5 Model theory

The high point of a first serious encounter with FOL is the proof of the completeness theorem. Introductory texts then usually discuss at least a couple of quick corollaries of the proof – the *compactness theorem* (which we've already met) and the *downward Löwenheim-Skolem theorem.* And so we take initial steps into what we can call Level 1 model theory. Further along the track we will encounter Level 3 model theory (I am thinking of the sort of topics covered in e.g. the later chapters of the now classic texts by Wilfrid Hodges and David Marker recommended in Part III). In between, there is a stretch of what we can think of as Level 2 theory – still relatively elementary, relatively accessible without too many hard scrambles, but going somewhat beyond the very basics.

Putting it like this in terms of 'levels' is of course only for the purposes of rough-and-ready organization: there are no sharp boundaries to be drawn. In a first mathematical logic course, though, you should certainly get your head around Level 1 model theory. Then tackle as much Level 2 theory as grabs your interest.

But what topics can we assign to these first two levels?

5.1 Elementary model theory: an overview

(a) Model theory is about mathematical structures and about how to characterize and classify them using formal languages. Put another way, it concerns the relationship between a mathematical theory (regimented as a collection of formal sentences) and the structures which 'realize' that theory (i.e. the structures which we can interpret the theory as being true of, i.e. the structures which provide a model for the theory).

It will help to be have in mind a sample range of theories and corresponding structures. For example, it is good to know just a little about theories of arithmetic, algebraic theories (like group theory or Boolean algebra), theories of various kinds of order, etc., and also to know just a little about some of the structures which provide models for these theories. Mathematicians will already be familiar with informally presented examples: philosophers will probably need to do a small amount of preparatory homework here (but the first reading recommendation in the next section should provide enough to start you off).

Here are some initial themes we'll need to explore:

(1) We'll be interested in relations between structures. One structure can be simply a substructure of another, or can extend another. Or we can map one structure to another in a way that preserves structural information – so, for example, a structure-preserving map can send one structure to a copy embedded inside another structure. In particular, we will be interested in the case where there's an isomorphism between structures, so that each is a replica of the other (as far as their structural features are concerned).

We will similarly be interested in relations between languages for describing structures – we can expand or reduce the non-logical resources of a language, potentially giving it greater or lesser expressive power. So we will also want to know something about the interplay between these expansions/reductions of structures and corresponding languages for them.

(2) How much can a language tell us about a structure? For a toy example, take the structure $(\mathbb{N}, <)$, i.e. the natural numbers equipped with their standard order relation. And consider the first-order formal language whose sole bit of non-logical vocabulary is a symbol for the order relation (let's re-use $<$ for this, with context making it clear that this now *is* an expression belonging to a formal language!). Then, note that we can e.g. define the successor relation over $\mathbb{N}$ in this language, using the formula

$$\mathsf{x} < \mathsf{y} \wedge \forall \mathsf{z}(\mathsf{x} < \mathsf{z} \to (\mathsf{z} = \mathsf{y} \vee \mathsf{y} < \mathsf{z}))$$

with the quantifier running over $\mathbb{N}$. For evidently a pair of numbers x, y satisfies this formula if y comes immediately after x in the ordering. And given we can define the successor relation, we can now e.g. define 0 as the number in the structure $(\mathbb{N}, <)$ which isn't a successor of anything.

Now take instead the structure $(\mathbb{Z}, <)$, i.e. all the integers, negative and positive, equipped with *their* standard order relation. And consider the corresponding formal language where $<$ gets re-interpreted accordingly. The same formula as before, but with the quantifier now running over $\mathbb{Z}$, also suffices to define the successor relation over the integers. But this time, we obviously can't define 0 as the integer which isn't a successor (all integers are successors!). And in fact no other expression from the formal language whose sole bit of non-logical vocabulary is the order-predicate $<$ will define the zero in $(\mathbb{Z}, <)$. Rather as you would expect, the ordering relation gives only the relative position of integers, but doesn't fix the zero.

OK, those were indeed trivial toy examples! But they illustrate a very important class of questions of the following form: which objects and relations in a particular structure can be pinned down, which can be defined, using expressions from a first-order language for the structure?

(3) Moving from what can be defined by particular expressions to the question of what gets fixed by a whole theory, we can ask how varied the models of a given theory can be. In many cases, quite different structures for interpreting a given language can be 'elementarily equivalent', meaning that they satisfy all the same sentences of the language. At the other

extreme, a theory like second-order Peano Arithmetic is *categorical* – its models will all 'look the same', i.e. are all isomorphic with each other. Categoricity is good when we can get it: but when is it available? We'll return to this in a moment.

(4) Instead of going from a theory to the structures which are its models, we can go from structures to theories. Given a class of structures, we can ask: is there a first-order theory for which just *these* structures are the models? Or given a particular structure, and a language for it with the right sort of names, predicates and functional expressions, we can look at the set of all the sentences in the language which are true of the structure. We can now ask, when can all those sentences be regimented into a nicely *axiomatized* theory? Perhaps we can find a finite collection of axioms which entails all those truths about the structure: or if a finite set of axioms is too much to hope for, perhaps we can at least get a set of axioms which are nicely disciplined in some other way. And when is the theory for a structure (i.e. the set of sentences true of the structure) *decidable,* in the sense that a computer could work out what sentences belong to the theory?

(b) Now, you have already met a pair of fundamental results linking semantic structures and sets of first-order sentences – the soundness and completeness theorems. And these lead to a pair of fundamental model-theoretic results. The first of these we've met before, at end of §3.1:

(5) *The compactness theorem* (a.k.a. the finiteness theorem). If every finite subset of a set of sentences Γ from a first-order language has a model, so does Γ.

For our second result, revisit a standard completeness proof for FOL, which shows that any syntactically consistent set of sentences from a first-order language (set of sentences from which you can't derive a contradiction) has a model. Look at the details of the proof: it gives an abstract recipe for building the required model. And assuming that we are dealing with normal first-order languages (with a countable vocabulary), you'll find that the recipe delivers a *countable* model – so in effect, our proof shows that a syntactically consistent set of sentences has a model whose domain is just (some or all) the natural numbers. From this observation we get

(6) *The downward Löwenheim-Skolem theorem.* Suppose a bunch of sentences Γ from a countable first-order language L has a model (however large); then Γ has a countable model.

Why so? Suppose Γ has a model. Then it is syntactically consistent in your favoured proof system (for if we could derive absurdity from Γ then, by the soundness theorem, Γ would semantically entail absurdity, i.e. would be semantically inconsistent after all and have no model). And since Γ is syntactically consistent then, by our proof of completeness, Γ has a countable model.

Note: compactness and the L-S theorem are both results about models, and don't themselves mention proof-systems. So you'd expect we ought to be able to prove them directly without appeal to the completeness theorem which mentions proof-systems. And we can!

(c) An easy argument shows that we can't consistently have (i) for each n a sentence $\exists n$ which is says that there are at least n things, (ii) a sentence $\exists\infty$ which is true in all and only infinite domains, and also (iii) compactness.[1] In the second-order case we can have (i) and (ii), so that rules out compactness. In the first-order case, we have (i) and (iii); hence

(7) There is no first-order sentence $\exists\infty$ which is true in all and only structures with infinite domains.

That's a nice mini-result about the limitations of first-order languages. But now let's note a second, much more dramatic, such result.

Suppose L_A is a formal first-order language for the arithmetic of the natural numbers. The precise details don't matter; but to fix ideas, suppose L_A's built-in non-logical vocabulary comprises the binary function expressions $+$ and $\times$ (with their obvious interpretations), the unary function expression $'$ (expressing the successor function), and the constant $\mathsf{0}$ (denoting zero). So note that L_A then has a sequence of expressions $\mathsf{0}, \mathsf{0}', \mathsf{0}'', \mathsf{0}''', \ldots$ which can serve as numerals, denoting 0, 1, 2, 3, ...

Now let the theory T_{true}, i.e. *true arithmetic*, be the set of *all* true L_A sentences. Then we can show the following:

(8) As well as being true of its 'intended model' – i.e. the natural numbers with their distinguished element zero and the successor, addition, and multiplication functions defined over them – T_{true} is *also* true of differently-structured, non-isomorphic, models.

This can be shown again by an easy compactness argument.

And this is really rather remarkable! Formal first-order theories are our standard way of regimenting informal mathematical theories: but now we find that even T_{true} – the set of *all* first-order L_A truths taken together – still fails to pin down a unique structure for the natural numbers.

(d) And, turning now to the L-S theorem, we find that things only get worse. Again let's take a dramatic example.

[1] Consider the infinite set of sentences

$$\Gamma =_{\text{def}} \{\exists 1, \exists 2, \exists 3, \exists 4, \ldots, \neg\exists\infty\}$$

Any *finite* subset $\Delta \subset \Gamma$ has a model (because there will be a maximum number n such that $\exists n$ is in Δ – and then all the sentences in Δ, which might include $\neg\exists\infty$, will be true in a structure whose domain contains exactly n objects). Compactness would then imply that Γ has a model. But that's impossible. No structure can have a domain which both *does* have at least n objects for every n and also *doesn't* have infinitely many objects. So compactness fails.

Suppose we aim to capture the set-theoretic principles we use as mathematicians, arriving at the gold-standard Zermelo-Fraenkel set theory with the Axiom of Choice, which we regiment as the first-order theory ZFC. Then:

(9) ZFC, on its intended interpretation, makes lots of infinitary claims about the existence of sets much bigger than the set of natural numbers. But the downward Löwenheim-Skolem theorem tells us that, all the same, assuming ZFC is consistent and has a model at all, it has an unintended countable model (despite the fact that ZFC has a theorem which on the intended interpretation says that there are uncountable sets). In other words, ZFC has an interpretation in the natural numbers. Hence our standard first-order formalized set theory certainly fails to uniquely pin down the wildly infinitary universe of sets – it doesn't even manage to pin down an uncountable universe.

What is emerging then, in these first steps into model theory, are some very considerable and perhaps unexpected(?) expressive limitations of first-order formalized theories. These limitations can be thought of as one of the main themes of Level 1 model theory.

(e) At Level 2, we can pursue this theme further, starting with the upward Löwenheim-Skolem theorem which tells us that if a theory has an infinite model it will also have models of all larger infinite sizes (as you see, then, you'll need some basic grip on the idea of the hierarchy of different cardinal sizes to make full sense of this sort of result). Hence

(10) The upward and downward Löwenheim-Skolem theorems tell us that first-order theories which have infinite models won't be categorical – i.e. their models won't all look the same because they can have domains of different infinite sizes. For example, try as we might, a first-order theory of arithmetic will always have non-standard models which 'look too big' to be the natural numbers with their usual structure, and a first-order theory of sets will always have non-standard models which 'look too small' to be the universe of sets as we intuitively conceive it.

But if we can't achieve full categoricity (*all* models looking the same), perhaps we can get restricted categoricity results for some theories (telling us that all models *of a certain size* look the same) – when is this possible?

An example you'll find discussed: the theory of dense linear orders is countably categorical (i.e. all its models of the size of the natural numbers are isomorphic – a lovely result due to Cantor); but it isn't categorical at the next infinite size up. On the other hand, theories of first-order arithmetic are not even countably categorical (even if we restrict ourselves to models in the natural numbers, there can be models which give deviant interpretations of successor, addition and multiplication).

How does that last claim square with the proof you often meet early in a maths course that a theory usually called 'Peano Arithmetic' *is* categorical? The answer is straightforward. As already indicated in (3) above, the version of Peano Arithmetic which is categorical is a *second-order* theory – i.e. a theory which quantifies not just over numbers but over numerical properties, and has a second-order induction principle. Going second-order makes all the difference in arithmetic, and in other theories too like the theory of the real numbers.But why? To understand what is going on here, you need to understand something about the contrast between first-order theories and second-order ones. (See our previous chapter, and follow up the readings if you didn't do so before.)

(f) Still at Level 2, there are results about which theories are *complete* in the sense of entailing either φ or $\neg\varphi$ for each relevant sentence φ, and how this relates to being categorical at a particular size. And there is another related notion of so-called model-completeness: but let's not pause over that.

Instead, let's mention just one more fascinating topic that you will encounter early in your model theory explorations:

(11) We can take a standard first-order theory of the natural numbers and use a compactness argument to show that it has a non-standard model which has an element c in the domain distinct from (and indeed greater than) zero or any of its successors. Similarly, we can take a standard first-order of the real numbers and use another compactness argument to show that it has a non-standard model with an element r in the domain such that that $0 < |r| < 1/n$ for any natural number n. So in this model, the non-standard real r is non-zero but smaller than any rational number, so is *infinitesimally* small. And indeed our model will have non-standard reals infinitesimally close to any standard real.

In this way, we can build up a model of *non-standard analysis* with infinitesimals (where e.g. a differential really *can* be treated as a ratio of infinitesimally small numbers – in just the sort of way that we all supposed wasn't respectable at all). Fascinating!

5.2 Main recommendations for beginning first-order model theory

A preliminary point. When exploring model theory you will very quickly encounter talk of different infinite cardinalities, and also occasional references to the Axiom of Choice. You need to be amiliar enough with these basic set-theoretic ideas (perhaps from the readings suggested back in Chapter 2).

Let's begin with a more expansive and very helpful overview (though you may not understand everything at this preliminary stage). For a bit more detail about the initial agenda of model theory, it is hard to beat

1. Wilfrid Hodges, 'Model Theory', in the *Stanford Encyclopaedia of Philosophy* at tinyurl.com/sepmodel.

Now, a number of the introductions to FOL that I noted in $3.4 have treatments of the Level 1 basics; I'll be recommending one in a moment, and will return to some of the others in the next section on parallel reading. Going just a little beyond, the very first volume in the prestigious and immensely useful Oxford Logic Guides series is Jane Bridge's short *Beginning Model Theory: The Completeness Theorem and Some Consequences* (Clarendon Press, 1977). This neatly takes us through some Level 1 and a few Level 2 topics. But the writing, though very clear, is also rather terse in an old-school way; and the book – not unusually for that publication date – looks like photo-reproduced typescript, which is nowadays really off-putting to read. What, then, are the more recent options?

2. I have already sung the praises of Derek Goldrei's *Propositional and Predicate Calculus: A Model of Argument* (Springer, 2005) for the accessibility of its treatment of FOL in the first five chapters. You should now read Goldrei's §§4.4 and 4.5 (on which I previously said you could skip), and then Chapter 6 'On some uses of compactness'.

   In a little more detail, §4.4 introduces some axiom systems describing various mathematical structures (partial orderings, groups, rings, etc.): this section could be particularly useful to philosophers who haven't really met the notions before. Then §4.5 introduces the notions of substructures and structure-preserving isomorphisms. After proving the compactness theorem in §6.1 (as a corollary of his completeness proof), Goldrei proceeds to use it in §§6.2 and 6.3 to show various theories can't be finitely axiomatized, or can't be nicely axiomatized at all. §6.4 introduces the Löwenheim-Skolem theorems and some consequences, and the following section introduces the notion of 'diagrams' and puts it to work. The final section, §6.6 considers issues about categoricity, completeness and decidability.

   All this is done with the same admirable clarity as marked out Goldrei's earlier chapters.

Goldrei goes quite slowly and doesn't get very far (it is Level 1 model theory). To take a further step (up to Level 2), here are two suggestions. Neither is quite ideal, but each has virtues. The first is

3. María Manzano, *Model Theory*, Oxford Logic Guides 37 (OUP, 1999). This book aims to be an introduction at the kind of levels we are currently concerned with. And standing back from the details, I do like the way that Manzano structures her book. The sequencing of chapters makes for a very natural path through her material, and the coverage seems very appropriate for a book at Levels 1 and 2. After chapters about structures (and mappings between them) and about first-order languages, she proves the completeness theorem again, and then has a sequence of chapters on various core model-theoretic notions and proofs.

Overall, Manzano's book should all be tolerably accessibly (especially if not your very first encounter with model theoretic ideas). However, it seems to me that the discussions at some points would have benefitted from rather more informal commentary, motivating various choices, and sometimes the symbolism is unncessarily heavy-handed. But overall, Manzano's text could work well enough as a follow-up to Goldrei. For more details, see tinyurl.com/manzanobook.

Another option is to look at the first two-thirds of the following book, which is explicitly aimed at undergraduate mathematicians, and is at approximately the same level of difficulty as Manzano:

4. Jonathan Kirby, *An Invitation to Model Theory* (CUP, 2019). As the blurb says, "The highlights of basic model theory are illustrated through examples from specific structures familiar from undergraduate mathematics." Now, one thing that usually isn't already familiar to most undergraduate mathematicians is any serious logic: so Kirby's book is an introduction to model theory that *doesn't* presuppose a previous FOL course. So he has to start with some rather speedy explanations in Part I about first-order languages and interpretations in structures.

   The book *is* then nicely arranged. Part II of the book is on 'Theories and Compactness', Part III on 'Changing Models', and Part IV on 'Characterizing Definable Sets'. (I'd say that some of the further Parts of the book, though, go a bit beyond what you need at this stage.)

   Kirby writes admirably clearly; but his book goes pretty briskly and would have been improved – at least for self-study – if he had slowed down for some more classroom asides. So I can imagine that some readers would struggle with parts of this short book if were treated as a sole introduction to model theory. However, again if you have read Goldrei, it should be very helpful as an alternative or complement to Manzano's book. For a little more about it, see tinyurl.com/kirbybooknote.

We noted that first-order theories behave differently from second-order theories where we have quantifiers running over all the properties and functions defined over a domain, as well as over the objects in the domain. For more on this see the readings on second-order logic suggested in §4.3.

5.3 Some parallel and slightly more advanced reading

I mentioned before that some other introductory texts on FOL apart from Goldrei's have sections or chapters beginning model theory.

Some topics are briefly touched on in §2.6 of Herbert Enderton's *A Mathematical Introduction to Logic* (Academic Press 1972, 2002), and there is discussion of non-standard analysis in his §2.8: but this is perhaps too little done too fast.

So I think the following suits our needs here better:

5. Dirk van Dalen *Logic and Structure* (Springer, 1980; 5th edition 2012), Chapter 3. This covers rather more model-theoretic material than Enderton and in more detail. You could read §3.1 for revision on the completeness theorem, then tackle §3.2 on compactness, the Löwenheim-Skolem theorems and their implications, before moving on to the action-packed §3.3 which covers more model theory including non-standard analysis again, and indeed touches on slightly more advanced topics like 'quantifier elimination'.

And there is also a nice chapter in another often-recommended text:

6. Richard E. Hodel, *An Introduction to Mathematical Logic** (originally published 1995; Dover reprint 2013). In Chapter 6, 'Mathematics and Logic', §6.1 discusses first-order theories, §6.2 treats compactness and the Löwenheim-Skolem theorem, and §6.3 is on decidable theories. Very clearly done.

For rather more detail, here is a recent book with an enticing title:

7. Roman Kossak, *Model Theory for Beginners: 15 Lectures** (College Publications 2021). As the title indicates, the fifteen chapters of this short book – just 138 pages – have their origin in introductory lectures, given to graduate students in CUNY.

   After initial chapters on structures and (first-order) languages, Chapters 3 and 4 are on definability and on simple results such as that ordering is not definable in the language for the integers with addition, $(\mathbb{Z}, +)$. Chapter 5 introduces the notion of 'types', and e.g. gives the back-and-forth proof conventionally attributed to Cantor that countable dense linearly ordered sets without endpoints are always isomorphic to the rationals in their natural order, $(\mathbb{Q}, <)$. Chapter 6 defines relations between structures like elementary equivalence and elementary extension, and establishes the so-called Tarski-Vaught test. Then Chapter 7 proves the compactness theorem, with Chapter 8 using compactness to establish some results about non-standard models of arithmetic and set theory.

   So there is a somewhat different arrangement of initial topics here, compared with books whose first steps in model theory are applications of compactness. The early chapters are indeed nicely done. However, I don't think that Kossak's Chapter 8 will be found an outstandingly clear first introduction to applications of compactness – it will probably be best read after e.g. Goldrei's nice final chapter in his logic text.

   Chapter 9 is on categoricity – in particular, countable categoricity. (Very sensibly, Kossak wants to keep his use of set theory in this book to a minimum; but he does have a section here looking at κ-categoricity for larger cardinals κ.) And now the book speeds up, and starts to require rather more of its reader, and eventually touches on what I think of as

Level 3 topics. Real beginners in model theory without much mathematical background might begin to struggle after the half-way mark in the book. But this is very nice addition to the introductory literature.

Thanks to the efforts of the respective authors to write very accessibly, the suggested main path into the foothills of model theory (from Chiswell & Hodges → Leary & Kristiansen → Goldrei → Manzano/Kirby/Kossack) is not at all a hard road to follow.

Now, we can climb up to the same foothills by routes involving rather tougher scrambles, taking in some additional side-paths and new views along the way. Here, then, is a suggestion for the more mathematical reader:

8. Shawn Hedman, *A First Course in Logic* (OUP, 2004). This covers a surprising amount of model theory. Ch. 2 tells you about structures and about relations between structures. Ch. 4 starts with a nice presentation of a Henkin completeness proof, and then pauses (as Goldrei does) to fill in some background about infinite cardinals etc., before going on to prove the Löwenheim-Skolem theorems and compactness theorems. Then the rest of Ch. 4 and the next chapter covers more introductory model theory, though already touching on a number of topics beyond the scope of Mansion's book (we are already at Level 2.5, perhaps!). Hedman so far could therefore serve as a rather tougher alternative to e.g. Manzano's treatment.

   Then Ch. 6 takes the story on a lot further, beyond what I'd regard as elementary model theory. For more, see tinyurl.com/hedmanbook.

Last but certainly not least, philosophers (but not just philosophers) will certainly want to tackle (parts of) the following book, which strikes me as a very impressive achievement:

9. Tim Button and Sean Walsh, *Philosophy and Model Theory** (OUP, 2018). This book both explains technical results in model theory, and also explores the appeals to model theory in various branches of philosophy, particularly philosophy of mathematics, but in metaphysics more generally, the philosophy of science, philosophical logic and more. So that's a very scattered literature that is being expounded, brought together, examined, inter-related, criticized and discussed. Button and Walsh don't pretend to be giving the last word on the many and varied topics they discuss; but they are offering us a very generous helping of first words and second thoughts. It's a large book because it is to a significant extent self-contained: model-theoretic notions get defined as needed, and many of the more significant results are proved.

   The philosophical discussion is done with vigour and a very engaging style. And the expositions of the needed technical results are usually exemplary (the authors have a good policy of shuffling some extended proofs into chapter appendices). They also say more about second-order logic and second-order theories than is usual.

> But I do rather suspect that, despite their best efforts, an amount of the material is more difficult than the authors fully realize: we soon get to tangle with some Level 3 model theory, and quite a lot of other technical background is presupposed. The breadth and depth of knowledge brought to the enterprise is remarkable: but it does make of a bumpy ride even for those who already know quite a lot. Philosophical readers of this Guide will probably find the book challenging, then, but should find at least the earlier parts fascinating. And indeed, with judicious skimming/skipping – the signposting in the book is excellent – mathematicians with an interest in some foundational questions should find a great deal of interest here too.

And that might already be about as far as many philosophers may want or need to go in this area. Many mathematicians, however, will want go further into model theory; so we pick up the story again in Part III.

5.4 A little history

The last book we mentioned has an historical appendix contributed by a now familiar author:

10. Wilfrid Hodges, 'A short history of model theory', in Button and Walsh, pp. 439–476.

Read the first six or so sections. Later sections refer to model theoretic topics a level up from our current more elementary concerns, so won't be very accessible at this stage.

For another piece that focuses on topics from the beginning of model theory, you could perhaps try R. L. Vaught's 'Model theory before 1945' in L. Henkin et al, eds, *Proceedings of the Tarski Symposium* (American Mathematical Society, 1974), pp. 153–172. You'll probably have to skim parts, but it will also give you some idea of the early developments. But here's something which is *much* more fun to read. Alfred Tarski was one of the key figures in that early history. And there is a very enjoyable and well-written biography, which vividly portrays the man, and gives a wonderful sense of his intellectual world, but also contains accessible interludes on his logical work:

11. Anita Burdman Feferman and Solomon Feferman, *Alfred Tarski, Life and Logic* (CUP, 2004).