

Teach Yourself Logic: Appendix

# Some Big Books on Mathematical Logic

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*Pass it on, . . . . That's the game I want you to learn. Pass it on.*

Alan Bennett, *The History Boys*

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# Introduction

The traditional menu for a first serious Mathematical Logic course is basic first order logic with some model theory, the basic theory of computability and related matters (like Gödel's incompleteness theorem), and introductory set theory. The Teach Yourself Logic Study Guide makes some topic-by-topic recommendations for reading yourself into some back knowledge. The plan in this Appendix is to look at some of the Big (or sometimes not so big) Books that aim to cover an amount of this core menu, to give an indication of what they cover and – more importantly – how they cover it, while commenting on style, accessibility, etc.

These wider-ranging books don't always provide the best introductions now available to this or that particular area: but they can still be very useful aids to widening and deepening your understanding and can reveal how topics from different areas fit together. I don't promise to discuss [eventually] every worthwhile Big Book, or to give a similar level of coverage to those I do consider. But I'm working on the principle that even a patchy guide, very much work in progress, might still be useful.

Sections start with a general indication of the coverage of the book, and some general remarks. Then there is usually, in small print, a more detailed description of contents, and more specific comments. Finally, there's a summary verdict. The books are listed in chronological order of first publication.

# 1 Kleene, 1952

First published sixty years ago, Stephen Cole Kleene's *Introduction to Metamathematics*\* (North-Holland, 1962; reprinted Ishi Press 2009: pp. 550) for a while held the field as a survey treatment of first-order logic (without going much past the completeness theorem), and a more in-depth treatment of the theory of computable functions, and Gödel's incompleteness theorems.

In a 1991 note about writing the book, Kleene notes that up to 1985, about 17,500 copies of the English version of his text were sold, as were thousands of various translations (including a sold-out first print run of 8000 of the Russian translation). So this is a book with a quite pivotal influence on the education of later logicians, and on their understanding of the fundamentals of recursive function theory and the incompleteness theorems in particular.

But it isn't just nostalgia that makes old hands continue to recommend it. Kleene's book remains particularly lucid and accessible: it is often discursive, pausing to explain the motivation behind formal ideas. It is still a pleasure to read (or at least, it ought to be a pleasure for anyone interested in logic enough to be reading this Appendix to the Guide!). And, modulo relatively superficial presentational matters, you'll probably be struck by a sense of familiarity when reading through, as aspects of his discussions evidently shape many later textbooks (not least my own Gödel book). The *Introduction to Metamathematics* remains a really impressive achievement: and not one to be admired only from afar, either.

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*Some details* Chs. 1–3 are introductory. There's a little about enumerability and countability (Cantor's Theorem); then a chapter on natural numbers, induction, and the axiomatic method; then a little tour of the paradoxes, and possible responses.

Chs. 4–7 are a gentle introduction to the propositional and predicate calculus and a formal system which is in fact first-order Peano Arithmetic (you need to be aware that the identity rules are treated as part of the arithmetic, not the logic). Although Kleene's official system is Hilbert-style, he shows that 'natural deduction' introduction and elimination rules can be thought of as derived rules

in his system, so it all quickly becomes quite user-friendly. (He doesn't at this point prove the completeness theorem for his predicate logic: as I said, things go quite gently at the outset!)

Ch. 8 starts work on 'Formal number theory', showing that his formal arithmetic has nice properties, and then defines what it is for a formal predicate to capture ('numeralwise represent') a numerical relation. Kleene then proves Gödel's incompleteness theorem, assuming a Lemma – eventually to be proved in his Chapter 10 – about the capturability of the relation ' $m$  numbers a proof [in Kleene's system] of the sentence with number  $n$ '.

Ch. 9 gives an extended treatment of primitive recursive functions, and then Ch. 10 deals with the arithmetization of syntax, yielding the Lemma needed for the incompleteness theorem.

Chs. 11-13 then give a nice treatment of general (total) recursive functions, partial recursive functions, and Turing computability. This is all very attractively done.

The last two chapters, forming the last quarter of the book, go under the heading 'Additional Topics'. In Ch. 14, after proving the completeness theorem for the predicate calculus without and then with identity, Kleene discusses the decision problem. And the final Ch. 15 discusses Gentzen systems, the normal form theorem, intuitionistic systems and Gentzen's consistency proof for arithmetic.

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*Summary verdict* Can still be warmly recommended as an enjoyable and illuminating presentation of this fundamental material, written by someone who was himself so closely engaged in the early developments back in the glory days. It should be entirely accessible if you have managed e.g. Chiswell and Hodges and then my Gödel book, and will enrich and broaden your understanding.

## 2 Mendelson, 1964, 2009

Elliot Mendelson's *Introduction to Mathematical Logic* (Van Nostrand, 1964: pp. 300) was first published in the distinguished and influential company of The University Series in Undergraduate Mathematics. It has been much used in graduate courses for philosophers since: a 4th edition was published by Chapman Hall in 1997 (pp. 440), with a slightly ex-

panded 5th edition being published in 2009. I will here compare the first and fourth editions, as these are the ones I know.

Even in the later editions this isn't, in fact, a very *big* Big Book (many of the added pages of the later editions are due to there now being answers to exercises): the length is kept under control in part by not covering a great deal, and in part by a certain brisk terseness. As the Series title suggests, the intended level of the book is upper undergraduate mathematics, and the book does broadly keep to that aim. Mendelson is indeed pretty clear; however, his style is of the times, and will strike many modern readers as dry and rather old-fashioned. (Some of the choices of typography are not wonderfully pretty either, and this can make some pages look as if they will be harder going than they really turn out to be.)

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*Some details* After a brief introduction, Ch. 1 is on the propositional calculus. It covers semantics first (truth-tables, tautologies, adequate sets of connectives), then an axiomatic proof system. The treatments don't change much between editions, and will probably only be of interest if you've never encountered a Hilbert-style axiomatic system before. The fine print of how Mendelson regards his symbolic apparatus is interesting: if you read him carefully, you'll see that the expressions in his formal systems are not sentences, not expressions of the kind that – on interpretation – can be true or false – but are *schemata*, what he calls statement forms. But this relatively idiosyncratic line about how the formalism is to be read, which for a while (due to Quine's influence) was oddly popular among philosophers, doesn't much affect the development.

Ch. 2 is on quantification theory, again in an axiomatic style. The fourth edition adds to the end of the chapter more sections on model theory: there is a longish section on ultra-powers and non-standard analysis, then there's (too brief) a nod to semantic trees, and finally a new discussion of quantification allowing empty domains. The extra sections in the fourth edition are a definite bonus: without them, there is nothing special to recommend this chapter, if you have worked through the suggestions in §2.2 of the Guide, and in particular, perhaps, the chapters in van Dalen's book.

Ch. 3 is titled 'Formal number theory'. It presents a formal version of first-order Peano Arithmetic, and shows you can prove some expected arithmetic theorems within it. Then Mendelson defines the primitive recursive and the (total) recursive functions, shows that these are representable (capturable) in PA.

It then considers the arithmetization of syntax, and proves Gödel's first incompleteness theorem and Rosser's improvement. The chapter then proves Church's Theorem about the decidability of arithmetic. One difference between editions is that the later proof of Gödel's theorem goes via the Diagonalization Lemma; another is that there is added a brief treatment of Löb's Theorem. At the time of publication of the original addition, this Chapter was a quite exceptionally useful guide thorough the material. But now – at least if you've read my Gödel book or the equivalent – then there is nothing to divert you here, except that Mendelson does go through *every* single stage of laboriously showing that the relation *m-numbers-a-PA-proof-of-the-sentence-numbered-n* is primitive recursive.

Ch. 4 is on set theory, and – unusually for a textbook – the system presented is NBG (von Neumann/Bernays/Gödel) rather than ZF(C). In the first edition, this chapter is under fifty pages, and evidently the coverage can't be very extensive and it also probably goes too rapidly for many readers. The revised edition doesn't change the basic treatment (much) but adds sections comparing NBG to a number of other set theories. So while this chapter certainly can't replace the introductions to set theory recommended in §??, it could be worth skimming briskly through the chapter in later editions to learn about NBG and other deviations from ZF.

The original Ch. 5 on effective computability starts with a discussion of Markov algorithms (again, unusual for a textbook), then treats Turing algorithms, then Herbrand-Gödel computability and proves the equivalence of the three approaches. There are discussions of recursive enumerability and of the Kleene-Mostowski hierarchy. And the chapter concludes with a short discussion of undecidable problems. In the later edition, the material is significantly rearranged, with Turing taking pride of place and other treatments of computability relegated to near the end of the chapter; also more is added on decision problems. Since the introductory texts mentioned in the Guide don't talk about Markov or Herbrand-Gödel computability, you might want to dip into the chapter briefly to round out your education!

I should mention the appendices. The first edition has a very interesting though brisk appendix giving a version of Schütte's variation on a Gentzen-style consistency proof for PA. Rather sadly, perhaps, this is missing from later editions. The fourth edition has instead an appendix on second-order logic.

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*Summary verdict* Moderately accessible and very important in its time, but there is now not so much reason to plough through this book end-to-end. It doesn't have the charm and readability of Kleene 1952, and



there are now better separate introductions to each of the main topics . You could *skim* the early chapters if you've never seen axiomatic systems of logic being used in earnest: it's good for the soul. The appendix that appears only in the first edition is interesting for enthusiasts. Look at the section on non-standard analysis in the revised editions. If set theory is your thing, you should dip in to get the headline news about NBG. And some might want to expand their knowledge of definitions of computation by looking at Ch. 5.

### 3 Shoenfield, 1967

Joseph R. Shoenfield's *Mathematical Logic* (Addison-Wesley, 1967: pp. 334) is officially intended as 'a text for a first-year [maths] graduate course'. It has, over the years, been much recommended and much used (a lot of older logicians first learnt their serious logic from it).

This book, however, is hard going – a significant step or two up in level from Mendelson – though the added difficulty in mode of presentation seems to me not always to be necessary. I recall it as being daunting when I first encountered it as a student. Looking back at the book after a very long time, and with the benefit of greater knowledge, I have to say I am not any more enticed: it is still a tough read.

So this book can, I think, only be recommended to hard-core mathmos who already know a fair amount and can cherry-pick their way through the book. It does have heaps of hard exercises, and some interesting technical results are in fact buried there. But whatever the virtues of the book, they don't include approachability or elegance or particular student-friendliness.

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*Some details* Chs. 1–4 cover first order logic, including the completeness theorem. It has to be said that the logical system chosen is rebarbative. The primitives are  $\neg$ ,  $\vee$ ,  $\exists$ , and  $=$ . Leaving aside the identity axioms, the axioms are the instances of excluded middle, instances of  $\varphi(\tau) \rightarrow \exists \xi \varphi(\xi)$ , and then there are five rules of inference. So this neither has the cleanness of a Hilbert system nor the naturalness of a natural deduction system. Nothing is said to motivate this

seemingly horrible choice as against others.

Ch. 5 is a brisk introduction to some model theory getting as far as the Ryll-Nardjewski theorem. I believe that the algebraic criteria for a first-order theory to admit elimination of quantifiers given here are original to Shoenfield. But this is surely all done very rapidly (unless you are using it as a terse revision course from quite an advanced base, going beyond what you will have picked up from the reading suggested in our §?? above).

Chs. 6–8 cover the theory of recursive functions and formal arithmetic. The take-it-or-leave-it style of presentation continues. Shoenfield defines the recursive functions as those got from an initial class by composition and regular minimization: again, no real motivation for the choice of definition is given (and e.g. the definition of the primitive recursive functions is relegated to the exercises). Unusually for a treatment at this sort of level, the discussion of recursion theory in Ch. 8 goes far enough to cover a Gödelian ‘Dialectica’-style proof of the consistency of arithmetic, though the presentation once more wins no prizes for accessibility.

Ch. 9 on set theory is perhaps the book’s real original *raison d’être*; in fact, it is a quarter of the whole text. The discussion starts by briskly motivating the ZF axioms by appeal to the conception of the set universe as built in stages (an approach that has become very common but at the time of publication was I think much less usually articulated); but this isn’t the place to look for an in depth development of that idea. For a start, there is Shoenfield’s own article ‘The axioms of set theory’, *Handbook of mathematical logic*, ed. J. Barwise, (North-Holland, 1977) pp. 321–344.

We get a brusque development of the elements of set theory inside ZF (and then ZFC), and something about the constructible universe. Then there is the first extended textbook presentation of Cohen’s 1963 independence results via forcing, published just four years previous to the publication of this book: set theory enthusiasts might want to look at this to help round out their understanding of the forcing idea. The discussion also touches on large cardinals.

This last chapter was in some respects a highly admirable achievement in its time: but it is equally surely not *now* the best place to start with set theory in general or forcing in particular, given the availability of later presentations.

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*Summary verdict* This is pretty tough going. Now surely only for *very* selective dipping into by already-well-informed enthusiasts.

## 4 Kleene, 1967

In the preface to his *Mathematical Logic*\* (John Wiley 1967, Dover reprint 2002: pp. 398), Stephen Cole Kleene writes

After the appearance in 1952 of my *Introduction to Metamathematics*, written for students at the first-year graduate level, I had no expectation of writing another text. But various occasions arose which required me to think about how to present parts of the same material more briefly, to a more general audience, or to students at an earlier educational level. These newer expositions were received well enough that I was persuaded to prepare the present book for undergraduate students in the Junior year.

You'd expect, therefore, that this later book would be more accessible, a friendlier read, than Kleene's remarkable *IM*. But in fact, this doesn't actually strike me as the case. I'd still recommend reading the older book, augmented by one chapter of this later 'Little Kleene'. To explain:

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*Some details* The book divides into two parts. The first part, 'Elementary Mathematical Logic' has three chapters. Ch. 1 is on the propositional calculus (including a Kalmár-style completeness proof). This presents a Hilbert-style proof system with an overlay of derived rules which look rather natural-deduction-like (but aren't the real deal) There is a lot of fussing over details in rather heavy-handed ways. I couldn't recommend anyone nowadays *starting* here, while if you've already read a decent treatment of the propositional calculus (and e.g. looked at Mendelson to see how things work in a Hilbert-style framework) you won't get much more out of this.

Much the same goes for the next two chapters. Ch. 2 gives an axiomatic version of the predicate calculus without identity, and Ch. 3 adds identity. (Note, a completeness proof doesn't come to the final chapter of the book). Again, these chapters are not done with a sufficiently light touch to make them a particularly attractive read now.

The second part of the book is titled 'Mathematical Logic and the Foundations of Mathematics'. Ch.4 is basically an abridged version of the opening three chapters of *IM*, covering the paradoxes, the idea of an axiomatic system,

introducing formal number theory. You might like to read in particular §§36–37 on Hilbert vs. Brouwer and ‘metamathematics’.

Ch. 5 is a sixty page chapter on ‘Computability and Decidability’. Kleene is now on his home ground, and he presents the material (some original to him) in an attractive and illuminating way, criss-crossing over some of the same paths trodden in later chapters of *IM*. In particular, he uses arguments for incompleteness and undecidability turning on use of the Kleene  $T$ -predicate (compare §33.7 of *IGT1* or §43.8 of *IGT2*). This chapter is certainly worth exploring.

Finally, the long Ch. 6 proves the completeness theorem for predicate logic by Beth/Hintikka rather than by Henkin (as we would now think of it, he in effect shows the completeness of a tree system for logic in the natural way). But nicer versions of this approach are available. The last few sections cover some supplementary material (on Gentzen systems, Herbrand’s Theorem, etc.) but again I think all of it is available more accessibly elsewhere

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*Summary verdict* Do read Chapter 5 on computability, incompleteness, decidability and closely related topics. This is nicely done, complements Kleene’s earlier treatment of the same material, and takes an approach which is interestingly different from what you will mostly see elsewhere.

## 5 Robbin, 1969

Joel W. Robbin’s *Mathematical Logic: A First Course\** (W. A Benjamin, 1969, Dover reprint 2006: pp. 212) is not exactly a ‘Big Book’. The main text is just 170 pages long. But it does range over both formal logic (first-order and second-order), and formal arithmetic, primitive recursive functions, and Gödelian incompleteness. Robbin, as you might guess, has to be quite brisk (in part he achieves brevity by leaving a lot of significant results to be proved as more or less challenging exercises). However, the book remains approachable and has some nice and unusual features for which it can be recommended.

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*Some details* Ch. 1 is on the propositional calculus. Robbin presents an axiomatic system whose primitives are  $\rightarrow$  and  $\perp$  – or rather, in his notation,  $\supset$  and  $\text{f}$ . The system, including the ‘dotty’ syntax which gives us wffs like  $p_1 \supset p_2 \text{ } \bullet \supset \bullet$ ,  $p_1 \supset \text{f}$ , is a version of Alzono Church’s system in his *Introduc-*

*tion to Mathematical Logic*, Vol. 1 (1944/1956), except that where Church lays down three specific wffs as axioms and has a substitution rule for deriving variant wffs, Robbin lays down three axiom schemas. [Perhaps I should say something about Church’s classic book in this Appendix: but that’s for another day.]

As in later chapters, Robbin buries some interesting results in the extensive exercises. Here’s one, pointed out to me by David Auerbach. Robbin defines negation in the obvious way from his two logical primitives, so that  $\sim \varphi =_{\text{def}} (\varphi \rightarrow \mathbf{f})$ . And then his three axiom schemas can all be stated in terms of  $\supset$  and  $\sim$ , and his one rule is modus ponens. This system is complete. However, if we take the alternative language with  $\supset$  and  $\sim$  *primitive*, then the same deductive system (with the same axioms and rules) is *not* complete. That’s a nice little surprise, and it is worth trying to work out just why it is true.

Ch. 2 briefly covers first-order logic, including the completeness theorem. Then Ch. 3 introduces what Robbin calls ‘First-order (Primitive) Recursive Arithmetic’ (RA). Robbin defines the primitive recursive functions, and then defines a language which has a function expression for each p.r. function  $f$  (the idea is to have a complex function expression built up to reflect a full definition of  $f$  by primitive recursion and/or composition ultimately in terms of the initial functions). RA has axioms for the logic plus axioms governing the expressions for the initial functions, and then there are axioms for dealing with complex functional expressions in terms of their constituents. RA also has all instances of the induction schema for open wffs of the language (so – for cognoscenti – this is a stronger theory than what is usually called Primitive Recursive Arithmetic these days, which normally has induction only for quantifier-free wffs).

Ch. 4 explores the arithmetization of syntax of RA. Since RA has every p.r. function built in, we don’t then have to go through the palaver of showing that we are dealing with a theory which can represent all p.r. functions (in the way we have to if we take standard PA as our base theory of interest). So in Ch. 5 Robbin can prove Gödel’s incompleteness theorem for RA in a more pain-free way.

Ch. 6 then turns to second-order logic, introduces a version of second-order  $\text{PA}_2$  with just the successor relation as primitive non-logical vocabulary. Robbin shows that all the p.r. functions can be explicitly defined in  $\text{PA}_2$ , so the incompleteness theorem carries over.

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*Summary verdict* Robbin’s book offers a different route through a rather different selection of material than is usual, accessibly written and still worth reading (you will be able to go though quite a bit of it pretty

rapidly if you are up to speed with the relevant basics. Look especially at Robbin’s Ch. 3 for the unusually detailed story about how to build a language with a function expression for every p.r. function, and the last chapter for how in effect to do the same in  $PA_2$ .

## 6 Enderton, 1972, 2002

The first edition of Herbert B. Enderton’s *A Mathematical Introduction to Logic* (Academic Press, 1972: pp. 295) rapidly established itself as much-used textbook among the mathematicians it was aimed towards. But it has also been used too in math. logic courses offered to philosophers. A second edition was published in 2002, and a glance at the section headings indicates much the same overall structure: but there are many local changes and improvements, and I’ll comment here on this later version of the book (which by now should be equally widely available in libraries). The author died in 2010, but his webpages live on, including one with his own comments on his second edition: <http://www.math.ucla.edu/~hbe/>.

Enderton’s text deals with first order-logic and a smidgin of model theory, followed by a look at formal arithmetic, recursive functions and incompleteness. A final chapter covers second-order logic and some other matters.

*A Mathematical Introduction to Logic* eventually became part of a logical trilogy, with the publication of the wonderfully lucid *Elements of Set Theory* (1977) and *Computability Theory* (2010). The later two volumes strike me as masterpieces of exposition, providing splendid ‘entry level’ treatments of their material. The first volume, by contrast, is *not* the most approachable first pass through its material. It is good (often *very* good), but I’d say at a notch up in difficulty from what you might be looking for in an *introduction* to the serious study of first-order logic and/or incompleteness.

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*Some details* After a brisk Ch. 0 (‘Some useful facts about sets’, for future reference), Enderton starts with a 55 page Ch. 1, ‘Sentential Logic’. Some might think this chapter to be slightly odd. For the usual motivation for separating

off propositional logic and giving it an extended treatment at the beginning of a book at this level is that this enables us to introduce and contrast the key ideas of semantic entailment and of provability in a formal deductive system, and then explain strategies for soundness and completeness proofs, all in a helpfully simple and uncluttered initial framework. But (except for some indications in final exercises) there is no formal proof system mentioned in Enderton's chapter.

So what does happen in this chapter? Well, we do get a proof of the expressive completeness of  $\{\wedge, \vee, \neg\}$ , etc. We also get an exploration (which can be postponed) of the idea of proofs by induction and the Recursion Theorem, and based on these we get proper proofs of unique readability and the uniqueness of the extension of a valuation of atoms to a valuation of a set of sentences containing them (perhaps not the most inviting things for a beginner to be pausing long over). We get a direct proof of compactness. And we get a first look at the ideas of effectiveness and computability.

The core Ch. 2, 'First-Order Logic', is over a hundred pages long, and covers a good deal. It starts with an account of first-order languages, and then there is a lengthy treatment of the idea of truth in a structure. This is pretty clearly done and mathematicians should be able to cope quite well (but does Enderton forget his officially intended audience on p. 83 where he throws in an unexplained commutative diagram?). Still, readers might sometimes appreciate rather more explanation (for example, surely it would be worth saying a bit more than that 'In order to define ' $\sigma$  is true in  $\mathfrak{A}$ ' for sentences  $\sigma$  and structures  $\mathfrak{A}$ , we will find it desirable [sic] first to define a more general concept involving wffs', i.e. satisfaction by sequences).

Enderton then at last introduces a deductive proof system (110 pages into the book). He chooses a Hilbert-style presentation, and if you are not already used to such a system, you won't get much of a feel for how they work, as there are very few examples before the discussion turns to metatheory (even Mendelson's presentation of a similar Hilbert system is here more helpful). Then, as you'd expect, we get the soundness and completeness theorems. The proof of the latter by Henkin's method *is* nicely chunked up into clearly marked stages, and again a serious mathematics student should cope well: but this is still not, I think, a 'best buy' among initial presentations.

The chapter ends with a little model theory – compactness, the LS theorems, interpretations between theorems – all rather briskly done, and there is an application to the construction of infinitesimals in non-standard analysis which is surely going to be *too* compressed for a first encounter with the ideas.

Ch. 3, 'Undecidability', is also a hundred pages long and again covers a great

deal. After a preview introducing three somewhat different routes to (versions of) Gödel's incompleteness theorem, we initially meet:

1. A theory of natural numbers with just the successor function built in (which is shown to be complete and decidable, and a decision procedure by elimination of quantifiers is given).
2. A theory with successor and the order relation (also shown to admit elimination of quantifiers and to be complete).
3. Presburger arithmetic (shown to be decidable by a quantifier elimination procedure, and shown not to define multiplication)
4. Robinson Arithmetic with exponentiation.

The discussion then turns to the notions of definability and representability. We are taken through a long catalogue of functions and relations representable in Robinson-Arithmetic-with-exponentiation, including functions for encoding and decoding sequences. Next up, we get the arithmetization of syntax done at length, leading as you'd expect to the incompleteness and undecidability results.

But we aren't done with this chapter yet. We get (sub)sections on recursive enumerability, the arithmetic hierarchy, partial recursive functions, register machines, the second incompleteness theorem for Peano Arithmetic, applications to set theory, and finally we learn how to use the  $\beta$ -function trick so we can get take our results to apply to any nicely axiomatized theory containing plain Robinson Arithmetic.

As is revealed by that quick description there really is a *lot* in Ch. 3. To be sure, the material here is not mathematically difficult in itself (indeed it is one of the delights of this area that the initial Big Results come so quickly). However, I do doubt that such an action-packed presentation is the best way to first meet this material. It would, however, make for splendid revision-consolidation-extension reading after tackling e.g. my Gödel book.

The final Ch. 4 is much shorter, on 'Second-Order Logic'. This goes *very* briskly at the outset. It again wouldn't be my recommended introduction for this material, though it could make useful supplementary reading for those wanting to get clear about the relation between second-order logic, Henkin semantics, and many-sorted first-order logic.

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*Summary verdict* To repeat, *A Mathematical Introduction to Logic* is good in many ways, but is – in my view – often a step or two more difficult in mode of presentation than will suit many readers wanting



an introduction to the material it covers. However, if you have already read an entry-level presentation of first order logic (e.g. Chiswell/Hodges) then you could read Chs 1 and 2 as revision/consolidation. And if you have already read an entry-level presentation on incompleteness (e.g. my book) then it could be well worth reading Ch. 3 as bringing the material together in a somewhat different way.

## 7 Ebbinghaus, Flum and Thomas, 1978, 1994

We now turn to consider H.-D. Ebbinghaus, J. Flum and W. Thomas, *Mathematical Logic* (Springer, 2nd end. 1994: pp. 289). This is the English translation of a book first published in German in 1978, and appears in a series ‘Undergraduate Texts in Mathematics’, which indicates the *intended* level: parts of the book, however, do seem more than a little ambitious for most undergraduates.

EFT’s book is often warmly praised and is (I believe) quite widely used. But revisiting it, I can’t find myself wanting to recommend it as a good place to start, certainly not for philosophers but I’m not sure I would recommend it for mathematicians either. The presentation of the core material on the syntax and semantics of first-order logic in the first half of the book is done more accessibly and more elegantly elsewhere. In the second half of the book, the chapters do range widely across interesting material. But again most of the discussions will go too quickly if you haven’t encountered the topics before, and – if you want revision/amplification of what you already know – you will mostly do better elsewhere.

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*Some details* The book is divided into two parts. EFT start Part A with a gentle opening chapter talking about a couple of informal mathematical theories (group theory, the theory of equivalence relations), giving a couple of simple informal proofs in those theories. They then stand back to think about what goes on in the proofs, and introduce the project of formalization. So far, so good.

Ch. 2 describes the syntax of first-order languages, and Ch. 3 does the semantics. The presentation goes at a fairly gentle pace, with some useful asides (e.g. on handling the many-sorted languages of informal mathematics using a

many-sorted calculus vs. use restricted quantifiers in a single-sorted calculus). EFT though do make quite heavy work of some points of detail.

Ch. 4 is called ‘A Sequent Calculus’. The version chosen strikes me as really not very nice. For a start (albeit a minor point that only affects readability), instead of writing a sequent as ‘ $\Gamma \vdash \varphi$ ’, or ‘ $\Gamma \Rightarrow \varphi$ ’, or even ‘ $\Gamma : \varphi$ ’, EFT just write an unpunctuated ‘ $\Gamma \varphi$ ’. Much more seriously, they adopt a system of rules which many would say runs together a classical logical rule for the connectives with a structural rule in an unprincipled way that hides e.g. the relation between classical and intuitionistic logic. [To be more specific, they introduce a classical ‘Proof by Cases’ rule that takes us from the sequents (in their notation)  $\Gamma \psi \varphi$  and  $\Gamma \neg\psi \varphi$  to  $\Gamma \varphi$ , and then then show that this almost immediate yields Cut as a derived rule. This seems to me to muddy waters that usual presentations of the sequent calculus strive to keep clear.]

Ch. 5 gives a Henkin completeness proof for first-order logic. For my money, there’s too much symbol-bashing and not enough motivating chat here. (I don’t think it is good exegetical policy to complicate matters as EFT do by going straight for a proof for the predicate calculus with identity: but they are not alone in this.)

Ch. 6 is briskly about The Löwenheim-Skolem Theorem, compactness, and elementarily equivalent structures (but probably fine if you’ve met this stuff before).

Ch. 7, ‘The Scope of First-Order Logic’ is really rather odd. It briskly argues that first-order logic is the logic for mathematics (readers of Shapiro’s book on second-order logic won’t be so quickly convinced!). The reason given is that we can reconstruct (nearly?) all mathematics in first-order ZF set theory – which the authors then proceed to give the axioms for. These few pages surely wouldn’t help if you have never seen the axioms before and don’t already know about the project of doing-maths-inside-set-theory.

Finally in Part A, there’s rather ill-written chapter on normal forms, on extending theories by definitions, and (badly explained) on what the authors call ‘syntactic interpretations’.

Part B of the book discusses a number of rather scattered topics. It kicks off with a nice little chapter on extensions of first-order logic, more specifically on second-order logic, on  $\mathcal{L}_{\omega_1\omega}$  [which allows infinitely long conjunctions and disjunctions], and  $\mathcal{L}_Q$  [logic with quantifier  $Qx$ , ‘there are uncountably many  $x$  such that ...’].

Then Ch. 10 is on ‘Limitations of the Formal Method’, and in under forty pages aims to talk about register machines, the halting problem for such ma-

chines, the undecidability of first-order logic, Trahtenbrot's theorem and the incompleteness of second-order logic, Gödel's incompleteness theorems, and more. This would just be far too rushed if you'd not seen this material before, and if you have then there are plenty of better sources for revising/consolidating/extending your knowledge.

Ch. 11 is by some way the longest in the book, on 'Free Models and Logic Programming'. This is material we haven't covered in this Guide. But again it doesn't strike me as a particular attractive introduction.

Ch. 12 is back to core model theory, Fraïssé's Theorem and Ehrenfeucht Games: but (I'm sorry – this is getting repetitious!) you'll again find better treatments elsewhere, this time in books dedicated to model theory.

Finally, there is an interesting (though quite tough) concluding chapter on Lindström's Theorems which show that there is a sense in which standard first-order logic occupies a unique place among logical theories.

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*Summary verdict* The core material in Part A of the book is covered better (more accessibly, more elegantly) elsewhere.

Of the supplementary chapters in Part B, the two chapters that stand out as worth looking at are perhaps Ch. 9 on extensions of first-order logic, and Ch. 13 (though not easy) on Lindström's Theorems.

## 8 Van Dalen, 1980, 2012

Dirk van Dalen's popular *Logic and Structure* (Springer: 263 pp. in the most recent edition) was first published in 1980, and has now gone through a number of editions. It is widely used and has a lot to recommend it. A very substantial chapter on incompleteness was added in the fourth edition in 2004. A fifth edition published in 2012 adds a further new section on ultraproducts. Comments here apply to these last two editions.

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*Some details* Ch. 1 on 'Propositional Logic' gives a presentation of the usual truth-functional semantics, and then a natural deduction system (initially with primitive connectives  $\rightarrow$  and  $\perp$ ). This is overall pretty clearly done – though really rather oddly, although van Dalen uses in his illustrative examples of deductions the usual practice of labelling a discharged premiss with numbers and using a matching label to mark the inference move at which that premiss is dis-

charged, he doesn't pause to explain the practice in the way you would expect. Van Dalen then gives a standard Henkin proof of completeness for this cut-down system, before re-introducing the other connectives into his natural deduction system in the last section of the chapter. Compared with Chiswell and Hodges, this has a somewhat less friendly, more conventional, mathematical look-and-feel: but this is still an accessible treatment, and will certainly be very readily manageable if you've read C&H first. (It should be noted that van Dalen can be surprisingly slapdash. For example, a tautology is defined to be an (object-language) proposition which is always true. But then the meta-linguistic schema  $\varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$  is said to be a tautology. He means the instances are tautologies; compare Mendelson who really *does* think tautologies are schemata. Again, van Dalen presents his natural deduction system, says he is going to give some 'concrete cases', but then presents not arguments in the object language, but schematic templates for arguments, written out using ' $\varphi$ 's and ' $\psi$ 's again.)

Ch. 2 describes the syntax of a first-order language, gives a semantic story, and then presents a natural deduction system, first adding quantifier rules, then adding identity rules. Overall, this is pretty clearly done (though van Dalen reuses variables as parameters, which isn't the nicest way of setting things up). The approach to the semantics is to consider an extension of a first order language  $L$  with domain  $A$  to an augmented language  $L^A$  which has a constant  $\bar{a}$  for every element  $a \in A$ ; and then we can say  $\forall x\varphi(x)$  is true if all  $\varphi(\bar{a})$  are true. Fine: though it would have been good if van Dalen had paused to say a little more about the pros and cons of doing things this way rather than the more common Tarskian way that students will encounter. (Let's complain some more about van Dalen's slapdash ways. For example, he talks about '*the* language of a similarity type' in §2.3, but gives examples of different languages of the same similarity type in §2.7. He fusses unclearly about different uses of the identity sign in §2.3, before going on to make use of the symbolism ' $:=$ ' in a way that isn't explained, and is different from the use made of it in the previous chapter. This sort of thing could upset the more pernicky reader.)

Ch. 3, 'Completeness and Applications', gives a 'pretty clear presentation of a Henkin-style completeness proof, and then the compactness and L-S theorems. The substantial third section on model theory goes rather more speedily, and you'll need some mathematical background to follow some of the illustrative examples. The final section newly added in the fifth edition on the ultraproduct construction speeds up again and is probably too quick to be useful to many.

Ch. 4 is quite short, on second-order logic. If you have already seen a presentation of the basic ideas, this quick presentation of the formalities could be

helpful.

Ch. 5 is on intuitionism – this is a particular interest of van Dalen’s, and his account of the BHK interpretation as motivating intuitionistic deduction rules, his initial exploration of the resulting logic, and his discussion of the Kripke semantics are quite nicely done (though again, this chapter will probably work better if you have already seen the main ideas in a more informal presentation before).

Ch. 6 is on proof theory, and in particular on the idea that natural deduction proofs (both classical and intuitionistic) can be normalized. Most readers will find a more expansive and leisurely treatment much to be preferred.

The final 50 page Ch. 7 *is* more leisurely. It starts by introducing the ideas of primitive recursive and partial recursive functions, and the idea of recursively enumerable sets, leading up to a proof that there exist effectively inseparable r.e. sets. We then turn to formal arithmetic, and prove that recursive functions are representable (because his version of PA does have the exponential built in, van Dalen doesn’t need to tangle with the  $\beta$ -function trick.). Next we get the arithmetization of syntax and proofs that the numerical counterparts of some key syntactic properties and relations are primitive recursive. Then, as you would expect, we get the diagonalization lemma, and that is used to prove Gödel’s first incompleteness theorem. We then get another proof relying on the earlier result that there are effectively inseparable r.e. sets, and going via the undecidability of arithmetic. The chapter finishes by announcing that there is a finitely axiomatized arithmetic strong enough to represent all recursive properties/relations, so the undecidability of arithmetic implies the undecidability of first-order logic. There’s nothing, however, about the second theorem: so most students who get this far will want that bit more more. However, this chapter is all done pretty clearly, could probably be managed by good students as a first introduction to its topics, and would be very good revision/consolidatory reading for those who’ve already encountered this material.

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*Summary verdict* Revisiting this book, I find it a rather patchily uneven read. Although intended for beginners in mathematical logic, the level of difficulty of the discussions rather varies, and the amount of more relaxed motivational commentary also varies. As noted there are occasional lapses where van Dalen’s exposition isn’t as tight as it could be. So this is probably best treated as a book to be read *after* you’ve had a first exposure to the material in the various chapters: but then it should indeed prove

pretty helpful for consolidating/expanding your initial understanding and then pressing on a few steps.

## 9 Prestel and Delzell, 1986, 2011

Alexander Prestell and Charles N. Delzell's *Mathematical Logic and Model Theory: A Brief Introduction* (Springer, 1986; English version 2011: pp. 193) is advertised as offering a 'streamlined yet easy-to-read introduction to mathematical logic and basic model theory'. Easy-to-read, perhaps, for those with a fair amount of mathematical background in algebra, for – as the Preface makes clear – the aim of the book is make available to interested mathematicians 'the best known model theoretic results in algebra'. The last part of the book develops a complete proof of Ax and Kochen's work on Artin's conjecture about Diophantine properties of  $p$ -adic number fields.

So this book is not really aimed at a likely reader of this Guide. Still, Ch. 1 is a crisp and clean 60 page introduction to first-order logic, that could be used as brisk and helpful revision material. And Ch. 2, 'Model constructions' gives a nice if pacey introduction to some basic model theoretic notions: again – at least for readers of this Guide – it could serve well to consolidate and somewhat extend ideas if you have already encountered at least some of this material before.

The remaining two chapters, 'Properties of Model Classes' and 'Model Theory of Several Algebraic Theories' are tougher going, and belong with the more advanced reading like Marker's book. But, for those who want to work through this material, it does strike me as well presented.

## 10 Johnstone, 1987

Peter T. Johnstone's *Notes on Logic and Set Theory* (CUP, 1987: pp. 111) is very short in page length, but very big in ambition. There is an introductory chapter on universal algebra, followed by chapters on propositional and first-order logic. Then there is a chapter on recursive functions

(showing that a function is register computable if and only if computable, and that such functions are representable in PA). That is followed by four chapters on set theory (introducing the axioms of ZF, ordinals, AC, and cardinal arithmetic). And there is a final chapter ‘Consistency and independence’ on Gödelian incompleteness and independence results in set theory.

This is a quite remarkably action-packed menu for such a short book. True, the story is filled out a bit by the substantive exercises, but still surely this isn’t the book to use for a first encounter with these ideas (even though it started life as notes for undergraduate lectures for the maths tripos). However, I would warmly recommend the book for revision/consolidation: its very brevity means that the Big Ideas get highlighted in a particularly uncluttered way, and particularly snappy proofs are given.

## 11 Hodel, 1995

Richard E. Hodel’s *An Introduction to Mathematical Logic*\* (PWS Publishing, 1995, reprinted Dover Publications, 2013: pp. 491) was originally launched into the world by a relatively obscure publisher, but has now been taken up and cheaply republished by Dover. I hadn’t heard of the book before a few people recommended it, commenting on the lack of any mention in earlier versions of this Guide.

The book covers first-order logic and some recursion theory, with a less usual – though not unique – feature being a full textbook treatment of Hilbert’s Tenth Problem (the one about whether there is an algorithm which tells us when a polynomial equation  $p(x) = 0$  has a solution in the integers).

So how does Hodel fare against his competitors? Overall, the book *is* pretty clearly written, though it does have a somewhat old-fashioned feel to it (you wouldn’t guess, for example, that it was written some twenty years after George Boolos and Richard Jeffrey’s *Computability and Logic*). And Hodel does give impressively generous sets of exercises throughout

the book – including some getting the student to prove significant results by guided stages. However, I can't really recommend his treatment of logic in the first half of the book.

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*Some details* Ch. 1 'Background' is an unusually wide-ranging introduction to ideas the budding logician should get her head round early. We get a first informal pass at the notions of a formal system and of an axiomatized system in particular, the idea of proof by induction, a few notions about sets, functions and relations, the idea of countability, the ideas of an algorithmically computable function and of effective decidability. We even get a first pass at the idea of a recursive function (and a look at Church's Thesis about how the informal idea of being computable relates to the idea of recursiveness). This is very lucidly done, and can be recommended.

Ch. 2 is on 'The language and semantics of propositional logic' and is again pretty clearly done.

Ch. 3 turns to formal deductive systems for propositional logic. Unfortunately, Hodel chooses to work primarily with Shoenfield's system (the primitive connectives are  $\neg$  and  $\vee$ , every instance of  $\neg A \vee A$  is an axiom, and there are four rules of inference). I *really* can't see the attraction of this system among all the competitors, or why it should be thought as especially appropriate as a starting point for beginners. It neither has the naturalness of a natural deduction system, nor the austere Bauhaus lines of one of the more usual Frege-Hilbert axiomatic systems. The chapter does also consider other systems of propositional logic in the concluding two sections, but goes too quickly to be very helpful. So this key chapter is not a success, it seems to me.

Ch. 4 on 'First-order languages', including Tarskian semantics, is again pretty clear (and could be helpful to a beginner who is first encountering the ideas and is looking for reading to augment another textbook). But as we'd expect given what's gone before, when we turn to Ch. 5 on 'First-order logic' we can a continuance of the discussion of a Shoenfield-style system (except that Hodel takes  $\forall$  rather than  $\exists$  as primitive). Which is, by my lights, much to be regretted. The ensuing discussion of completeness, while perhaps a little laboured, is carefully structured with a good amount of signposting. But there are better presentations of first-order logic overall.

Ch. 6 is called 'Mathematics and logic' and touches on an assortment of topics about first-order theories and their limitations (and a probably rather-too-hasty look at set theory as an example).

The next two chapters form quite a nice unit. Ch. 7 discusses 'Incompleteness,



undecidability and indefinability'. Recursive functions are defined Shoenfield-style as those arising from a certain class of initial functions by composition and regular minimization, which eases the proof that all recursive functions are representable (though doesn't do much to make recursiveness seem a natural idea to beginners). Then, by making the informal assumption that certain intuitively decidable relations are recursive, Hodel proves Gödel's incompleteness theorem, Church's Theorem and Tarski's Theorem. The next chapter fills in enough detail about recursive functions and relations to show how to lift that informal assumption. This seems all pretty clearly done, even if not a first-choice for real beginners.

Then Ch. 9 extends the treatment of computability by showing that the functions computable by an unlimited register machine are just the recursive ones: but, again this sort of thing is done at least as well in other books.

Finally, Ch. 10 deals with Hilbert's tenth problem (so the first five sections of Ch. 8 and then this chapter form a nice, stand-alone treatment of the negative solution of Hilbert's problem).

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*Summary verdict* Beginners could all usefully read Hodel's opening chapter, which is better than usual in setting the scene, and supplying the student with a useful toolkit of preliminary ideas. The presentation of first-order logic in Chs. 2-6 is based around an unattractive formal system, and while the discussion of the usual meta-theoretic results is pretty clear, it doesn't stand out from the good alternatives: so overall, I wouldn't recommend this as your first encounter with serious logic.

Students, however, might find Chs. 7 and 8 provide a nice complement to other discussions of Gödelian incompleteness and Church and Tarski's Theorems. While more advanced students could revise their grip on basic definitions and results by (re)reading §§8.1-8.5 and then enjoy tackling Ch. 10 on Hilbert's Tenth Problem.

## 12 Goldstern and Judah, 1995

Half of Martin Goldstern and Haim Judah's *The Incompleteness Phenomenon: A New Course in Mathematical Logic* (A.K. Peters, 1995: pp. 247) is a treatment of first-order logic. The rest of the book is two long

chapters (as it happens, of just the same length), one on model theory, one mostly on incompleteness and with a little on recursive functions. So the emphasis on incompleteness in the title is somewhat misleading: the book is at least equally an introduction to some model theory. I have had this book recommended to me more than once, but I seem to be immune to its supposed charms (I too often don't particularly like the way that it handles the technicalities): your mileage may vary.

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*Some details* Ch. 1 starts by talking about inductive proofs in general, then gives a semantic account of sentential and then first-order logic, then offers a Hilbert-style axiomatic proof system.

Very early on, the authors introduce the notion of  $\mathcal{M}$ -terms and  $\mathcal{M}$ -formulae. An  $\mathcal{M}$ -term (where  $\mathcal{M}$  is model for a given first-order language  $\mathcal{L}$ ) is built up from  $\mathcal{L}$ -constants,  $\mathcal{L}$ -variables *and/or elements of the domain of  $\mathcal{M}$* , using  $\mathcal{L}$ -function-expressions; an  $\mathcal{M}$ -formula is built up from  $\mathcal{M}$ -terms in the predictable way. Any half-awake student is initially going to balk at this. Re-reading the set-theoretic definitions of expressions as tuples, she will then realize that the apparently unholy mix of bits of language and bits of some mathematical domain in an  $\mathcal{M}$ -term is *not* actually incoherent. But she will right wonder what on earth is going on and *why*: our authors don't pause to explain why we might want to do things like this at the very outset. (A good student who knows other presentations of the basics of first-order semantics should be able to work out after the event what is going on in the apparent trickery of Goldstern and Judah's sort of story: but I really can't recommend *starting* like this, without a good and expansive explanation of the point of the procedure.)

Ch. 2 gives a Henkin completeness proof for the first-order deductive system given in Ch. 1. This has nothing special to recommend it, as far as I can see: there are many more helpful expositions available. The final section of the chapter is on non-standard models of arithmetic: Boolos and Jeffrey (Ch. 17 in their third edition) do this more approachably.

Ch.3 is on model theory. There are three main sections, 'Elementary substructures and chains', 'ultra products and compactness', and 'Types and countable models'. So this chapter – less than sixty pages – aims quite high to be talking e.g. about ultraproducts and about omitting types. You could indeed usefully read it after working through e.g. Manzano's book: but I certainly don't think this chapter makes for an accessible and illuminating first introduction to serious model theory.

Ch. 4 is on incompleteness, and the approach here is significantly more gentle than the previous chapter. Goldstern and Judah make things rather easier for themselves by adopting a version of Peano Arithmetic which has exponentiation built in (so they don't need to tangle with Gödel's  $\beta$  function). And they only prove a semantic version of Gödel's first incompleteness theorem, assuming the soundness of PA. The proof here goes as by showing directly that – via Gödel coding – various syntactic properties and relations concerning PA are expressible in the language of arithmetic with exponentiation (in other words, they don't argue that those properties and relations are primitive recursive and then show that PA can express all such properties/relations).

How well, how accessibly, is this done? The authors hack through eleven pages (pp. 207–217) of the arithmetization of syntax, but the motivational commentary is brisk and yet the proofs aren't completely done (the authors still leave to the reader the task of e.g. coming up with a predicate satisfied by Gödel numbers for induction axioms). So this strikes the present reader as really being neither one thing nor another – neither a treatment with all the details nailed down, nor a helpfully discursive treatment with a lot of explanatory arm-waving. And in the end, the diagonalization trick seems to be just pulled like a rabbit out of the hat. After proving incompleteness, they prove Tarski's theorem and the unaxiomatizability of the set of arithmetic truths.

To repeat, the authors assume PA's soundness. They don't say anything about why we might want to prove the syntactic version of the first theorem, and don't even mention the second theorem which we prove by formalizing the syntactic version. So this could well leave students a bit mystified when they come across other treatments.

The book ends by noting that the relevant predicates in the arithmetization of syntax are  $\Sigma_1$ , and then *defines* a set as being recursively enumerable if it is expressible by a  $\Sigma_1$  predicates (so now talk of recursiveness etc. does get into the picture). But really, if you want to go down this route, this is surely all much better handled in Leary and Kristiansen's book.

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*Summary verdict* The first two chapters of this book can't really be recommended either for making a serious start on first-order logic or for revision. The third chapter could be used for a brisk revision of some model theory if you have already done some reading in this area. The final chapter about incompleteness (which the title of the book might lead you to think will be a high point) isn't the most helpful introduction

in this style – go for Leary and Kristiansen (2015) instead – and on the other hand doesn't go far enough for revision/consolidatory purposes.

## 13 Forster, 2003

Thomas Forster's *Logic, Induction and Sets* (CUP, 2003: pp. x + 234) is rather quirky, and some readers will enjoy it for just that reason. It is based on a wide-ranging lecture course given to mathematicians who – such being the oddities of the Cambridge tripos syllabus – at the beginning of the course already knew a good deal of maths but very little logic. The book is very bumpily uneven in level, and often goes skips forward very fast, so I certainly wouldn't recommend it as an 'entry level' text on mathematical logic for someone wanting a conventionally systematic approach. But it is often intriguing.

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*Some details* Ch. 1 is called 'Definitions and notations' but is rather more than that, and includes some non-trivial exercises: but if you are dipping into later parts of the book, you can probably just consult this opening chapter on a need-to-know basis.

Ch. 2 discusses 'Recursive datatypes', defined by specifying a starter-pack of 'founders' and some constructors, and then saying the datatype is what you can get from the founders by applying and replying the constructors (and nothing else). The chapter considers a range of examples, induction over recursive datatypes, well-foundedness, well-ordering and related matters (with some interesting remarks about Horn clauses too).

Ch. 3 is on partially ordered sets, and we get a lightning tour through some topics of logical relevance (such as the ideas of a filter and an ultrafilter).

Chs. 4 and 5 deal slightly idiosyncratically with propositional and predicate logic, and could provide useful revision material (there's a slip about theories on p. 70, giving two non-equivalent definitions).

Ch. 6 is on 'Computable functions' and is another lightning tour, touching on quite a lot in just over twenty pages (getting as far as Rice's theorem). Again, could well be useful to read as revision, especially if you want to highlight again the Big Ideas and their interrelations.

Ch. 7 is on 'Ordinals'. Note that Forster gives us the elements of the theory of transfinite ordinal numbers *before* turning to set theory in the next chapter.

It's a modern doctrine that ordinals just *are* sets, and that the basic theory of ordinals is part of set theory; and in organizing his book as he does, Forster comes nearer than most to getting the correct conceptual order into clear focus (though even he wobbles sometimes, e.g. at p. 182). However, the chapter could have been done more clearly.

Ch. 8 is called 'Set Theory' and is perhaps the quirkiest of them all – though not because Forster is here banging the drum for non-standard set theories (surprisingly given his interests, he doesn't). But the chapter is oddly structured, so for example we get a quick discussion of models of set theory and the absoluteness of  $\Delta_0$  properties *before* we actually encounter the ZFC axioms. The chapter is probably only for those, then, who already know the basics.

Ch. 9 comprises answers to some of the earlier exercises – exercises are indeed scattered through the book, and some of them are rather interesting.

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*Summary verdict* Different from the usual run of textbooks, not a good choice for beginners. However, if you already have encountered some of the material in one way or the other, Forster's book could very well be worth looking through for revision and/or to get some new perspectives.

## 14 Hedman, 2004

Shawn Hedman's *A First Course in Logic* (OUP, 2004: pp. xx + 214) is subtitled 'An Introduction to Model Theory, Proof Theory, Computability and Complexity'. So there's no lack of ambition in the coverage! And I do like the general tone and approach at the outset. So I wish I could be more enthusiastic about the book in general. But, as we will see, it is decidedly patchy both in terms of the level of the treatment of various topics, and in terms of the quality of the exposition.

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*Some details* After twenty pages of mostly rather nicely done 'Preliminaries' – including an admirably clear couple of pages the  $\mathbf{P} = \mathbf{NP}$  problem, Ch. 1 is on 'Propositional Logic'. On the negative side, we could certainly quibble that Hedman is a bit murky about object-language vs meta-language niceties. The treatment of induction half way through the chapter isn't as clear as it could be. Much more importantly, the chapter offers a particularly ugly formal deductive system. It is in fact a (single conclusion) sequent calculus, but with

proofs constrained to be a simple linear column of wffs. So – heavens above! – we are basically back to Lemmon’s *Beginning Logic* (1965). Except that the rules are not as nice as Lemmon’s (thus Hedman’s  $\wedge$ -elimination rule only allows us to extract a left conjunct; so we need an additional  $\wedge$ -symmetry rule to get from  $P \wedge Q$  to  $Q$ ). I can’t begin to think what recommended this system to the author out of all the possibilities on the market. On the positive side, there’s quite a nice treatment of a resolution calculus for wffs in CNF form, and a proof that this is sound and complete. This gives Hedman a completeness proof for derivations in his original calculus with a finite number of premisses, and he gives a compactness proof to beef this up to a proof of strong completeness.

Ch. 2, ‘Structures and first-order logic’ should really be called ‘Structures and first-order languages’, and deals with relations between structures (like embedding) and relations between structures and languages (like being a model for a sentence). I’m not sure I quite like its way of conceiving of a structure as always some  $\mathcal{V}$ -structure, i.e. as having an associated first-order vocabulary  $\mathcal{V}$  which it is the interpretation of – so structures for Hedman are what some would call labelled structures. But otherwise, this chapter is clearly done.

Ch. 3 is about deductive proof systems for first-order logic. The first deductive system offered is an extension of the bastardized sequent calculus for propositional logic, and hence is equally horrible. Somehow I sense that Hedman just isn’t much interested in standard proof-systems for logic. His heart is in the rest of the chapter, which moves towards topics of interest to computer scientists, about Skolem normal form, the Herbrand method, unification and resolution, so-called ‘SLD-resolution’, and Prolog – interesting topics, but not on *my* menu of basics to be introduced at this very early stage in a first serious logic course. The discussions seem quite well done, and will be accessible to an enthusiast with an introductory background (e.g. from Chiswell and Hodges) and who has read the section of resolution in the first chapter.

Ch. 4 is on ‘Properties of first-order logic’. The first section is a nice presentation of a Henkin completeness proof (for countable languages). There is then a long aside on notions of infinite cardinals and ordinals (Hedman has a policy of introducing background topics, like the idea of an inductive proof, and now these set theoretic notions, only when needed: but it can break the flow). §4.3 can use the assumed new knowledge about non-countable infinities to beef up the completeness proof, give upwards and downwards LS theorems, etc., again done pretty well. §§4.4–4.6 does some model theory under the rubrics ‘Amalgamation of structures’. ‘Preservation of formulas’ and ‘Amalgamation of vocabularies’: this already gets pretty abstract and uninviting, with not enough motivating

examples. §4.7 is better on ‘The expressive power of first-order logic’.

The next two chapters, ‘First order theories’ and ‘Models of countable theories’, give a surprisingly (I’d say, unrealistically) high level treatment of some model theory, going well beyond e.g. Manzano’s book, eventually talking about saturated models, and even ending with ‘A touch of stability’. This hardly chimes with the book’s prospectus as being a *first* course in logic. The chapters, however, could be useful for someone who wants to push onwards, after a first encounter with some model theory.

Ch. 6 comes sharply back to earth: an excellent chapter on ‘Computability and complexity’ back at a sensibly introductory level. It begins with a well done review of the standard material on primitive recursive functions, recursive functions, computing machines, semi-decidable decision problems, undecidable decision problems. Which is followed by a particularly clear introduction to ideas about computational complexity, leading up to the notion of **NP**-completeness. An excellent chapter.

Sadly, the following Ch. 8 on the incompleteness theorems again isn’t very satisfactory as a first pass through this material. In fact, I doubt whether a beginning student would take away from this chapter a really clear sense of what the key big ideas are, or of how to distinguish the general results from the hack-work needed to show that they apply to this or that particular theory. And things probably aren’t helped by proving the first theorem initially by Boolos’s method rather than Gödel’s. Still, just because it gives an account of Boolos’s proof, this chapter *can* be recommended as supplementary reading for those who have already seen some standard treatments of incompleteness.

The last two chapters ratchet up the difficulty again. Ch. 9 goes ‘Beyond first-order logic’ by speeding through second-order logic, infinitary logics (particularly  $\mathcal{L}_{\omega_1\omega}$ ), fixed-point logics, and Lindström’s theorem, all in twenty pages. This will probably go too fast for those who haven’t encountered these ideas before. It should be noted that the particularly brisk account of second-order logic gives a non-standard syntax and says nothing about Henkin vs full semantics. The treatment of fixed-point logics (logics that are ‘closed under inductive definitions’) is short on motivation and examples. But enthusiasts might appreciate the treatment of Lindström’s theorem.

Finally, Ch. 10 is on finite model theory and descriptive complexity. Beginners doing a first course in logic will again find this quite tough going.

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*Summary verdict* A very uneven book in level, with sections that work well at an introductory level and other sections which will only be happily

managed by considerably more advanced students. An uneven book in coverage too. By my lights, this couldn't be used end-to-end as a course text: but in the body of the Guide, I've recommended parts of the book on particular topics.

## 15 Hinman, 2005

Now for a really *big* Big Book – Peter G. Hinman's *Fundamentals of Mathematical Logic* (A. K. Peters, 2005: pp. 878).

The author says the book was written over a period of twenty years, as he tried out various approaches 'to enable students with varying levels of interest and ability to come to a deep understanding of this beautiful subject'. But I suspect that you will need to be mathematically quite strong to really cope with this book: whatever Hinman's intentions for a wider readership, this is not for the fainthearted.

The book's daunting size is due to its very wide coverage rather than a slow pace – so after a long introduction to first-order logic (or more accurately, to its model theory) and a discussion of the theory of recursive functions and incompleteness and related results, there follows a *very* substantial survey of set theory, and then lengthy essays on more advanced model theory and on recursion theory. As too often, proof theory is the poor relation here – indeed Hinman is very little interested in deductive systems for logic, which don't make an appearance until over two hundred pages into the book.

Let me mention at the outset what strikes me as a pretty unfortunate global notational convention, which might puzzle casual browsers or readers who want to start some way through the book, Given the two-way borrowing of notation between informal mathematics and the formal languages in which logicians regiment that mathematics, it is good to have some way of visually distinguishing the formal from the informal (so we don't just rely on context). One common method is font selection. Thus, even in an informal context, we may snappily say that addition commutes by writing e.g.  $\forall x \forall y x + y = y + x$ ; the counterpart wff for expressing this



in a fully formalized language may then be, e.g.,  $\forall x \forall y x + y = y + x$ . But instead of using sans serif or boldface for formal wffs, or another font selection, Hinman prefers using an ‘(informal) mathematical sign with a dot over it to represent a formal symbol in a formal language which denotes the informal object’, so he’d write  $\forall x \forall y x \dot{+} y \dot{=} y \dot{+} x$  for the formal wff. As you can imagine, this convention eventually leads to really nasty rashes of dots – for example, to take a relatively tame example from p. 459, we get

$$\dot{\bigcup} x \dot{=} \{z: \exists v[v \dot{\in} x \wedge z \dot{\in} v]\}$$

(note how even opening braces in formal set-former notation get dotted). This dottiness quite surely isn’t a happy choice!

*Some details* Hinman himself in his Preface gives some useful pointers to routes through the book, depending on your interests.

The Introduction gives a useful and approachable overview of some key notions tied up with the mathematical logician’s project of formalization (and talks about a version of Hilbert’s program as setting the scene for some early investigations).

Ch. 1 is on ‘Propositional Logic and other fundamentals’. §§1.1, 1.3 and 1.4 are devoted to the language of propositional logic, and give the usual semantics, define the notion tautological entailment and explore its properties, giving a proof of the compactness theorem. But note, there is no discussion at all here – or in the other sections of this chapter – of a proof-system for propositional logic.

§1.2 is a rather general treatment of proofs by induction and the definition of functions by recursion (signposted as skippable at this early stage – and indeed the generality doesn’t make for a particularly easy read for a section so early in the book). §§1.6 and 1.7 also cover more advanced material, mainly introducing ideas for later use: the first briskly deals e.g. with ultrafilters and ultraproducts (we get another take on compactness), and the second relates compactness to topological ideas and also introduces the idea of a Boolean algebra.

Ch. 2, ‘First-order logic’, presents the syntax and semantics of first-order languages, and then talks about first-order structures (isomorphisms, embeddings, extensions, etc), and proves the downward L-S theorem. We then get a general discussion of theories (thought of as sets of sentences closed under *semantic* consequence), and an extended treatment of some examples (the theory of equality,

the theory of dense linear orders, and various strengths of arithmetic). There's some quite sophisticated stuff here, including discussion of quantifier elimination. But there is still no discussion yet of a proof-system for first-order logic, so the chapter could as well, if not better, have been called 'Elements of model theory'.

Ch. 3, 'Completeness and compactness', starts with a compactness proof for countable languages. Then we at last have a *very* brisk presentation of an old-school axiomatic system for first-order logic (I told you that Hinman is not interested in proof-systems!), and a proof of completeness using the Henkin construction that has already been used in the compactness proof.

We next get – inter alia – an algebraic proof of compactness for first-order consequence via ultraproducts, and a return to Boolean algebras and e.g. the Rasiowa-Sikorski theorem (§3.3); an extension of the compactness and completeness results to uncountable languages (§3.4); and some heavy-duty applications of compactness (§3.5).

Finally in this action-packed chapter, we have some rather unfriendly treatments of higher-order logic (§3.6) and infinitary logic (§3.7).

Let's pause for breath. We are now a bit over 300 pages into the book. Things have already got pretty tough. The book is not quite a relentless march along a chain of definitions/theorems/corollaries; there are just enough pauses for illustrations and helpful remarks en route to make it a bearable. But Hinman does have a taste for going straight for abstractly general formulations (and his notational choices can sometimes be unhappy too). So as indicated in my preamble, the book will probably only appeal to mathematicians already used to this sort of fairly hardcore approach. In sum, therefore, I'd only recommend the first part of the book to the mathematically minded who already know their first-order logic and a bit of model theory; but such readers might then find it quite helpful as a beginning/mid-level model theory resource.

On we go. Next we have two chapters (almost 150 pages between them) on recursive functions, Gödelian incompleteness, and related matters. Perhaps it is because these topics are conceptually easier, more 'concrete', than what's gone before, or perhaps it is because the topics are closer to Hinman's heart, but these chapters seem to me to work better as an introduction to their topics. In particular, while not my favourite treatment, Ch. 4 is clear, very sensibly structured, and should be accessible to anyone with some background in logic and who isn't put off by a certain amount of mathematical abstraction. The chapter opens with informal proofs of the undecidability of consistent extensions of  $\mathbf{Q}$ , the first incompleteness theorem and Tarski's theorem on the undefinability of truth (as well as taking a first look at the second incompleteness theorem). These informal

proofs depend on the hypothesis that effectively calculable functions are expressible or the hypothesis that such functions are representable (we don't yet have a formal story about these functions). Unsurprisingly, given I do something in the same ball-park in my Gödel book, I too think this is a good way to start and to motivate the ensuing development. There follows, as you'd expect, the necessary account of the effectively calculable in terms of recursiveness, and then we get proofs that recursive functions can be expressed/represented in arithmetic, leading on to formal versions of the theorems about undecidable and incompleteness. This presentation takes a different-enough path through the usual ideas to be worth reading even if you've already encountered the material a couple of times before.

Ch. 5 is called 'Topics in definability' and, unlike the previous rather tightly organised chapter, is something of a grab-bag of topics. §5.1 says something about the arithmetical hierarchy; §5.2 discusses inter alia the indexing of recursive functions and the halting problem; §5.3 explains how the second incompleteness theorem is proved, and – while not attempting a full proof – there is rather more detail than usual about how you can show that the HBL derivability conditions are satisfied in PA. Then §5.4 gives more evidence for Church's Thesis by considering a couple of other characterisations of computability (by equation manipulation and by abstract machines) and explains why they again pick out the recursive functions. §5.5 discusses 'Applications to other languages and theories' (e.g. the application of incompleteness to a theory like ZF which is not initially about arithmetic). These various sections are all relatively clearly done.

Pausing for breath again, we might now try to tackle Ch. 6 on set theory (whose 200 pages amount by themselves to an almost-stand-alone book). The menu covers the basics of ZF, the way we can construct mathematics inside set theory, ordinals and cardinals, then models and independence proofs, the constructible universe, models and forcing, large cardinals and determinacy. But even from the outset, this does seem quite relentlessly hard going, too short on motivation and illustrations of concepts and constructions. Dense, to say the least. The author says of the chapter that his particular mode of presentation means that 'for each of the instances where one wants to verify that something is a class model – the intuitive universe of sets  $V$ , the constructible universe  $L$  and a forcing extension  $m[G]$  – . . . the proofs . . . exhibit more of the underlying unity.' So enthusiasts who know their set theory might want to do a fast read of the chapter to see if they can glean new insights. But I can't recommend this as a way into set theory when compared with the standard set theory texts mentioned in §§? and §§? of this Guide.

Ch. 7 returns to more advanced model theory for another 80 pages, getting as far as Morley's theorem. Again, if you want a more accessible initial treatment, you'll go for Hodges's *Shorter Model Theory*. And then why not tackle Marker's book if you are a graduate mathematician?

Finally, there's another equally long chapter on recursion theory. The opening sections on degrees and Turing reducibility are pretty approachable. The rest of the chapter gets more challenging but (at least compared with the material on model theory and set theory) should still be tolerably accessible to those willing to put in the work.

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*Summary verdict* It is *very* ambitious to write a book with this range and depth of coverage (as it were, an expanded version of Shoenfield, forty years on – but now when there is already a wealth of textbooks on the various areas covered, at various levels of sophistication). After such a considerable labour from a good logician, it seems very churlish to say it, but the treatments of, respectively, (i) first-order logic, (ii) model theory, (iii) computability theory and incompleteness, and (iv) set theory aren't as good as the best of the familiar stand-alone textbooks on the four areas. And I can't see that these shortcomings are balanced by any conspicuous advantage in having the accounts in a single text, rather than a handful of different ones. Still, the text should be in any university library, as enthusiasts might well find parts of it quite useful supplementary/reference material. Chapters 4, 5 and 8 on computability and recursion work the best.

## 16 Chiswell and Hodges, 2007

Ian Chiswell and Wilfrid Hodges's *Mathematical Logic* (OUP, 2007: pp. 249) is very largely focused on first-order logic, only touching on Church's undecidability theorem and Gödel's first incompleteness theorem in a Postlude after the main chapters. So it is perhaps stretching a point to include it in a list of texts which cover more than one core area of the mathematical logic curriculum. Still, I wanted to comment on this book at some length, without breaking the flow of the Guide (where it is warmly

recommended in headline terms), and this is the obvious place to do so.

Let me highlight three key features of the book, the first one not particularly unusual (though it still marks out this text from quite a few of the older, and not so old, competitors), the second very unusual but extremely welcome, the third a beautifully neat touch:

1. Chiswell and Hodges (henceforth C&H) present natural deduction proof systems and spend quite a bit of time showing how such formal systems reflect the natural informal reasoning of mathematicians in particular.
2. Instead of dividing the treatment of logic into two stages, propositional logic and quantificational logic, C&H take things in *three* stages. First, propositional logic. Then we get the quantifier-free part of first-order logic, dealing with properties and relations, functions, and identity. So at this second stage we get the idea of an interpretation, of truth-in-a-structure, and we get added natural deduction rules for identity and the handling of the substitution of terms. At both these first two stages we get a Hintikka-style completeness proof for the given natural deduction rules. Only at the third stage do quantifiers get added to the logic and satisfaction-by-a-sequence to the semantic apparatus. Dividing the treatment of first order logic into stages like this means that a lot of key notions get first introduced in the less cluttered contexts of propositional and/or quantifier-free logic, and the novelties at the third stage are easier to keep under control. This does make for a great gain in accessibility.
3. The really cute touch is to introduce the idea of polynomials and diophantine equations early – in fact, while discussing quantifier-free arithmetic – and to state (without proof!) Matiyasevich’s Theorem. Then, in the Postlude, this can be appealed to for quick proofs of Church’s Theorem and Gödel’s Theorem.

This is all done with elegance and a light touch – not to mention photos of major logicians and some nice asides – making an admirably attractive

introduction to the material.

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*Some details* C&H start with almost 100 pages on the propositional calculus. Rather too much of a good thing? Perhaps, if you have already done a logic course at the level of my intro book or Paul Teller's. Still, you can easily skim and skip. After Ch. 2 which talks about informal natural deductions in mathematical reasoning, Ch. 3 covers propositional logic, giving a natural deduction system (with some mathematical bells and whistles along the way, being careful about trees, proving unique parsing, etc.). The presentation of the formal natural deduction system is not exactly my favourite in its way representing discharge of assumptions (I fear that some readers might be puzzled about vacuous discharge and balk at Ex. 2.4.4 at the top of p. 19): but apart from this little glitch, this is done well. The ensuing completeness proof is done by Hintikka's method rather than Henkin's.

After a short interlude, Ch. 5 treats quantifier-free logic. The treatment of the semantics without quantifiers in the mix to cause trouble is very nice and natural; likewise at the syntactic level, treatment of substitution goes nicely in this simple context. Again we get a soundness and Hintikka-style completeness proof for an appropriate natural deduction system.

Then, after another interlude, Ch 7 covers full first-order logic with identity. Adding natural deduction rules (on the syntactic side) and a treatment of satisfaction-by-finite- $n$ -tuples (on the semantic side) all now comes very smoothly after the preparatory work in Ch. 5. The Hintikka-style completeness proof for the new logic builds very nicely on the two earlier such proofs: this is about as accessible as it gets in the literature, I think. The chapter ends with a look at the Löwenheim-Skolem theorems and 'Things that first-order logic cannot do'.

Finally, as explained earlier, material about diophantine equations introduced naturally by way of examples in earlier chapters is used in a final Postlude to give us undecidability and incompleteness results very quickly (albeit assuming Matiyasevich's Theorem).

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*Summary verdict* C&H have written a very admirably readable and nicely structured introductory treatment of first-order logic that can be warmly recommended. The presentation of the syntax of their type of (Gentzen-Prawitz) natural deduction system is perhaps done a trifle better elsewhere (Tennant's freely available *Natural Logic* mentioned in §?? gives a full dress version). But the core key sections on soundness and com-

pleteness proofs and associated metalogical results are second to none for their clarity and accessibility.

## 17 Leary and Kristiansen, 2015

Christopher C. Leary and Lars Kristiansen's *A Friendly Introduction to Mathematical Logic* (Milne Library 2015: pp. 364) is the second, significantly expanded, edition of a fine book originally just authored by Leary (Prentice Hall, 2000: pp. 218). The book is now available at a very attractive price; the main differences between the editions are a long new chapter on computability theory, and some 75 pages of solutions to exercises.

So how friendly is *A Friendly Introduction?* – meaning, of course, ‘friendly’ by the standard of logic books! I do like the tone a great deal (without being the least patronizing, it is indeed relaxed and inviting), and the level of exposition seems to me to be very well-judged for an introductory course. The book is officially aimed mostly at mathematics undergraduates without assuming any particular background knowledge. But as the Preface notes, it should also be accessible to logic-minded philosophers who are happy to work at following rather abstract arguments (and, I would add, who are also happy to skip over just a few inessential elementary mathematical illustrations).

What does the book cover? Basic first-order logic (up to the L-S theorems), the incompleteness theorems, and some computability theory. But by being so tightly focused, this book rarely seems to rush at what it *does* cover: the pace is pretty even. The authors do opt for a Hilbertian axiomatic system of logic, with fairly brisk explanations. (If you'd never seen before a serious formal system for first-order logic this could initially make for a somewhat dense read: if on the other hand you have been introduced to logic by trees or seen a natural deduction presentation, you would perhaps welcome a paragraph or two explaining the advantages for present purposes of the choice of an axiomatic approach here.) But the clarity is indeed exemplary.

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*Some details* Ch. 1, ‘Structures and languages’, starts by talking of first-order languages (The authors make the good choice of not starting over again with propositional logic, but assume that most readers will know their truth-tables so just give quick revision). The chapter then moves on to explaining the idea of first order structures, and truth-in-a-structure. There is a good amount of motivational chat as we go through, and the exercises – as elsewhere in the book – seem particularly well-designed to aid understanding. (The solutions to exercises added to the new edition makes the book even more suitable for self-study.)

Ch. 2, ‘Deductions’, introduces an essentially Hilbertian logical system and proves its soundness: it also considers systems with additional non-logical axioms. The logical primitives are ‘ $\vee$ ’, ‘ $\neg$ ’, ‘ $\forall$ ’ and ‘ $=$ ’. Logical axioms are just the identity axioms, an axiom-version of  $\forall$ -elimination (and its dual,  $\exists$ -introduction): the inference rules are  $\forall$ -introduction (and its dual) and a rule which allows us to infer  $\varphi$  from a finite set of premisses  $\Gamma$  if it is an instance of a tautological entailment. I don’t think this is the friendliest ever logical system (and no doubt for reasons of brevity, the authors don’t pause to consider alternative options); but it certainly is not horrible either. If you take it slowly, the exposition here should be quite manageable even for the not-very-mathematical.

Ch. 3, ‘Completeness and compactness’, gives a nice version of a Henkin-style completeness theorem for the described deductive system, then proves compactness and the upward and downward Lwenheim-Skolem theorems (the latter in the version ‘if  $L$  is a countable language and  $\mathfrak{B}$  is an  $L$ -structure, then  $\mathfrak{B}$  has a countable elementary substructure’ [the proof might be found *just* a bit tricky though]). So there is a little model theory here as well as the completeness proof: and you could well read this chapter without reading the previous ones if you are already reasonably up to speed on structures, languages, and deductive systems. And so, in a hundred pages, we wrap up what is indeed a pretty friendly introduction to FOL.

Ch. 4, ‘Incompleteness, from two points of view’ is a helpful bridge chapter, outlining the route ahead, and then defining  $\Sigma$ ,  $\Pi$  and  $\Delta$  wffs (no subscripts in their usage, and exponentials are atomic – maybe a footnote would have been wise, to help students when they encounter other uses). Then in Ch. 5, ‘Syntactic Incompleteness – Groundwork’, the authors (re)introduce the theory they call  $N$ , a version of Robinson Arithmetic with exponentiation built in. They then show that (given a scheme of Gdel coding) that the usual numerical properties and relations involved in the arithmetization of syntax – such as, ultimately,  $Prf(m, n)$ , i.e.  $m$  codes for an  $N$ -proof of the formula numbered  $n$  – can be



represented in  $N$ . They do this by the direct method. That is to say, instead of [like my *IGT*] showing that those properties/relations are (primitive) recursive, and that  $N$  can represent all (primitive) recursive relations, they directly write down  $\Delta$  wffs which represent them. This inevitably gets more than a bit messy: but they have a very good stab at motivating every step working up to showing that  $N$  can express  $Prf(m, n)$  by a  $\Delta$  wff. If you want a full-dress demonstration of this result, then this is one of the most user-friendly available.

Ch. 6, ‘The Incompleteness Theorems’, is then pretty short: but all the groundwork has been done to enable the authors now to give a brisk but very clear presentation, at least after they have proved the Diagonalization Lemma. I did complain that, in the first edition, the proof of the Lemma was slightly too rabbit-out-of-a-hat for my liking. This edition I think notably softens the blow (one of many such small but significant improvements, as well as the major additions). And with the Lemma in place, the rest of the chapter goes very nicely and accessibly. We get the first incompleteness theorem in its semantic version, the undecidability of arithmetic, Tarski’s theorem, the syntactic version of incompleteness and then Rosser’s improvement. Then there is nice section giving Boolos’s proof of incompleteness echoing the Berry paradox. Finally, the second theorem is proved by assuming (though not proving) the derivability conditions.

The newly added Ch. 7, ‘Computability theory’ starts with a very brief section on historical origins, mentioning Turing machines etc.: but we then settle to exploring the  $\mu$ -recursive functions. We get some way, including the S-m-n theorem and a full-dress proof of Kleene’s Normal Form Theorem (with due apologies for the necessary hacking though details) and meet the standard definition of the set  $K = \{x \mid x \in W_x\}$  where  $W_x$  is the domain of the computable function with index  $x$ . The uncomputability of  $K$  is then used, in the usual sort of way, to prove the undecidability of the *Entscheidungsproblem*, to re-prove the incompleteness theorem, and in tackling Hilbert’s 10th problem. This is all nicely done in the same spirit and with the same level of accessibility as the previous chapters.

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*Summary verdict* If you have already briefly met a formally presented deductive system for first-order logic, and some account of its semantics, then you’ll find the opening two chapters of this book very manageable (if you haven’t they’ll be a bit more work). The treatment of completeness etc. in Ch. 3 would make for a nice stand-alone treatment even if you don’t read the first two chapters. Or you could just start the book by reading 2.8 (where  $N$  is first mentioned), and then read the excellent

ensuing chapters on incompleteness and computability with a lot of profit. *A Friendly Introduction* is indeed in many ways a very unusually likeable introduction to the material it covers, and has a great deal to recommend it.