Wa)$	(Supp)
(4)	$\quad\quad (Pb \land \neg Wb)$	(Supp)
(5)	$\quad\quad\quad a = b$	(Supp)
(6)	$\quad\quad\quad Wa$	(∧E 3)
(7)	$\quad\quad\quad Wb$	(=E 5, 6)
(8)	$\quad\quad\quad \neg Wb$	(∧E 4)
(9)	$\quad\quad\quad \bot$	(Abs 7, 8)
(10)	$\quad\quad \neg a = b$	(RAA 5–9)

(11)	$\mid\ \mid\ \mid\ \mathsf{Pa}$	($\wedge$E 3)
(12)	$\mid\ \mid\ \mid\ \mathsf{Pb}$	($\wedge$E 4)
(13)	$\mid\ \mid\ \mid\ (\mathsf{Pa} \wedge \mathsf{Pb})$	($\wedge$I 11, 12)
(14)	$\mid\ \mid\ \mid\ ((\mathsf{Pa} \wedge \mathsf{Pb}) \wedge \neg\, \mathsf{a} = \mathsf{b})$	($\wedge$I 13, 10)
(15)	$\mid\ \mid\ \mid\ \exists \mathsf{y}((\mathsf{Pa} \wedge \mathsf{Py}) \wedge \neg\, \mathsf{a} = \mathsf{y})$	($\exists$I 14)
(16)	$\mid\ \mid\ \mid\ \exists \mathsf{x} \exists \mathsf{y}((\mathsf{Px} \wedge \mathsf{Py}) \wedge \neg\, \mathsf{x} = \mathsf{y})$	($\exists$I 15)
(17)	$\mid\ \mid\ \exists \mathsf{x} \exists \mathsf{y}((\mathsf{Px} \wedge \mathsf{Py}) \wedge \neg\, \mathsf{x} = \mathsf{y})$	($\exists$E 2 4–16)
(18)	$\mid\ \exists \mathsf{x} \exists \mathsf{y}((\mathsf{Px} \wedge \mathsf{Py}) \wedge \neg\, \mathsf{x} = \mathsf{y})$	($\exists$E 1 3–17)

(7) *Anyone who loves Jo is a logician. Why? Because only one person loves Jo. And some logician loves Jo.*

Translation (using J for *loves Jo*): $\exists \mathsf{x}(\mathsf{Jx} \wedge \forall \mathsf{y}(\mathsf{Jy} \to \mathsf{y} = \mathsf{x}))$, $\exists \mathsf{x}(\mathsf{Lx} \wedge \mathsf{Jx})$ $\therefore$ $\forall \mathsf{x}(\mathsf{Jx} \to \mathsf{Lx})$

Here's the intuitive reasoning:

> Suppose Alex is the only person who loves Jo – so anyone who loves Jo is Alex. And suppose Bertie is a logician who loves Jo (since he loves Jo, Bertie must be Alex again).
>
> OK: now take *anyone*, Charlie, who loves Jo. Then Charlie is Alex again, so is Bertie again, so is a logician. Which is what we said : anyone who loves Jo is a logician (that doesn't depend on our choice of Alex and Bertie, and we are given that there are such people).

Now formalize that intuitive reasoning!

(1)	$\mid\ \exists \mathsf{x}(\mathsf{Jx} \wedge \forall \mathsf{y}(\mathsf{Jy} \to \mathsf{y} = \mathsf{x}))$	(Prem)
(2)	$\mid\ \underline{\exists \mathsf{x}(\mathsf{Lx} \wedge \mathsf{Jx})}$	(Prem)
(3)	$\mid\ \mid\ \underline{(\mathsf{Ja} \wedge \forall \mathsf{y}(\mathsf{Jy} \to \mathsf{y} = \mathsf{a}))}$	(Supp – with a view to using $\exists$E on 1)
(4)	$\mid\ \mid\ \forall \mathsf{y}(\mathsf{Jy} \to \mathsf{y} = \mathsf{a})$	($\wedge$E 3)
(5)	$\mid\ \mid\ \mid\ \underline{(\mathsf{Lb} \wedge \mathsf{Jb})}$	(Supp – with a view to using $\exists$E on 2)
(6)	$\mid\ \mid\ \mid\ \mathsf{Lb}$	($\wedge$E 5)
(7)	$\mid\ \mid\ \mid\ \mathsf{Jb}$	($\wedge$E 5)
(8)	$\mid\ \mid\ \mid\ (\mathsf{Jb} \to \mathsf{b} = \mathsf{a})$	($\forall$E 4)
(9)	$\mid\ \mid\ \mid\ \mathsf{b} = \mathsf{a}$	(MP 7, 8)
(10)	$\mid\ \mid\ \mid\ \mid\ \underline{\mathsf{Jc}}$	(Supp – with a view to proving Lc)
(11)	$\mid\ \mid\ \mid\ \mid\ (\mathsf{Jc} \to \mathsf{c} = \mathsf{a})$	($\forall$E 4)
(12)	$\mid\ \mid\ \mid\ \mid\ \mathsf{c} = \mathsf{a}$	(MP 10, 11)
(13)	$\mid\ \mid\ \mid\ \mid\ \mathsf{b} = \mathsf{c}$	(=E 12, 9)
(14)	$\mid\ \mid\ \mid\ \mid\ \mathsf{Lc}$	(=E 13, 6)
(15)	$\mid\ \mid\ \mid\ (\mathsf{Jc} \to \mathsf{Lc})$	(CP 10–14)
(16)	$\mid\ \mid\ \mid\ \forall \mathsf{x}(\mathsf{Jx} \to \mathsf{Lx})$	($\forall$I 15 – NB c is in no live assumption)
(17)	$\mid\ \mid\ \forall \mathsf{x}(\mathsf{Jx} \to \mathsf{Lx})$	($\exists$E 2 5–16)
(18)	$\mid\ \forall \mathsf{x}(\mathsf{Jx} \to \mathsf{Lx})$	($\exists$E 1 3–17)

(8) *For any number, there's a larger one. There is no number which is larger than itself. So for any number, there's a distinct number which is larger than it.*

Take the domain of quantification, obviously enough, to be numbers. We could use e.g. L to translate 'larger than'. But why don't we use the conventional '$>$', written infix (remember we do allow departures from the default predicate-first ordering in writing wffs, and this seems a natural candidate case).

Then the inference gets translated

$$\forall x \exists y\, y > x, \neg \exists x\, x > x \;\therefore\; \forall x \exists y(\neg y = x \wedge y > x)$$

As we said before, by all means, write $y \neq x$ for $\neg y = x$, and similarly for other negated identities.

How are we going to start the proof? We'll instantiate the first premiss, which gives us an existential quantification at line (3). We then do the obvious thing – in other words, we instantiate *that*.

And what to we want to aim for? Presumably the conclusion is going to be derived by universally generalizing something like $\exists y(\neg y = a \wedge y > a)$, and we'd expect to get that by existentially generalizing something like $(\neg b = a \wedge b > a)$.

We those thoughts in mind, the proof then falls out very easily!

(1)	$\forall x \exists y\, y > x$	(Prem)
(2)	$\neg \exists x\, x > x$	(Prem)
(3)	$\exists y\, y > a$	($\forall$E 1)
(4)	$\quad b > a$	(Supp)
(5)	$\qquad b = a$	(Supp)
(6)	$\qquad a > a$	(=E 5, 4)
(7)	$\qquad \exists x\, x > x$	($\exists$I 6)
(8)	$\qquad \bot$	(Abs 7, 2)
(9)	$\quad \neg\, b = a$	(RAA 5–8)
(10)	$\quad (\neg\, b = a \wedge b > a)$	($\wedge$I 9, 4)
(11)	$\quad \exists y(\neg y = a \wedge y > a)$	($\exists$I 10)
(12)	$\exists y(\neg y = a \wedge y > a)$	($\exists$E 3 4–11)
(13)	$\forall x \exists y(\neg y = x \wedge y > x)$	($\exists$I 13)

(9) *Exactly one person admires Frank. All and only those who admire Frank love him. Hence exactly one person loves Frank.*

The translation is straightforward. Note, we might as well use simply A for *admires Frank*, and L for *loves Frank*. If you wrote the likes of Axf instead of Ax, no harm done of course!

$$\exists x(Ax \wedge \forall y(Ay \rightarrow y = x)),\ \forall x((Ax \rightarrow Lx) \wedge (Lx \rightarrow Ax)) \;\therefore\; \exists x(Lx \wedge \forall y(Ly \rightarrow y = x))$$

Given the existential premiss and the existential conclusion, we'll (by now!) expect a proof that looks overall like this:

$\exists x(Ax \wedge \forall y(Ay \rightarrow y = x))$	(Prem)
$\forall x((Ax \rightarrow Lx) \wedge (Lx \rightarrow Ax))$	(Prem)
$\quad (Aa \wedge \forall y(Ay \rightarrow y = a))$	(Supp)
$\quad \vdots$	
$\quad (La \wedge \forall y(Ly \rightarrow y = a))$	($\wedge$I 7, 16)
$\quad \exists x(Lx \wedge \forall y(Ly \rightarrow y = x))$	($\exists$I)
$\exists x(Lx \wedge \forall y(Ly \rightarrow y = x))$	($\exists$E)

So we need to prove a conjunction in the subproof:

i We will get La from Aa from our supposition and conditional $(Aa \rightarrow La)$ that we get from instantiating the second premiss.

ii We will get $\forall y(Ly \rightarrow y = a)$ by generalizing on some singular conditional $(Lb \rightarrow b = a)$, and we will get that conditional by assuming its antecedent Lb and deriving its consequent $b = a$.

It's stage (ii) that requires a moment's thought – but look at the inputs we have to play with. The second premiss again gives us $(Lb \rightarrow Ab)$ and hence Ab; and the as-yet-unused second conjunct of our supposition will give us $(Ab \rightarrow b = a)$ and hence $b = a$.

So putting everything together we get

(1)	$\exists x(\mathsf{A}x \wedge \forall y(\mathsf{A}y \rightarrow y = x))$	(Prem)
(2)	$\forall x((\mathsf{A}x \rightarrow \mathsf{L}x) \wedge (\mathsf{L}x \rightarrow \mathsf{A}x))$	(Prem)
(3)	$(\mathsf{Aa} \wedge \forall y(\mathsf{A}y \rightarrow y = a))$	(Supp)
(4)	Aa	(∧E 3)
(5)	$((\mathsf{Aa} \rightarrow \mathsf{La}) \wedge (\mathsf{La} \rightarrow \mathsf{Aa}))$	(∀E 2)
(6)	$(\mathsf{Aa} \rightarrow \mathsf{La})$	(∧E 5)
(7)	La	(MP 4, 6)
(8)	Lb	(Supp)
(9)	$((\mathsf{Ab} \rightarrow \mathsf{Lb}) \wedge (\mathsf{Lb} \rightarrow \mathsf{Ab}))$	(∀E 2)
(10)	$(\mathsf{Lb} \rightarrow \mathsf{Ab})$	(∧E 9)
(11)	Ab	(MP 8, 10)
(12)	$\forall y(\mathsf{A}y \rightarrow y = a)$	(∧E 3)
(13)	$(\mathsf{Ab} \rightarrow b = a)$	(∀E 12)
(14)	$b = a$	(MP 11, 13)
(15)	$(\mathsf{Lb} \rightarrow b = a)$	(CP 8–14)
(16)	$\forall y(\mathsf{L}y \rightarrow y = a)$	(∀I 15)
(17)	$(\mathsf{La} \wedge \forall y(\mathsf{Lb} \rightarrow b = a))$	(∧I 7, 16)
(18)	$\exists x(\mathsf{L}x \wedge \forall y(\mathsf{L}y \rightarrow y = x))$	(∃I 17)
(19)	$\exists x(\mathsf{L}x \wedge \forall y(\mathsf{L}y \rightarrow y = x))$	(∃E 1, 3–18)

(10) *The present King of France is bald. Bald men are sexy. Hence whoever is a present King of France is sexy.*

Translation: $\exists x((\mathsf{K}x \wedge \forall y(\mathsf{K}y \rightarrow y = x)) \wedge \mathsf{B}x),\ \forall x(\mathsf{B}x \rightarrow \mathsf{S}x)\ \therefore\ \forall x(\mathsf{K}x \rightarrow \mathsf{S}x)$

(1)	$\exists x((\mathsf{K}x \wedge \forall y(\mathsf{K}y \rightarrow y = x)) \wedge \mathsf{B}x)$	(Prem)
(2)	$\forall x(\mathsf{B}x \rightarrow \mathsf{S}x)$	(Prem)
(3)	$((\mathsf{Ka} \wedge \forall y(\mathsf{K}y \rightarrow y = a)) \wedge \mathsf{Ba})$	(Supp)
(4)	$(\mathsf{Ka} \wedge \forall y(\mathsf{K}y \rightarrow y = a))$	(∀E 3)
(5)	Ba	(∀E 3)
(6)	$\forall y(\mathsf{K}y \rightarrow y = a)$	(∧E 4)
(7)	Kb	(Supp)
(8)	$(\mathsf{Kb} \rightarrow b = a)$	(∀E 5)
(9)	$b = a$	(∧E 9)
(10)	Bb	(=E 6,)
(11)	$(\mathsf{Bb} \rightarrow \mathsf{Sb})$	(∀E 2)
(12)	Sb	(MP 10, 11)
(13)	$(\mathsf{Kb} \rightarrow \mathsf{Sb})$	(CP 7–12)
(14)	$\forall x(\mathsf{K}x \rightarrow \mathsf{S}x)$	(∀I 13)
(15)	$\forall x(\mathsf{K}x \rightarrow \mathsf{S}x)$	(∃E 1, 3–14)

(11) *Someone is a logician. But no one is the only logician. Therefore there at least two logicians.*

The only translational issue is how to render the second premiss. We can read that as *It isn't the case that there is someone who is a logician while no one else is*, so the argument can be rendered

$$\exists x\mathsf{L}x,\ \neg\exists x(\mathsf{L}x \wedge \forall y(\mathsf{L}y \rightarrow y = x))\ \therefore\ \exists x\exists y((\mathsf{L}x \wedge \mathsf{L}y) \wedge \neg x = y).$$

The derivation naturally starts by instantiating with first premiss, with a view to using (∃E) like this:

	$\exists x Lx$	(Prem)
	$\neg\exists x(Lx \wedge \forall y(Ly \rightarrow y = x))$	(Prem)
	$\quad La$	(Supp)
	$\quad \vdots$	
	$\quad \exists x\exists y(Lx \wedge Ly) \wedge \neg x = y)$	
	$\exists x\exists y(Lx \wedge Ly) \wedge \neg x = y)$	($\exists$E)

And how are we going to use the second premiss? A now familiar trick when dealing with a negated existential, $\neg\exists x\alpha(x))$, assume something β that allows us to prove $\alpha(a)$, hence $\exists x\alpha(x))$, hence absurdity – establishing $\neg\beta$.

So that suggests starting our proof like this:

(1)	$\exists x Lx$	(Prem)
(2)	$\neg\exists x(Lx \wedge \forall y(Ly \rightarrow y = x))$	(Prem)
(3)	$\quad La$	(Supp)
(4)	$\quad\quad \forall y(Ly \rightarrow y = a))$	(Supp)
(5)	$\quad\quad (La \wedge \forall y(Ly \rightarrow y = a))$	($\wedge$I 3, 4)
(6)	$\quad\quad \exists x(Lx \wedge \forall y(Ly \rightarrow y = x))$	($\exists$I 5)
(7)	$\quad\quad \bot$	(Abs 6, 2)
(8)	$\quad \neg\forall y(Ly \rightarrow y = a))$	(RAA 4–7)

Now, at this stage in the game, let's cut ourselves some slack and allow ourselves some short cuts. We'll use the derived rule which allows us to argue from $\neg\forall x$ to $\exists x\neg$. That gives us an existential quantification to instantiate, and we can use PL reasoning to unpack the result, continuing the proof like this:

(9)	$\quad \exists y\neg(Ly \rightarrow y = a))$	($\neg\forall$ 8)
(10)	$\quad\quad \neg(Lb \rightarrow b = a)$	(Supp)
(11)	$\quad\quad Lb$	(PL 10)
(12)	$\quad\quad \neg b = a$	(PL 10)

And now we just need to put together the ingredients as lines (3), (11) and (12) to complete the proof like this. Just note the order in which we need to arrange the conjuncts.

(13)	$\quad\quad (Lb \wedge La)$	($\wedge$I)
(14)	$\quad\quad (Lb \wedge La) \wedge \neg b = a)$	($\wedge$I)
(15)	$\quad\quad \exists y(Lb \wedge Ly) \wedge \neg b = y)$	($\exists$I)
(16)	$\quad\quad \exists x\exists y(Lx \wedge Ly) \wedge \neg x = y)$	($\exists$I)
(17)	$\quad \exists x\exists y(Lx \wedge Ly) \wedge \neg x = y)$	($\exists$E)
(18)	$\exists x\exists y(Lx \wedge Ly) \wedge \neg x = y)$	($\exists$E)

(b) *The following wffs are alternative renderings of a claim of the form* The F is G*:*

(R) $\exists x((Fx \wedge \forall y(Fy \rightarrow y = x)) \wedge Gx)$

(R′) $\exists x\forall y((Fy \leftrightarrow y = x) \wedge Gx)$

(R″) $(\{\exists x Fx \wedge \forall x\forall y((Fx \wedge Fy) \rightarrow x = y)\} \wedge \forall x(Fx \rightarrow Gx))$.

We claimed that from each wff we can derive the other two using a $QL^=$ *proof. Give at least three of the six required proofs. Remember, for us an expression of the form* $(\alpha \leftrightarrow \beta)$ *simply abbreviates the corresponding expression* $((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$*.*

(I) First we show that (R) implies (R′).

Evidently, we want a proof shaped like this:

$\exists x((Fx \land \forall y(Fy \to y = x)) \land Gx)$	(Prem)
$\quad ((Fa \land \forall y(Fy \to y = a)) \land Ga)$	(Supp)
$\quad \vdots$	
$\quad \exists x \forall y(((Fy \to y = x) \land (y = x \to Fy)) \land Gx)$	
$\exists x \forall y(((Fy \to y = x) \land (y = x \to Fy)) \land Gx)$	(∃E)
$\exists x \forall y((Fy \leftrightarrow y = x) \land Gx)$	(Abbreviating using ↔)

And how is the subproof going to end?

Surely with an existential quantification on *some* dummy name – and since we already have Ga, surely we are going to be using the dummy name a, like this:

$\quad ((Fa \land \forall y(Fy \to y = a)) \land Ga)$	(Supp)
$\quad \vdots$	
$\quad \forall y(((Fy \to y = a) \land (y = a \to Fy)) \land Ga)$	
$\quad \exists x \forall y(((Fy \to y = x) \land (y = x \to Fy)) \land Gx)$	(∃I)

And where are we going to get that universally quantified wff from? Presumably by quantifying on some other dummy name (not a of course, as that's in a supposition, and so we can't universally quantify on *that*). So we'll expect the subproof to go

$\quad ((Fa \land \forall y(Fy \to y = a)) \land Ga)$	(Supp)
$\quad \vdots$	
$\quad (((Fb \to b = a) \land (b = a \to Fb)) \land Ga)$	(b is in no live assumption)
$\quad \forall y(((Fy \to y = a) \land (y = a \to Fy)) \land Ga)$	(∀I)
$\quad \exists x \forall y(((Fy \to y = x) \land (y = x \to Fy)) \land Gx)$	(∃I)

And *now* it is clear what we have to do. We disassemble the conjunction in our supposition at the top of the subproof, giving us lines (3) to (6) below. Then we aim to derive the three simple conjuncts in that target wff, i.e. in what becomes line (12). Let's signal the conjuncts by starring them when we reach them in turn!

(1)	$\exists x((Fx \land \forall y(Fy \to y = x)) \land Gx)$	(Prem)
(2)	$\quad ((Fa \land \forall y(Fy \to y = a)) \land Ga)$	(Supp)
(3)	$\quad (Fa \land \forall y(Fy \to y = a))$	(∧E 2)
(4)	$\quad Ga^*$	(∧E 2)
(5)	$\quad Fa$	(∧E 3)
(6)	$\quad \forall y(Fy \to y = a)$	(∧E 3)
(7)	$\quad (Fb \to b = a)^*$	(∀E 6)
(8)	$\quad\quad b = a$	(Supp)
(9)	$\quad\quad Fb$	(=E 8, 5)
(10)	$\quad (b = a \to Fb)^*$	(CP 8–9)
(11)	$\quad ((Fb \to b = a) \land (b = a \to Fb))$	(∧I 7, 10)
(12)	$\quad (((Fb \to b = a) \land (b = a \to Fb)) \land Ga)$	(∧I 11, 4)
(13)	$\quad \forall y(((Fy \to y = a) \land (y = a \to Fy)) \land Ga)$	(∀I 12 – NB b is in no live assumption)
(14)	$\quad \exists x \forall y(((Fy \to y = x) \land (y = x \to Fy)) \land Gx)$	(∃I 13)
(15)	$\exists x \forall y(((Fy \to y = x) \land (y = x \to Fy)) \land Gx)$	(∃E 1, 2–14)
(16)	$\exists x \forall y((Fy \leftrightarrow y = x) \land Gx)$	(Abbreviating using ↔)

(II) Next, we show that (R) implies (R″). And evidently, we want a proof shaped like this:

$\exists x((Fx \wedge \forall y(Fy \rightarrow y = x)) \wedge Gx)$	(Prem)
$\quad ((Fa \wedge \forall y(Fy \rightarrow y = a)) \wedge Ga)$	(Supp)
$\quad \vdots$	
$\quad (\{\exists xFx \wedge \forall x\forall y((Fx \wedge Fy) \rightarrow x = y)\} \wedge \forall x(Fx \rightarrow Gx))$	
$(\{\exists xFx \wedge \forall x\forall y((Fx \wedge Fy) \rightarrow x = y)\} \wedge \forall x(Fx \rightarrow Gx))$	(∃E)

So, after disassembling that conjunctive supposition, we have three new conjuncts to establish!

(i) $\exists xFx$ trivially follows from Fa

(ii) The strategy for proving $\forall x\forall y((Fx \wedge Fy) \rightarrow x = y)$ is to aim to establish the conditional $((Fb \wedge Fc) \rightarrow b = c)$, with new dummy names b and$c$ so we can universally generalize. And, like any conditional, the natural way of trying to establish *that* is by a conditional proof.

(iii) The strategy for proving $\forall x(Fx \rightarrow Gx)$ is to aim to establish the conditional $(Fd \rightarrow Fd)$, with a new dummy name d so we can universally generalize. And, again, the natural way of trying to establish *that* conditional is by a conditional proof.

So here goes, again starring the desired conjuncts when we reach them, after which we have to set off on the next stage of the proof:

(1)	$\exists x((Fx \wedge \forall y(Fy \rightarrow y = x)) \wedge Gx)$	(Prem)
(2)	$\quad ((Fa \wedge \forall y(Fy \rightarrow y = a)) \wedge Ga)$	(Supp)
(3)	$\quad (Fa \wedge \forall y(Fy \rightarrow y = a))$	(∧E 2)
(4)	$\quad Ga$	(∧E 2)
(5)	$\quad Fa$	(∧E 3)
(6)	$\quad \forall y(Fy \rightarrow y = a)$	(∧E 3)
(7)	$\quad \exists xFx^*$	(∃I 5)
(8)	$\quad\quad (Fb \wedge Fc)$	(Supp)
(9)	$\quad\quad Fb$	(∧E 8)
(10)	$\quad\quad (Fb \rightarrow b = a)$	(∀E 6)
(11)	$\quad\quad b = a$	(MP 9, 10)
(12)	$\quad\quad Fc$	(∧E 8)
(13)	$\quad\quad (Fc \rightarrow c = a)$	(∀E 6)
(14)	$\quad\quad c = a$	(MP 12, 13)
(15)	$\quad\quad b = c$	(=E 14, 11)
(16)	$\quad ((Fb \wedge Fc) \rightarrow b = c)$	(CP 8 – 15)
(17)	$\quad \forall y((Fb \wedge Fy) \rightarrow b = y)$	(∀I 16)
(18)	$\quad \forall x\forall y((Fx \wedge Fy) \rightarrow x = y)^*$	(∀I 17)
(19)	$\quad\quad Fd$	(Supp)
(20)	$\quad\quad (Fd \rightarrow d = a)$	(∀E 6)
(21)	$\quad\quad d = a$	(MP 19, 20)
(22)	$\quad\quad Gd$	(=E 21, 4)
(23)	$\quad (Fd \rightarrow Gd)$	(CP 19–22)
(24)	$\quad \forall x(Fx \rightarrow Gx)^*$	(∀I 23)
(25)	$\quad \{\exists xFx \wedge \forall x\forall y((Fx \wedge Fy) \rightarrow x = y)\}$	(∧I 7, 18)
(26)	$\quad (\{\exists xFx \wedge \forall x\forall y((Fx \wedge Fy) \rightarrow x = y)\} \wedge \forall x(Fx \rightarrow Gx))$	(∧I 25, 24)
(27)	$(\{\exists xFx \wedge \forall x\forall y((Fx \wedge Fy) \rightarrow x = y)\} \wedge \forall x(Fx \rightarrow Gx))$	(∃E 1 2–26)

So were are done. (Of course, we could have re-used b or c as the dummy variable introduced at line (19): using a new name is just always a good policy to avoid accidental tangles!)

(III) Next, we show that (R′) implies (R). So we want a proof shaped like this:

$\exists x\forall y((Fy \leftrightarrow y = x) \wedge Gx)$	(Prem)
$\exists x\forall y(((Fy \rightarrow y = x) \wedge (y = x \rightarrow Fy)) \wedge Gx)$	(Unpacking the abbreviation $\leftrightarrow$)
$\quad \forall y(((Fy \rightarrow y = a) \wedge (y = a \rightarrow Fy)) \wedge Ga)$	(Supp)
$\quad \vdots$	
$\quad ((Fa \wedge \forall y(Fy \rightarrow y = a)) \wedge Ga)$	
$\quad \exists x((Fx \wedge \forall y(Fy \rightarrow y = x)) \wedge Gx)$	($\exists$I)
$\exists x((Fx \wedge \forall y(Fy \rightarrow y = x)) \wedge Gx)$	($\exists$E using unpacked premiss)

Here the proof starts by unpacking the abbreviation for the two-way conditional, and then we set off on a proof by ($\exists$E) in the usual way, aiming for the desired conclusion (R). But (r) which is an existential quantification which we are surely going to establish by establishing an instance, as suggested.

So our target now conjoins three wffs:

(i) Fa follows because our supposition implies $(a = a \rightarrow Fa)$ (just instantiate the quantifier with a!), and the antecedent is trivially true.

(ii) $\forall y(Fy \rightarrow y = a)$ can be extracted from the supposition. Though to do this, we'll have to take an instance of the quantification (with some new dummy name), and disassemble before quantifying gain.

(iii) Ga we can just extract from the supposition.

So with that plan, here's a full proof, again punctuated by starring the desired conjunctions:

(1)	$\exists x\forall y((Fy \leftrightarrow y = x) \wedge Gx)$	(Prem)
(2)	$\exists x\forall y(((Fy \rightarrow y = x) \wedge (y = x \rightarrow Fy)) \wedge Gx)$	(Unpacking the abbreviation $\leftrightarrow$)
(3)	$\quad \forall y(((Fy \rightarrow y = a) \wedge (y = a \rightarrow Fy)) \wedge Ga)$	(Supp)
(4)	$\quad (((Fa \rightarrow a = a) \wedge (a = a \rightarrow Fa)) \wedge Ga)$	($\forall$E 3)
(5)	$\quad ((Fa \rightarrow a = a) \wedge (a = a \rightarrow Fa))$	($\wedge$E 4)
(6)	$\quad (a = a \rightarrow Fa)$	($\wedge$E 5)
(7)	$\quad a = a$	(=I)
(8)	$\quad Fa^*$	(MP 7, 6)
(9)	$\quad (((Fb \rightarrow b = a) \wedge (b = a \rightarrow Fb)) \wedge Ga)$	($\forall$E 3)
(10)	$\quad ((Fb \rightarrow b = a) \wedge (b = a \rightarrow Fb))$	($\wedge$E 9)
(11)	$\quad (Fb \rightarrow b = a)$	($\wedge$E 10)
(12)	$\quad \forall y(Fy \rightarrow y = a)^*$	($\forall$I 11)
(13)	$\quad Ga^*$	($\wedge$E 4)
(14)	$\quad (Fa \wedge \forall y(Fy \rightarrow y = a))$	($\wedge$I 8, 12)
(15)	$\quad ((Fa \wedge \forall y(Fy \rightarrow y = a)) \wedge Ga)$	($\wedge$I 14, 13)
(16)	$\quad \exists x((Fx \wedge \forall y(Fy \rightarrow y = x)) \wedge Gx)$	($\exists$I 15)
(17)	$\exists x((Fx \wedge \forall y(Fy \rightarrow y = x)) \wedge Gx)$	($\exists$E 2, 3–16)

(IV) We show that (R′) implies (R″). So we want a proof shaped like this:

$\exists x\forall y((Fy \leftrightarrow y = x) \wedge Gx)$	(Prem)
$\exists x\forall y(((Fy \rightarrow y = x) \wedge (y = x \rightarrow Fy)) \wedge Gx)$	(Unpacking the abbreviation $\leftrightarrow$)
$\quad \forall y(((Fy \rightarrow y = a) \wedge (y = a \rightarrow Fy)) \wedge Ga)$	(Supp)
$\quad \vdots$	
$\quad (\{\exists xFx \wedge \forall x\forall y((Fx \wedge Fy) \rightarrow x = y)\} \wedge \forall x(Fx \rightarrow Gx))$	
$(\{\exists xFx \wedge \forall x\forall y((Fx \wedge Fy) \rightarrow x = y)\} \wedge \forall x(Fx \rightarrow Gx))$	($\exists$E)

Again our target conjoins three wffs. How are we going to prove them?

(i) Fa follows as in (III). So that entails $\exists \mathsf{xFx}$, as wanted.

(ii) As in (II), we prove $\forall \mathsf{x} \forall \mathsf{y}((\mathsf{Fx} \land \mathsf{Fy}) \to \mathsf{x} = \mathsf{y})$ by establishing the conditional $((\mathsf{Fb} \land \mathsf{Fc}) \to \mathsf{b} = \mathsf{c})$, with new dummy names b and c so we can universally generalize. By our universally quantified supposition, Fb implies $\mathsf{b} = \mathsf{a}$, Fc implies $\mathsf{c} = \mathsf{a}$. So if we have Fb *and* Fc the two identities will give us $\mathsf{b} = \mathsf{c}$.

(iii) As in (II), we prove $\forall \mathsf{x}(\mathsf{Fx} \to \mathsf{Gx})$ by establishing the conditional $(\mathsf{Fd} \to \mathsf{Gd})$, with a new dummy name d so we can universally generalize. By our supposition, Fd implies $\mathsf{d} = \mathsf{a}$. And that identity together with Ga gives us Gd.

So with that plan, here's a full proof (long, but not difficult if we break it into stages):

(1)	$\exists x \forall y((Fy \leftrightarrow y = x) \land Gx)$	(Prem)
(2)	$\exists x \forall y(((Fy \to y = x) \land (y = x \to Fy)) \land Gx)$	(Unpacking the abbreviation $\leftrightarrow$)
(3)	$\forall y(((Fy \to y = a) \land (y = a \to Fy)) \land Ga)$	(Supp)
(4)	$(((Fa \to a = a) \land (a = a \to Fa)) \land Ga)$	($\forall$E 3)
(5)	$((Fa \to a = a) \land (a = a \to Fa))$	($\land$E 4)
(6)	$(a = a \to Fa)$	($\land$E 5)
(7)	$a = a$	(=I)
(8)	Fa	(MP 7, 6)
(9)	$\exists xFx^*$	($\exists$I 8)
(10)	$(Fb \land Fc)$	(Supp)
(11)	Fb	($\land$E 10)
(12)	$(((Fb \to b = a) \land (b = a \to Fb)) \land Ga)$	($\forall$E 3)
(13)	$((Fb \to b = a) \land (b = a \to Fb))$	($\land$E 9)
(14)	$(Fb \to b = a)$	($\land$E 10)
(15)	$b = a$	(MP 11, 14)
(16)	Fc	($\land$E 10)
(17)	$(((Fc \to c = a) \land (c = a \to Fc)) \land Ga)$	($\forall$E 3)
(18)	$((Fc \to c = a) \land (c = a \to Fc))$	($\land$E 17)
(19)	$(Fc \to c = a)$	($\land$E 18)
(20)	$c = a$	(MP 16, 19)
(21)	$b = c$	(=E 20, 15)
(22)	$((Fb \land Fc) \to b = c)$	(CP 10–21)
(23)	$\forall y((Fb \land Fy) \to b = y)$	($\forall$I 22)
(24)	$\forall x \forall y((Fx \land Fy) \to x = y)^*$	($\forall$I 23)
(25)	Fd	(Supp)
(26)	$(((Fd \to d = a) \land (d = a \to Fd)) \land Ga)$	($\forall$E 3)
(27)	$((Fd \to d = a) \land (d = a \to Fd))$	($\land$E 26)
(28)	$(Fd \to d = a)$	($\land$E 27)
(29)	$d = a$	(MP 25, 28)
(30)	Ga	($\land$E 26)
(31)	Gd	(=E 29, 30)
(32)	$(Fd \to Gd)$	(CP 25–31)
(33)	$\forall x(Fx \to Gx)^*$	($\forall$I 32)
(34)	$\{\exists xFx \land \forall x \forall y((Fx \land Fy) \to x = y)\}$	($\land$I 2, 24)
(35)	$(\{\exists xFx \land \forall x \forall y((Fx \land Fy) \to x = y)\} \land \forall x(Fx \to Gx))$	($\land$I 34, 33)
(36)	$(\{\exists xFx \land \forall x \forall y((Fx \land Fy) \to x = y)\} \land \forall x(Fx \to Gx))$	($\exists$E 2 3–35)

(V) We show that (R$''$) implies (R). So by now we will expect a proof shaped like this! –

$$\begin{array}{ll}
(\{\exists x Fx \wedge \forall x \forall y((Fx \wedge Fy) \to x = y)\} \wedge \forall x(Fx \to Gx)) & \text{(Prem)} \\
\exists x Fx & \text{(extracting conjuncts!)} \\
\forall x \forall y((Fx \wedge Fy) \to x = y) & \text{(extracting conjuncts!)} \\
\forall x(Fx \to Gx) & \text{(extracting conjuncts!)} \\
\quad Fa & \text{(Supp)} \\
\quad \vdots & \\
\quad ((Fa \wedge \forall y(Fy \to y = a)) \wedge Ga) & \\
\quad \exists x((Fx \wedge \forall y(Fy \to y = x)) \wedge Gx) & (\exists\text{I}) \\
\exists x((Fx \wedge \forall y(Fy \to y = x)) \wedge Gx) & (\exists\text{E})
\end{array}$$

Again our target conjoins three wffs. How are we going to prove them?

(i) Fa is immediate

(ii) To prove $\forall y(Fy \to y = a)$ we want to establish the conditional $(Fb \to b = a)$, with the new dummy name b so we can universally generalize. So we assume Fb and aim for $b = a$. Obviously we use the doubly universally quantified conjunct we've already extracted to do the job, and we just stay alert to which of the dummy names a and b we need to instantiate which variable to get the identity the desired way around.

(iii) To derive Ga we just instantiate $\forall x(Fx \to Gx)$ and then apply modus ponens.

Which gives us the following proof:

$$\begin{array}{lll}
(1) & (\{\exists x Fx \wedge \forall x \forall y((Fx \wedge Fy) \to x = y)\} \wedge \forall x(Fx \to Gx)) & \text{(Prem)} \\
(2) & \{\exists x Fx \wedge \forall x \forall y((Fx \wedge Fy) \to x = y)\} & (\wedge\text{E } 1) \\
(3) & \exists x Fx & (\wedge\text{E } 2) \\
(4) & \forall x \forall y((Fx \wedge Fy) \to x = y) & (\wedge\text{E } 2) \\
(5) & \forall x(Fx \to Gx) & (\wedge\text{E } 1) \\
(6) & \quad Fa^* & \text{(Supp)} \\
(7) & \quad\quad Fb & \text{(Supp)} \\
(8) & \quad\quad (Fb \wedge Fa) & (\wedge\text{I } 7, 6) \\
(9) & \quad\quad \forall y((Fb \wedge Fy) \to b = y) & (\forall\text{E } 4) \\
(10) & \quad\quad ((Fb \wedge Fa) \to b = a) & (\forall\text{E } 9) \\
(11) & \quad\quad b = a & (\text{MP } 8 – 10) \\
(12) & \quad (Fb \to b = a) & (\text{CP } 7–11) \\
(13) & \quad \forall y(Fy \to y = a)^* & (\forall\text{I } 12) \\
(14) & \quad (Fa \to Ga) & (\forall\text{E } 5) \\
(15) & \quad Ga^* & (\text{MP } 6, 14) \\
(16) & \quad (Fa \wedge \forall y(Fy \to y = a)) & (\wedge\text{I } 6, 13) \\
(17) & \quad ((Fa \wedge \forall y(Fy \to y = a)) \wedge Ga) & (\wedge\text{I } 16, 15) \\
(18) & \quad \exists x((Fx \wedge \forall y(Fy \to y = x)) \wedge Gx) & (\exists\text{I } 17) \\
(19) & \exists x((Fx \wedge \forall y(Fy \to y = x)) \wedge Gx) & (\exists\text{E } 3, 6–18)
\end{array}$$

(VI) Finally we need to show that (R$''$) implies (R$'$).

By now we will expect a proof of the following shape. We first disassemble the conjunctive premiss, and use an instance of the existential component $\exists x Fx$ to set off on a proof by ($\exists$E). At the end of the proof, we want to get to a wff of the form $\exists x \forall y \varphi(x, y)$ – which will involve a universal quantification which we then existentially quantify, like this:

$(\{\exists x Fx \wedge \forall x \forall y((Fx \wedge Fy) \rightarrow x = y)\} \wedge \forall x(Fx \rightarrow Gx))$	(Prem)
$\exists x Fx$	(extracting conjuncts!)
$\forall x \forall y((Fx \wedge Fy) \rightarrow x = y)$	(extracting conjuncts!)
$\forall x(Fx \rightarrow Gx)$	(extracting conjuncts!)
$\quad \mid\ Fa$	(Supp)
$\quad \mid\ \vdots$	
$\quad \mid\ (((Fb \rightarrow b = a) \wedge (b = a \rightarrow Fb)) \wedge Ga)$	
$\quad \mid\ \forall y(((Fy \rightarrow y = a) \wedge (y = a \rightarrow Fy)) \wedge Ga)$	(∀I)
$\quad \mid\ \exists x \forall y(((Fy \rightarrow y = x) \wedge (y = x \rightarrow Fy)) \wedge Gx)$	(∃I)
$\exists x \forall y(((Fy \rightarrow y = x) \wedge (y = x \rightarrow Fy)) \wedge Gx)$	(∃E)
$\exists x \forall y((Fy \leftrightarrow y = x) \wedge Gx)$	(Abbreviating using ↔)

Filling in the details is now pretty straightforward – once more, we use stars to signal when we reach each desired initial target which we are going to conjoin and quantify.

(1)	$(\{\exists x Fx \wedge \forall x \forall y((Fx \wedge Fy) \rightarrow x = y)\} \wedge \forall x(Fx \rightarrow Gx))$	(Prem)
(2)	$\{\exists x Fx \wedge \forall x \forall y((Fx \wedge Fy) \rightarrow x = y)\}$	(∧E 1)
(3)	$\exists x Fx$	(∧E 2)
(4)	$\forall x \forall y((Fx \wedge Fy) \rightarrow x = y)$	(∧E 2)
(5)	$\forall x(Fx \rightarrow Gx)$	(∧E 1)
(6)	$\quad \mid\ Fa$	(Supp)
(7)	$\quad \mid \quad \mid\ Fb$	(Supp)
(8)	$\quad \mid \quad \mid\ (Fb \wedge Fa)$	(∧I 7, 6)
(9)	$\quad \mid \quad \mid\ \forall y((Fb \wedge Fy) \rightarrow b = y)$	(∀E 4)
(10)	$\quad \mid \quad \mid\ ((Fb \wedge Fa) \rightarrow b = a)$	(∀E 9)
(11)	$\quad \mid \quad \mid\ b = a$	(MP 8 –10)
(12)	$\quad \mid\ (Fb \rightarrow b = a)^*$	(CP 7–11)
(13)	$\quad \mid \quad \mid\ b = a$	(Supp)
(14)	$\quad \mid \quad \mid\ Fb$	(=E 13, 6)
(15)	$\quad \mid\ (b = a \rightarrow Fb)^*$	(CP 13–14)
(16)	$\quad \mid\ (Fa \rightarrow Ga)$	(∀E 5)
(17)	$\quad \mid\ Ga^*$	(MP 6, 16)
(18)	$\quad \mid\ ((Fb \rightarrow b = a) \wedge (b = a \rightarrow Fb))$	(∧I 12, 15)
(19)	$\quad \mid\ (((Fb \rightarrow b = a) \wedge (b = a \rightarrow Fb)) \wedge Ga)$	(∧I 18, 17)
(20)	$\quad \mid\ \forall y(((Fy \rightarrow y = a) \wedge (y = a \rightarrow Fy)) \wedge Ga)$	(∀I 19)
(21)	$\quad \mid\ \exists x \forall y(((Fy \rightarrow y = x) \wedge (y = x \rightarrow Fy)) \wedge Gx)$	(∃I 20)
(22)	$\exists x \forall y(((Fy \rightarrow y = x) \wedge (y = x \rightarrow Fy)) \wedge Gx)$	(∃E 3, 6–21)
(23)	$\exists x \forall y((Fy \leftrightarrow y = x) \wedge Gx)$	(Abbreviating using ↔)

(c) *Outline a proof that 'one and two makes three' (i.e. show that if* There is one F *and* There are two Gs *(where* None of the Fs are Gs*), then* There are three things which are F-or-G*).*

We want a proof which starts and finishes like this:

$\exists x(Fx \wedge \forall y(Fy \rightarrow y = x))$	(There is one *F*)
$\exists x \exists y(\{(Gx \wedge Gy) \wedge x \neq y\} \wedge \forall z\{Gz \rightarrow (z = x \vee z = y)\})$	(There are two *G*s)
$\neg \exists x(Fx \wedge Gx)$	(No *F* are *G*)
$\vdots$	
$\exists x \exists y \exists z(\{(([Fx \vee Gx] \wedge [Fy \vee Gy]) \wedge [Fz \vee Gz]) \wedge ((x \neq y \wedge y \neq z) \wedge z \neq x)\}$ $\wedge \forall w\{[Fw \vee Gw] \rightarrow ((w = x \vee w = y) \vee w = z)\})$	(There are three *F-or-G*s)

We have three existential quantifications at the beginning of the proof, so we'll expect to instantiate these with the aim of using (∃E) three times. Our target is a triple existential quantification at the end of the proof, so we'll expect to derive that by a triple use of (∃E). So that suggests we want a proof of the overall shape:

$$
\begin{array}{llllr}
| & & & \exists x(Fx \land \forall y(Fy \to y = x)) & \text{(Prem)} \\
| & & & \exists x\exists y(\{(Gx \land Gy) \land x \neq y\} \land \forall z\{Gz \to (z = x \lor z = y)\}) & \text{(Prem)} \\
| & & & \neg\exists x(Fx \land Gx) & \text{(Prem)} \\
| & | & & (Fa \land \forall y(Fy \to y = a)) & \text{(Supp)} \\
| & | & | & \exists y(\{(Gb \land Gy) \land b \neq y\} \land \forall z\{Gz \to (z = b \lor z = y)\}) & \text{(Supp)} \\
| & | & | \;| & (\{(Gb \land Gc) \land b \neq c\} \land \forall z\{Gz \to (z = b \lor z = c)\}) & \text{(Supp)} \\
| & | & | \;| & \vdots & \\
| & | & | \;| & (\{(([Fa \lor Ga] \land [Fb \lor Gb]) \land [Fc \lor Gc]) \land ((a \neq b \land b \neq c) \land c \neq a)\} & \\
| & | & | \;| & \quad \land \forall w\{[Fw \lor Gw] \to ((w = a \lor w = b) \lor w = c)\}) & (**) \\
| & | & | \;| & \exists z(\{(([Fa \lor Ga] \land [Fb \lor Gb]) \land [Fz \lor Gz]) \land ((a \neq b \land b \neq z) \land z \neq a)\} & \\
| & | & | \;| & \quad \land \forall w\{[Fw \lor Gw] \to ((w = a \lor w = b) \lor w = z)\}) & (\exists\text{I on } c) \\
| & | & | \;| & \exists y\exists z(\{(([Fa \lor Ga] \land [Fy \lor Gy]) \land [Fz \lor Gz]) \land ((a \neq y \land y \neq z) \land z \neq a)\} & \\
| & | & | \;| & \quad \land \forall w\{[Fw \lor Gw] \to ((w = a \lor w = y) \lor w = z)\}) & (\exists\text{I on } b) \\
| & | & | \;| & \exists x\exists y\exists z(\{(([Fx \lor Gx] \land [Fy \lor Gy]) \land [Fz \lor Gz]) \land ((x \neq y \land y \neq z) \land z \neq x)\} & \\
| & | & | \;| & \quad \land \forall w\{[Fw \lor Gw] \to ((w = x \lor w = y) \lor w = z)\}) & (\exists\text{I on } a) \\
| & | & | & \longleftarrow & (\exists\text{E}) \\
| & | & & \longleftarrow & (\exists\text{E}) \\
| & & & \longleftarrow & (\exists\text{E})
\end{array}
$$

where at the end we bring down the same long wff each time!

So now we just need to think about how we can get the different conjuncts of (**).

1. We already have Fa available, so that gives $(Fa \lor Ga)$; we already have Gb available, so that gives $(Fb \lor Gb)$; we already have Gc available, so that gives $(Fc \lor Gc)$.

2. Suppose $a = b$. We have Fa, and also have Gb and hence Ga. But that gives us $(Fa \land Ga)$ and hence $\exists x(Fx \land Gx)$ contradicting the third premiss. Hence $a \neq b$. We can show $c \neq a$ similarly. And we already have the third negated identity $b \neq c$.

3. To show the remaining conjunct $\forall w\{[Fw \lor Gw] \to ((w = a \lor w = b) \lor w = c)\}$, assume $[Fd \lor Gd]$ and aim for $((d = a \lor d = b) \lor d = c)$ (so we can use CP and then ∀I). That is simply a matter of making use of $\forall y(Fy \to y = a)$ and $\forall z\{Gz \to (z = b \lor z = c)\}$, and we are done.

A pain to write out in full, but the strategies are simple enough!