21 A very short word about sets

Before moving on, we should quickly say something about sets.

More advanced formal logic texts quite often have a preliminary chapter explaining some core set-theoretic ideas which they are later going to make extensive use of. This is *not* a rather belated version of that kind of chapter! In fact, we take the opposite direction in this book, and will rather downplay ideas about sets. We are not going to religiously avoid *all* mention of sets; but for us, such talk is an occasionally useful façon de parler which can, however, be translated away.

Rather than explain this point repeatedly at various scattered places in the next few chapters, let's quickly get it out of the way now, to avoid some distracting digressions later.

21.1 'Virtual sets'

(a) Consider again the informal notion of deductive entailment and its formal counterpart for PL arguments, the notion of tautological entailment. As we introduced these ideas in §§1.3 and 2.1, and then in §14.2, entailment is a relation between perhaps *many* premisses on the one side and a single conclusion on the other. And this is surely the natural way to think about the relation. Yet it is very common for logicians to write as if entailment always relates a *single* thing to a conclusion.

The thought is this. The premisses of an argument may be many, but the *set* of them is one thing. And logicians usually prefer to say that entailment relates a *set* of premisses, one thing, to a conclusion.

However, this claim about sets can be taken in two ways. In the decaffeinated version, the talk about a set of premisses can be understood as merely a stylistic variant of talking about the premisses, plural, and it commits us to no more. This sort of apparent reference to a set can be translated away without loss of content – the sets here are merely *virtual* sets (to borrow the word used by the logician/philosopher W.V.O. Quine).

In the fully caffeinated version of the claim, by contrast, a set of premisses is taken to be a real entity in its own right – in other words, the set is supposed to be something over and above those sentences which it has as its members. And on a standard view about sets, this move up from some premisses to the set of them is just the beginning: the set-forming operation (it is assumed) can be repeatedly iterated, so that we can have not only Xs and sets-of-Xs, but also sets-of-sets-of-Xs, sets-of-sets-of-sets-of-Xs, . . . , new entities without limit.

Now, when we want to talk about perhaps many Xs at the same time, it can indeed be grammatically convenient to talk as if about a single thing – about, as we say, a collection, class, or set of Xs. But in many elementary contexts, this can be happily construed in the decaffeinated way, just as a helpful façon de parler, and it will be quite unnecessary to take on a commitment to any sets as additional entities in their own right, let alone any commitment to a full hierarchy of them (a point famously emphasized by Quine). This applies in the present case: we can understand talk about a set of premisses entailing a conclusion as just another way of saying that the premisses, plural, entail the conclusion.

One remark. It is a basic feature of the standard view of (real) sets that the order in which the elements of a set are given is irrelevant; and it makes no difference if an element is mentioned more than once, it is still the same set. But as with sets, so with plurals. When we say some premisses entail a conclusion, we take the order in which premisses are presented as irrelevant – i.e. we still have the same premisses, plural, however they are listed; and it makes no difference either if a premiss is mentioned more than once.

(b) Let's mention a couple of other cases where logicians typically invoke sets, but where this set-talk can again be construed in the decaffeinated way.

(1) We will say that the objects which satisfy a property-ascribing predicate expression (i.e. the objects which the predicate is true of) are the *extension* of the predicate. Then a simple sentence 'n is F' is true just when the object picked out by the name 'n' is in (i.e. is one among) the extension of the predicate '(is) F'.

    But it is more usual to put it this way: the extension of a predicate is redefined as the *set* of objects which satisfy the predicate. And then 'n is F' is true just when the object picked out by 'n' is in (i.e. is a member of) the extension of 'F'.

    However, the notion of set here is doing no substantive work. In other words, the sets here can be treated as virtual sets and talk of extensions-as-sets can be translated back into plural talk of their members.

(2) Take a sentence of the form 'Everyone is F'. To understand the claim, we need to know who counts as 'everyone' – everyone in the whole wide world? everyone in the university? just those in the logic class? The objects the generalizing expression are supposed to be ranging over are, we will say, the intended *domain of quantification*.

    It is more usual, though, to define the domain of quantification as being a *set*, the set of things we are generalizing over. Then it is said that 'Everyone is F' is true when whatever is in (is a member of) the relevant domain-as-set satisfies 'F'.

    But here too, the notion of a set is doing no substantive work, and talk of domains-as-single-sets can be translated back into talk of domains-as-many-objects. So the claim becomes that 'Everyone is F' is true when whatever is in (is one among) the objects-which-are-the-relevant-domain satisfies 'F'. Set-domains are virtual sets again.

Another remark. It is also a basic feature of the standard view of (real) sets that it is entirely determinate what is and what isn't a member. So, in particular, if extensions or domains are treated as real sets, their membership will be entirely determinate. There can't be borderline cases: any object is definitely a member or definitely not a member of a given extension or domain.

But what goes for extensions-as-real-sets can again go for extensions-as-many-objects. It is entirely open to us to stipulate that we are only going to be interested in predicates whose extensions are determinate – i.e. for a given predicate F, any object is either definitely one of the Fs or it definitely isn't. Indeed, better to be explicit about the determinacy assumption than just smuggling it in without comment by taking extensions to be real sets. Similarly for domains.

(c) This isn't to say we gain nothing by using the set idiom in the decaffeinated way. In some cases, we gain ease and smoothness of expression (ordinary language plural idioms can in more complex cases seem clumsy or forced); and this convenience is certainly to be valued. We are just stressing that using the set idiom for ease of expression is one thing, taking on a full commitment to sets-as-objects-in-their-own-right is something else (and as we'll see later, there are some cases where – for technical reasons – we positively want to avoid that extra commitment).

It is rather good policy, then, to avoid unnecessary set-talk so long as we can easily and smoothly do so. This will help us to get clearer about just when, in more advanced logical investigations, we *do* genuinely take on serious set-theoretic commitments that can't be just wished away as plural talk in thin disguise. Which is why we will largely adopt the set-avoiding policy in this book – and why the residual set idiom which we do make use of is to be construed in the non-committal way.

21.2 'Tuples'

Standard sets are *unordered*. But it will prove useful to talk about some *ordered* sets of objects – in particular, about *finite* sequences, or *tuples* as they are called for a reason which will become clear in a moment. This section indicates why we might want to use this residual set idiom, but notes that a low-caffeine interpretation again suffices.

(a) Expanding a bit on our previous usage, we'll now use 'predicate' for an expression that combines with one *or more* names (or other appropriate expressions) to form a sentence – so expresses a property or perhaps a relation between two or more things.

Take a simple relational predicate like 'loves'. When is a sentence of the form 'm loves n' true? When the people denoted by 'm' and 'n' taken in that order satisfy the relational predicate 'loves', i.e. when the first of the *ordered pair* loves the second. 'Ordered' because obviously enough the order matters here: life sadly being as it is, 'm loves n' might be true while 'n loves m' is false.

Take next take a case with a relational predicate which can combine with three names to form a sentence. For example, when is a sentence of the form 'l prefers m to n' true? When the triple of people denoted by 'l','m', 'n', taken in that order, satisfies the relational predicate, i.e. when the first of them prefers the second to the third.

Next come predicates satisfied by ordered quadruples – as, for example, in attributions of proportion, 'k is to l as m is to n'. Less naturally, we can cook up predicative expressions taking five names (or other suitable expressions) to form a sentence: such an expression will be satisfied by an ordered quintuple. And so it goes.

It is useful to introduce some standard jargon. We've spoken of individual objects, and then ordered pairs, triples, quadruples, quintuples; and we can go on, with sextuples, etc. Taking 'ordered' as understood, such items are rather naturally called *tuples*: tuples of k objects are, of course, k-tuples. (For convenience, we can stipulate that '1-tuples' are just individual objects.)

Say that a predicate expression taking k names etc. to form a sentence is a k-place predicate. Then a k-place predicate may be satisfied by zero, one, or many k-tuple(s).

(b) So, when talking about what k-place predicates are true of, it is useful to be able to talk snappily about k-tuples of objects. But what are tuples?

Well, as with unordered sets, we *could* take finite tuples to be entities in their own right. And indeed, we can model such tuples in standard fully-caffeinated set theories (though there are multiple different ways of doing this, equally effective, so no one of them can be said to reveal what tuples really are). However taking tuples this seriously is overkill again. Given our modest purposes, there is no need to take the ordered pair of Romeo and Juliet, for example, as some special entity, over and above the two people Romeo and Juliet with an order relation between them. Likewise the ordered triple of Romeo, Juliet and Rosaline can be taken to be just the three people with an order relation between *them*. And so on.

However, we won't fuss greatly about this. The point at which the real philosophical worries kick in concerning over-commitment to sets, unordered or ordered, is in moving from the modest finite cases to the wildly infinite cases. If you are happy to think of finite tuples (which are all we will want) as tame entities in their own right, be my guest. The claim is only that, for the purposes of this book, you don't really need to buy into even that much commitment to non-virtual sets.

21.3 Summary

Talk of a set of Xs can be understood in an anodyne way, as a mere device for speaking of the X's themselves. Or it can be understood as committing us to an entity in its own right, over and above the Xs.

At our introductory level, we can construe set talk (e.g. in talking domains or extensions) in the first way – so for us, domains and extensions are merely virtual sets.

In general, though, we will avoid talk of sets: the main exception is when we speak of what satisfy predicates as 'tuples', i.e. finite ordered sets. But these tuples too can be understood in an noncommittal way.