# Preface

This is not another textbook on mathematical logic: it is a Study Guide, a book *about* textbooks on mathematical logic. Its purpose is to enable you to locate the best resources for teaching yourself various areas of logic. Inevitably, given the breadth of its coverage, the Guide is quite long: but don't let that scare you off! There is a good deal of signposting and there are also explanatory overviews to enable you to pick your way through and choose the parts which are most relevant to you.

*Beginning Mathematical Logic* is a descendant of my much-downloaded *Teach Yourself Logic*. The new title highlights that the Guide focuses on the core mathematical logic curriculum and a few closely related topics. It also signals that I do not try to cover advanced material in much detail.

The first chapter says more about who the Guide is intended for, what it covers, and how to use it. But let me note straightaway that most of the reading recommendations do indeed point to published *books*. True, there are quite a lot of on-line lecture-notes that university teachers have made available. Some of these are excellent. However, they do tend to be terse, and often *very* terse (as entirely befits material originally intended to support a lecture course). They are therefore usually not as helpful as fully-worked-out book-length treatments for students needing to teach themselves.

So where can you find the titles mentioned here? I suppose I ought to pass over the issue of downloading books from certain well-known and extremely well-stocked copyright-infringing PDF repositories. That's between you and your conscience (though the books are all available to be sampled there). Anyway, many prefer to work from physical books. Most of these titles should in fact be held by any large-enough university library which has been trying over the years to maintain core collections in mathematics and philosophy (and if the local library is too small, books should be borrowable through some inter-library loans system).

Since I'm not assuming that you will be buying the recommended books, I have *not* made cost or being currently in print a significant consideration. However, I have marked with a star* books that are available new or second-hand at a reasonable price (or at least are unusually good value for the length and/or importance of the book). When e-copies of books are freely and legally available, links are provided. Where articles or encyclopaedia entries have been recommended, these can almost always be freely downloaded, and again I give links.

Before I retired from the University of Cambridge, it was my greatest good fortune to have secure, decently paid, university posts for forty years in leisurely times, with almost total freedom to follow my interests wherever they meandered. Like most of my contemporaries, for much of that time I didn't really appreciate how extraordinarily lucky I was. In writing this Study Guide and making it readily available, I am trying to give a little back by way of heartfelt thanks. I hope you find it useful.[1]

[1] Many, many thanks are due to all those who commented on versions of *Teach Yourself Logic* over more than a decade. Further comments and suggestions for future editions of this revised Guide will always be gratefully received.

# Contents

# 1 The Guide, and how to use it

Who is this Study Guide for? What does it cover? At what level? How should the Guide be used? And what background knowledge do you need, in order to make use of it? This preliminary chapter explains.

## 1.1 Who is the Guide for?

It is a depressing phenomenon. Relatively few mathematics departments now have courses on mathematical logic. And serious logic is taught less and less in philosophy departments too.

Yet logic itself is, of course, no less exciting and rewarding a subject than it ever was. So how is knowledge to be passed on if there are not enough courses, or indeed if there are none at all? It seems that many will need to teach themselves from books, either solo or by organizing their own study groups (local or online).

In a way, this is perhaps no real hardship; there are some wonderful books written by great expositors out there. But *what* to read and work through? Logic books can have a *very* long shelf life, and you shouldn't at all dismiss older texts when starting out on some topic area. There's more than a sixty year span of publications to select from, which means that there are hundreds of good books to choose from.

That's why students – whether mathematicians or philosophers – wanting to begin learning some logic need a Study Guide if they are to find their way around the very large literature old and new, with the aim of teaching themselves enjoyably and effectively.

There are other students too who will have interests in areas of logic, e.g. theoretical linguists and computer scientists. But this Guide isn't written with them much in mind.

## 1.2 The Guide's structure: "Est omnis divisa in partes tres"

There is another preliminary chapter after this one, Chapter 2 on *'naive' set theory*, which reviews the concepts and constructions typically taken for granted in quite elementary mathematical writing (not just in texts about logic).Then the main Guide, like Gaul, divides into three parts.

In headline terms,

*Part I* covers the now standard core mathematical logic curriculum at an introductory level.

*Part II* is by way of an interlude – we pause to look sideways at a range of other logical topics, while still remaining at roughly the same level of mathematical difficulty.

*Part III* then gives pointers forward to more advanced treatments of some of the areas introduced in Part I.

The chapters in Parts I and II carve up the broad field of logic in a pretty conventional way: but of course, even these 'horizontal' divisions into different subfields can in places be a little arbitrary. And the 'vertical' divisions between the entry-level coverage in Parts I and the further explorations of the same areas in Part III are necessarily going to be rather more arbitrary – but some such divisions are surely necessary if chapters are not to become too unwieldy and too daunting.

## 1.3 Part I: Core topics

The standard menu in a mathematical logic course has remained fairly fixed ever since e.g. Elliott Mendelson's justly famous *Introduction to Mathematical Logic* (1st edn., 1964). And this menu is reflected in the chapters in Part I.

*Chapter 3* discusses *classical first-order logic* (FOL), which is at the fixed centre of any mathematical logic course. The remaining chapters all depend on this one, as we discuss the use of classical logic in building formal theories, or consider extensions and variants.

Now, there is one extension of FOL that it is worth knowing just a little about straight away (in order to understand some themes touched on in the next few chapters). So:

*Chapter 4* goes beyond first-order logic by briefly looking at *second-order logic*. (Second-order languages have more ways of forming general propositions than first-order ones.)

You can then start on the topics of the following three chapters in whichever order you choose:

*Chapter 5* introduces a modest amount of *model theory* which, roughly speaking, explores how formal theories relate to the structures they are about.

*Chapter 6* looks at one particular kind of formal theory, i.e. *formal arithmetics*, and relatedly explores the theory of computable functions. We arrive at proofs of epochal results such as Gödel's incompleteness theorems.

*Chapter 7* is on *set theory* proper – starting fairly informally, examining basic notions of cardinals and ordinals, constructions of number systems in set theory, the role of the axiom of choice, etc. We then look at the standard formal axiomatization, first-order ZFC, and nod towards alternatives.

What else should you encounter when beginning serious logic? There is a whole area which is under-represented in many textbooks:

*Chapter 8* says something about *proof theory*. OK, that label is pretty unhelpful given that most areas of logic deal with proofs! – but it conventionally points to a cluster of issues about the structure of proofs and the consistency of theories, etc.

We will find that considerations from proof theory highlight one particular subsystem of FOL which is also important for independent reasons. Hence, to end Part I,

*Chapter 9* says more about *intuitionist logic*, which drops the classical principle that, whatever proposition we take, either it or its negation is true.

## 1.4 Part II: Other logics

In developing intuitionist logic, we meet a new way of thinking about the meanings of the logical operators, using so-called 'possible world semantics'. This takes us into the territory we explore next:

*Chapter 10* discusses *modal logics*, which deal with various notions of necessity and possibility.

Now, return to the standard version of FOL. This can can be criticized in various ways – for example, (i) it allows certain arguments to count as valid even when the premisses are irrelevant to the conclusion; (ii) it is not neutral about existence assumptions; and (iii) it can't cope naturally with terms denoting more than one thing like 'Russell and Whitehead' and 'the roots of the quintic equation $E$'. So:

*Chapter 11* discusses a number of deviations from FOL. These include so-called relevant logics (where we impose stronger requirements on the relevance of premisses to conclusions for valid arguments), free logics (i.e. logics free of existence assumptions, where we no longer presuppose that e.g. names in a formal language actually name something), and plural logics (where we can e.g. cope with plural terms).

These other logics are perhaps mostly of interest to philosophers – though any logician interested in the foundations of mathematics, say, should want to know more about the pros and cons of dealing with talk about many things by using set theory vs second-order logic vs plural logic.

## 1.5 Part III: Going further

This is primarily a Guide to *beginning* mathematical logic. And indeed, the recommended introductory readings in the first two parts won't take you *very* far. But they should be enough to put you in a position from which you can venture into rather more advanced work under your own steam. Still, in Part III, I suggest readings for those who do want to further explore some of the mainstream topics of Part I.

I won't going into any more details here. But, very roughly, if the earlier Parts are at undergraduate level, Part III is at graduate level. And I don't pretend that the coverage remains systematic: it now follows my own interests, and just records some of what, over the years, I have found interesting and/or helpful.

## 1.6 How chapters in Part I are structured

Each chapter in Part I starts with one or more overviews of its topic area(s). These overviews are intended to give helpful pointers to the coverage of the ensuing chapter; but if their necessarily brisk headlines sometimes mystify, feel very free to skim or skip as much you like.

Overviews are followed by a list of main recommended texts for the chapter's topic(s), put into what strikes me as a sensible reading order.

I then offer some suggestions for alternative/additional reading at about the same level or only another half a step up in difficulty/sophistication. And because it is always quite illuminating to know just a little of it, background history of a topic, most chapters end with a few brisk suggestions for reading on that.

## 1.7 Strategies for self-teaching from logic books

As I said in the Preface, one reason for the length of this Guide is its breadth of coverage. But there is another reason, connected to a point which I now really want to highlight:

> I very strongly recommend tackling a new area of logic by reading a series of books which *overlap* in level (with the next one covering some of the same ground and then pushing on from the previous one), rather than trying to proceed by big leaps.

In fact, I probably can't stress this bit of advice too much (which, in my experience, applies equally to getting to grips with any new area of mathematics). This approach will really help to reinforce and deepen understanding as you encounter and re-encounter the same material, coming at it from somewhat different angles, with different emphases.

Exaggerating only a little, there are many instructors who say 'This is the textbook we are using/here is my set of notes: take it or leave it'. But you will *always* gain from looking at a variety of treatments, perhaps at slightly different levels. The multiple overlaps in coverage in the reading lists in later chapters, which help make the Guide as long as it is, are therefore fully intended. They also mean that you should always be able to find the options that best suit your degree of mathematical competence and your preferences for textbook style.

To repeat: you will certainly miss a lot if you concentrate on just one text in a given area, especially at the outset. Yes, do very carefully read one or two central texts, chosing books that work for you. But do also cultivate the crucial habit of judiciously skipping and skimming through a number of other works so that you can build up a good overall picture of an area seen from various angles of approach.

While we are talking about strategies for self-teaching, I suppose I should add a quick remark on the question of doing exercises.

> Mathematics is, as they say, not merely a spectator sport: so you should try some of the exercises in the books as you read along, in order to check and reinforce comprehension. On the other hand, don't obsess about this, and do concentrate on the exercises that look interesting and/or might deepen understanding.

Note that some authors have the irritating(?) habit of burying quite important results among the exercises, mixed in with routine homework. It is therefore always a good policy to skim through the exercises in a book even if you don't plan to work on answers to very many of them.

## 1.8 Choices, choices

What has guided my choices of texts to recommend?

Different people find different expository styles congenial. What is agreeably discursive for one reader might be irritatingly verbose and slow-moving for another. For myself, I do particularly like books that are good at explaining the ideas behind the various formal technicalities while avoiding needless early complications, excessive hacking through routine detail, or misplaced 'rigour'. So I prefer a treatment that highlights intuitive motivations and doesn't rush too fast to become too abstract: this is surely what we particularly want in books to be used for self-study. (There's a certain tradition of masochism in older maths writing, of going for brusque formal abstraction from the outset with little by way of explanatory chat: this is quite unnecessary in other areas, and just because logic is all about formal theories, that doesn't make it any more necessary here.)

The selection of readings in the following chapters reflects these tastes. But overall, while I have no doubt been opinionated, I don't think that I have been very idiosyncratic: indeed, in many respects I have probably been really rather conservative in my choices. So nearly all the readings I recommend will very widely be agreed to have significant virtues (even if other logicians would have different preference-orderings).

## 1.9 So what do you need to bring to the party?

There is no specific knowledge you need before tackling e.g. the main recommended books on FOL. And none of the more introductory books recommended in other chapters in Part I and Part II presupposes very much 'mathematical maturity'. So mathematics students from mid-year undergraduates up should be able to just dive in and explore.

What about philosophy students without any mathematical background? It will certainly help to have done an introductory logic course based on a book at the level of my own *Introduction to Formal Logic** (2nd edition, CUP, 2020; corrected version now freely downloadable from logicmatters.net/ifl), or Nicholas Smith's excellent *Logic: The Laws of Truth* (Princeton UP 2012). And non-mathematicians should broaden their proof-writing skills by also looking at an introductory book like one of the following:

1. Daniel J. Velleman, *How to Prove It: A Structured Approach** (CUP, 3rd edition, 2019). From the Preface: "Students ... often have trouble the first time that they're asked to work seriously with mathematical proofs, because they don't know 'the rules of the game'. What is expected of you if you are asked to prove something? What distinguishes a correct proof from an incorrect one? This book is intended to help students learn the answers to these questions by spelling out the underlying principles involved in the construction of proofs." There are chapters on the propositional connectives and quantifiers, and on key informal proof-strategies for using them; there are chapters on relations and functions, a chapter on mathematical induction, and a final chapter on infinite sets (countable vs. uncountable sets).

   This is a truly excellent student text; at least skip and skim through the book, taking what you need (perhaps paying especial attention to the chapter on mathematical induction).

2. Joel David Hamkins, *Proof and the Art of Mathematics** (MIT Press, 2020) From the blurb: "This book offers an introduction to the art and craft of proof-writing. The author ... presents a series of engaging and compelling mathematical statements with interesting elementary proofs. These proofs capture a wide range of topics ... The goal is to show students and aspiring mathematicians how to write [informal!] proofs with elegance and precision." A less conventional text than Velleman's, with a different emphasis. Attractively written (though it has to be said rather uneven in level and tone). Readers without much of a mathematical background at all could still well enjoy this, and will learn a good deal, e.g. about proofs by induction. Lots of striking and memorable examples.

# 2 A very little informal set theory

Notation, concepts and constructions from entry-level set theory are often presupposed in elementary mathematical texts – including some of the introductory logic texts mentioned in the following chapters, even before we get round to officially studying set theory itself. If the absolute basics aren't already familiar to you, it is worth pausing to get acquainted at a very early stage.

In §2.1, then, I note what you should ideally know about sets here at the outset. It isn't a lot! For now, we proceed 'naively' – i.e. we proceed quite informally, and will just *assume* that the various constructions we talk about are permitted, etc. §2.2 gives recommended readings on basic informal set theory for those who need them. In §2.3 I point out that, while the use of set-talk in elementary contexts is conventional, it many cases it can in fact be eliminated without serious loss.

## 2.1 Sets: a checklist of some basics

(a) So what elementary ideas *should* you be familiar with, given our limited current purposes? Let's have a quick checklist (there should be no surprises for mathematicians!):

(i) A set is a collection of objects, treated as a single object. $A$ and $B$ count as one and the same set if and only if whatever is a member of $A$ is a member of $B$ and vice versa (that's the *extensionality* principle).

(ii) Notation. We use the likes of '$\{a, b, c, d\}$' to denote the set whose members are $a, b, c, d$. And we use the likes of '$\{x \mid x \text{ is } F\}$' to denote the set of things (in some domain) which are $F$.

Membership is symbolized by '$\in$'; the subset relation is symbolized by '$\subseteq$', so $A \subseteq B$ is true just when for all $x$, if $x \in A$, then $x \in B$.

The membership and subset relations need to be sharply distinguished from each other (the beginning of set-theoretic wisdom!). And note in particular that the singleton set $\{a\}$ is to be distinguished from its sole member $a$: thus $a \in \{a\}$ and $\{a\} \subseteq \{a\}$, but not $a = \{a\}$ and not $a \subseteq \{a\}$.

(iii) If $A$, $B$ are sets, so are their union, intersection and their powersets.

If the intersection $A \cap B$ is always to exist, then we have to allow a set which contains no members (since $A$ and $B$ might not overlap). By extensionality, the empty set $\emptyset$ is unique.

The powerset of $A$, $\mathcal{P}(A)$, is the set whose members are all and only the subsets of $A$. Note this assumes that sets are indeed things which can be members of other sets.

(iv) Sets are in themselves unordered. But we often need to work with ordered pairs, ordered triples, ordered quadruples, ..., *tuples* more generally. We use '$\langle a, b\rangle$' – or often simply '$(a, b)$' – for the ordered pair, first $a$, then $b$. So, while $\{a, b\} = \{b, a\}$, by contrast $\langle a, b\rangle \neq \langle b, a\rangle$.

We can define ordered pairs using unordered sets in various ways: all we need is some definition which ensures that $\langle a, b\rangle = \langle a', b'\rangle$ if and only if $a = a'$ and $b = b'$. The following is standard: $\langle a, b\rangle =_{\text{def}} \{\{a\}, \{a, b\}\}$.

Once we have ordered pairs available, we can use them to define ordered triples: $\langle a, b, c\rangle$ can be defined as first the pair $\langle a, b\rangle$, then $c$, i.e. as $\langle\langle a, b\rangle, c\rangle$. Then the quadruple $\langle a, b, c, d\rangle$ can be defined as $\langle\langle a, b, c\rangle, d\rangle$. And so it goes.

(v) The Cartesian product $A \times B$ of the sets $A$ and $B$ is the set whose members are all the ordered pairs whose first member is in $A$ and whose second member is in $B$. So $A \times B$ is $\{\langle x, y\rangle \mid x \in A \;\&\; y \in B\}$. Cartesian products of $n$ sets are defined as sets of $n$-tuples, again in the obvious way.

(vi) If $R$ is a binary relation between members of the set $A$ and members of the set $B$, then its extension is the set of ordered pairs $\langle x, y\rangle$ (with $x \in A$ and $y \in B$) such that $x$ is $R$ to $y$. So the extension of $R$ is a subset of $A \times B$.

Similarly, the extension of an $n$-place relation is the set of $n$-tuples of things which stand in that relation. In the unary case, where $P$ is a property defined over some set $A$, then we can simply say that the extension of $P$ is the set of members of $A$ which are $P$.

For many mathematical purposes, we treat properties and relations extensionally; i.e. we regard properties with the same extension as being the same property, and likewise for relations. Indeed, we can often simply treat a property (relation) as if it simply *is* its extension.

(vii) The extension (or graph) of a unary function $f$ which sends members of $A$ to members of $B$ is the set of ordered pairs $\langle x, y\rangle$ (with $x \in A$ and $x \in B$) such that $f(x) = y$. Similarly for $n$-place functions. For many purposes, we treat functions extensionally, regarding functions with the same extension as the same. Again we often treat a function as if it *is* its extension, i.e. we identify a function with its graph.

(viii) Relations can, for example, be reflexive, symmetric, transitive; equivalence relations are all three. Note that if $\equiv$ is an equivalence relation defined over some set, it partitions that set into equivalence classes (we never say 'equivalence sets'!) of objects standing in that relation. If $[x]$ is the equivalence class (with respect to $\equiv$) containing $x$, then $[x] = [y]$ if and only if $x \equiv y$.

(ix) Two sets are equinumerous just if we can match up their members one-to-one, i.e. when there is a one-to-one function, a bijection, between the sets. A set is countably infinite if and only if it is equinumerous with the natural numbers.

And here we get to the first exciting claim – *there are infinite sets which are not countably infinite*. A simple example is the set of infinite binary strings. Why so? if we take any countably infinite list of such strings, we can always define another infinite binary string which differs from the first string on our list in the first place, differs from the second in the second place, the third in the third place, etc., so cannot appear anywhere in our given list.

This is just the beginning of a story about how sets can have different infinite 'sizes' or cardinalities. But at this stage you need to know little more than that bald fact: further elaboration can wait.

(x) There's one further, rather less elementary, idea that you should perhaps also meet sooner rather than later, so that you recognize any passing references to it. This is the Axiom of Choice. In one version, this says that, given an infinite family of sets, there is a choice function – i.e. a function which 'chooses' a single member from each set in the family. Bertrand Russell's toy example: given an infinite collection of pairs of socks, there is a function which chooses one sock from each pair.

Note that while other principles for forming new sets (e.g. unions, powersets) determine what the members of the new set are, Choice just tells us that there *is* a set (the extension of the choice function) which plays a certain role, without specifying its members.

At this stage you basically just need to know that Choice is a principle which is implicitly or explicitly invoked in many mathematical proofs. But you should also know that it is independent of other basic set-theoretic principles (and there are set theories in which it doesn't hold) – which is why we often explicitly note when, in more advanced logical theory, a result does indeed depend on Choice.

(b) An important observation before proceeding.

You've probably met Russell's Paradox. Say a set is *normal* if it isn't a member of itself. The set of musketeers $\{\text{Athos}, \text{Porthos}, \text{Aramis}\}$ is not another musketeer and so is not a member of itself. Again, the set of prime numbers isn't itself a prime number, so also isn't a member of itself. These are normal sets. Now we ask: is there a Russell set $R$ whose members are all and only the normal sets?

No. For if there were, it would be a member of itself if and only if wasn't – think about it! – which is impossible. The putative set $R$ is, in some sense, 'too big' to exist. Hence, if we overshoot and naively suppose that for *any* property (including the property of being a normal set) there is a set which is its extension, we get into deep trouble.

Now, some people use 'naive set theory' to mean, specifically, a theory which makes that simple but hopeless assumption that any property at all has a set as its extension. As we've just seen, naive set theory in *this* sense is inconsistent.

And here we need to avoid getting entangled in one of those rather annoying terminological divergences. For many others use 'naive set theory' just to mean set theory developed informally, without rigorous axiomatization, but guided by unambitious low-level principles. In this different second sense, we have been proceeding naively in this chapter – and hopefully we remain on track for developing a consistent story! Thus, we were careful in (vi) to assign extensions just to those properties and relations that are defined over domains we are already given as sets. True, our story so far is silent about exactly which putative sets *are* the kosher ones – i.e. are not 'too big' to be to be problematic. However, important though it is, we can leave this topic until Chapter 7 when we turn to set theory proper. Low-level practical uses of sets in 'ordinary' mathematics seem remote from such problems; hopefully, we can continue to proceed naively for now in elementary contexts.

## 2.2 Recommendations on informal basic set theory

You only need a *very* modest mathematical background for the ideas on our checklist to be already entirely familiar; and if they are, you can skip over these first reading suggestions. But non-mathematicians (or mathematicians who have never seen a systematic presentation) should find one of the following to be exactly what they need:

1. Tim Button, *Set Theory: An Open Introduction* (Open Logic Project), Chapters 1–5. Available at tinyurl.com/opensettheory. Read Chapter 1 for some interesting background. Chapter 2 introduces basic notions like subsets, powersets, unions, intersections, pairs, tuples, Cartesian products. Chapter 3 is on relations (treated as sets). Chapter 4 is on functions. Chapter 5 is on the size of sets, countable vs uncountable sets, Cantor's Theorem. At this stage in his book, Button is proceeding naively in our second sense, with the promise that everything he does can be replicated in the rigorously axiomatized theory he introduces later.

   Button writes, here as elsewhere, with very admirable clarity. So this is warmly recommended.

2. David Makinson, *Sets, Logic and Maths for Computing* (Springer, 3rd edn 2020), Chapters 1 to 3. This is exceptionally clear and very carefully written for students without much mathematical background. Chapter 1 reviews basic facts about sets. Chapter 2 is on relations. Chapter 3 is on functions. This too can be warmly recommended (though you might want to supplement it by following up his reference to Cantor's Theorem).

Now, Makinson doesn't mention the Axiom of Choice at all. While Button does eventually get round to Choice in his Chapter 16; but the treatment there depends on the set theory developed in the intervening chapters, so isn't appropriate for us just now. Instead, the following two pages should be enough for the present:

3. Timothy Gowers et al. eds, *The Princeton Companion to Mathematics* (Princeton UP, 2008), §III.1: The Axiom of Choice.

We return to set theory in Chapter 7. But let me mention a classic short text that – whether you are a philosopher or mathematician – you could usefully and enjoyably read in advance:

4. Paul Halmos, *Naive Set Theory** (1960: republished by Martino Fine Books, 2011).

The purpose of this famous book, Halmos says in his Preface, is "to tell the beginning student ... the basic set-theoretic facts of life, and to do so with the minimum of philosophical discourse and logical formalism". Again he is proceeding naively in our second sense. True he tells us about some official axioms as he goes along, but he doesn't explore the development of set theory inside a resulting formal theory: this is informally written in an unusually conversational style for a maths book, concentrating on the motivation for various concepts and constructions. You could read the first fifteen – very short – chapters now, leaving the rest for later.

Finally, it is still well worth seeking out this famous discussion where we meet Russell's infinite collection of socks:

5. Bertrand Russell, *An Introduction to Mathematical Philosophy*** (1919), Chapter XII, 'Selections and the Multiplicative Axiom'. Available at tinyurl.com/russellimp.

The 'Multiplicative Axiom' is Russell's name for a version of the Axiom of Choice. (In fact, the whole of Russell's lovely book remains after all these years a wonderful read if you have any interest in the foundations of mathematics!)

## 2.3 Virtual classes, real sets

An afterword. According to Cantor, a set is a unity, a single thing in itself over and above its members. But if *that* is the guiding idea, then it is worth noting that *a great deal of elementary informal set talk in mathematics is really no more than a façon de parler*. Yes, it is a useful and familiar idiom for talking about many things at once; but in many elementary contexts apparent talk of a set doesn't really carry any serious commitment to there being any *additional* object, a set, over and above those many things. On the contrary, in such contexts, apparent talk about a *set of $F$s* can very often be paraphrased away into direct talk about those $F$s, without any loss of content.

Here is just one example, relevant for us. It is usual to say something like this: (1) "A set of formulas $\Gamma$ logically entails the formula $\varphi$ if and only if any valuation which makes every member of $\Gamma$ true makes $\varphi$ true too". Don't worry for now about the talk of valuations: just note that the reference to a *set* of formulas and it *members* is doing no work here. It would do just as well to say

(2) "Some formulas $G$ logically entail $\varphi$ if and only if every valuation which makes those formulas $G$ all true makes $\varphi$ true too". The set version (1) adds nothing important to the plural version (2).

When set talk can be paraphrased away like this, we are only dealing with – as they say – mere *virtual classes*.

One source for this description is W.V.O. Quine's famous discussion in the opening chapter of his *Set Theory and its Logic* (1963):

> Much ... of what is commonly said of classes with the help of '$\in$' can be accounted for as a mere manner of speaking, involving no real reference to classes nor any irreducible use of '$\in$'. ... [T]his part of class theory ... I call the virtual theory of classes.

You will eventually find that this same usage plays an important role in set theory in some treatments of so-called 'proper classes' as distinguished from sets. For example, in his standard book *Set Theory* (1980), Kenneth Kunen writes

> Formally, proper classes do not exist, and expressions involving them must be thought of as abbreviations for expressions not involving them.

The distinction being made is an old one. Here is Paul Finsler, writing in 1926 (as quoted by Luca Incurvati, in his *Conceptions of Set*):

> It would surely be inconvenient if one always had to speak of many things in the plural; it is much more convenient to use the singular and speak of them as a class. ... A class of things is understood as being the things themselves, while the set which contains them as its elements is a single thing, in general distinct from the things comprising it. ... Thus a set is a genuine, individual entity. By contrast, a class is singular only by virtue of linguistic usage; in actuality, it almost always signifies a plurality.

Finsler writes 'almost always', I take it, because a class term may in fact denote just one thing, or even – perhaps by misadventure – none.

Nothing hangs on the particular terminology here, 'classes' vs 'sets'. What matters (or will eventually matter) is the distinction between non-committal, eliminable, talk – talk of merely virtual sets/classes/pluralities (whichever idiom we use) – and uneliminable talk of sets as entities in their own right.

It is a tough question exactly when and why, at various points as we get into the serious study of logic, we do eventually get committed to working with sets as genuine entities. Now certainly is the time to pause over the issue! I just thought it worth flagging up that, at least when beginning logic, you *can* take a lot of set talk just as a pretty non-committal light-weight dispensable idiom, simply there to conveniently enable us to talk in the singular about many things at once.