8 Intuitionistic logic

In the briefest headline terms, intuitionistic logic is what you get if you drop the classical principle that $\neg\neg A$ implies A (or equivalently drop the law of excluded middle which says that $A \lor \neg A$ always holds). But why would we want to do *that*? And what further consequences for our logic does that have?

8.1 A formal system

(a) To fix ideas, it will help to have in front of us a particular natural deduction system, presented in a Gentzen style you should be familiar with.

We assume the propositional connectives $\land, \lor, \to$ and $\bot$ are built in. The three binary connectives are then governed by the usual pairs of elimination and introduction rules.

As will be familiar, each elimination rule in effect just undoes an application of the corresponding introduction rule – putting it roughly, for each binary connective $\diamond$, its elimination rule allows us to argue onwards from $A \diamond B$ to a conclusion that we could *already* have derived from what was required to derive $A \diamond B$ by its introduction rule.

And as for the absurdity constant, we take this to be governed by the rule that given $\bot$ we can derive anything – *ex falso, quodlibet*.

Finally, what about negation? One option is to treat $\neg A$ as simply an abbreviation for $A \to \bot$. The introduction and elimination rules for the conditional on the left then immediately yield the special cases on the right:

$$(\to\text{I})\ \frac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \to B} \qquad (\to\text{E})\ \frac{A \qquad A \to B}{B} \qquad (\neg\text{I})\ \frac{\begin{array}{c}[A]\\ \vdots\\ \bot\end{array}}{\neg A} \qquad (\neg\text{E})\ \frac{A \qquad \neg A}{\bot}$$

Alternatively, we can take the right-hand pair to be the introduction and elimination rules governing a primitive built-in negation connective. Nothing hangs on this choice.

We then define IPL, intuitionistic propositional logic (in its natural deduction version), to be the logic governed these rules.

The described rules are of course all rules of classical logic too. However, the intuitionistic system is strictly weaker in the sense that the following classically acceptable principles are *not* derived rules of our intuitionistic logic:

$$\text{(DN)}\ \frac{\neg\neg A}{A} \qquad \text{(LEM)}\ \frac{}{A \vee \neg A} \qquad \text{(CR)}\ \begin{array}{c}[\neg A]\\ \vdots\\ \bot\\ \hline A\end{array}$$

DN allows us to drop double negations. LEM is the Law of Excluded Middle, which permits us to infer $A \vee \neg A$ whenever we want, from no assumptions. CR is the classical reductio rule. And these three rules are equivalent in the sense that adding any one of them to intuitionistic propositional logic enables us to prove all the same conclusions; each way, we get back full classical propositional logic.

(b) If only for brevity's sake, we will largely be concentrating on propositional logic in the two introductory overviews which follow. But we should note what it takes to get intuitionistic predicate logic in natural deduction form.

Technically, it's very straightforward. Just as the rules for $\wedge$ and $\vee$ are the same in classical and intuitionist logic, the rules for generalized conjunctions and generalized disjunctions remain the same too. In other words, to get intuitionistic predicate logic we simply add to IPL the same pair of introduction and elimination rules for $\forall$ and $\exists$ as for classical logic.

But note, because of the different background propositional logic – in particular, because of the different rules concerning negation – these familiar quantifier rules don't have the same implications in the intuitionistic setting. For example $\exists x A(x)$ is no longer equivalent to $\neg\forall x \neg A(x)$. More about this below.

8.2 Overview: why intuitionistic logic?

(a) A little experimentation quickly suggests that we indeed cannot derive an instance of excluded middle like $\mathsf{P} \vee \neg\mathsf{P}$ in IPL. But how can we *prove* that this is underivable?

There's a proof-theoretic argument. We examine the structure of proofs in IPL, and thereby show that we can only prove $A \vee B$ as a theorem (i.e. from no premisses) if there is a proof of A or a proof of B. Since neither P nor $\neg\mathsf{P}$ is a theorem of intuitionistic logic (with P atomic), it follows that $\mathsf{P} \vee \neg\mathsf{P}$ isn't a theorem either.

Alternatively, there's a semantic argument. We find some new, non-classical, way of interpreting IPL as a formal system, an interpretation on which the intuitionistic rules of inference are still acceptable, but on which the double negation rule etc. are clearly *not* acceptable. It will then follow that buying into IPL can't by itself commit us to those classical rules. How might this new interpretation go?

It is natural to think of a correct assertion as one that corresponds to some realm of facts (whatever that means exactly). But suppose just for a moment that we instead think of correctness as a matter of being *warranted*, where we understand this in the following strong sense: A is warranted if and only if there is an informal proof which provides a direct certification for A's correctness.

Then here is a reasonably natural four-part story about how to characterize the connectives in this new framework (it's a rough version of what's called the BHK – Brouwer-Heyting-Kolmorgorov – interpretation):

(i) $(A \wedge B)$ is warranted iff (if and only if) A and B are both warranted.
(ii) While there may be other ways of arriving at a disjunction, the direct and ideally informative way of certifying a disjunction's correctness is by establishing one or other disjunct. So we will count $(A \vee B)$ as warranted iff at least one disjunct is certified to be correct, i.e. iff there is a warrant for A or a warrant for B.
(iii) A warranted conditional $(A \to B)$ must be one that, together with the warranted assertion A, will enable us to derive another warranted assertion B by using modus ponens. Hence $(A \to B)$ is directly warranted iff there is a way of converting any warrant for A into a warrant for B.
(iv) $\neg A$ is warranted iff we have a warrant for ruling out A because it leads to something absurd (given what else is warranted).
(v) $\bot$ is never warranted.

Then, in keeping with this approach, we will think of a reliable inference as one that takes us from warranted premisses to a warranted conclusion.

Now, in this framework, the familiar *introduction* rules for the connectives will *still* be acceptable, for they will evidently be warrant-preserving (given our interpretation of the connectives). But as we said, the various elimination rules in effect just 'undo' the effects of the introduction rules: so they should come for free along with the introduction rules. Finally, we can still endorse EFQ – the plausible thought is that if, *per impossible*, the absurd is warrantedly assertible, then all hell breaks loose, and anything goes.

Hence, regarded now as warrant-preserving rules, all our IPL rules can remain in place. However:

1. DN will *not* be acceptable in this framework. We might have a warrant for ruling out being able actually to rule out A, so we can warrantedly assert $\neg\neg A$. But that doesn't put us in a position to warrantedly assert A. We might just have to remain neutral about A

2. Likewise LEM will *not* be acceptable. On the present understanding of the connectives, $(A \vee \neg A)$ would be correct, i.e. directly warranted, just if there is a warrant for A or a warrant for ruling out A. But again, must there always be a way of justifiably deciding a conjecture A in the relevant area of inquiry one way or the other? Some things may be beyond our ken.

Again, for similar reasons, CR is *not* acceptable either in this framework: but I won't keep mentioning this third rule.

In sum, then, if we want a propositional logic suitable as a framework for regimenting arguments which preserve warranted assertability, we should stick with the core rules of IPL – and shouldn't endorse those further distinctively classical laws.

But be very careful here! It is, for example, one thing to stand back from endorsing the law of excluded middle. It would be something else entirely actually to *deny* some instance of the law. In fact, it is an easy exercise to show that, even in IPL, any outright negation of an instance – i.e. any sentence of the form $\neg(A \vee \neg A)$ – entails absurdity!

(b) The double negation rule DN which is usually built into natural deduction systems for classical logic is an outlier, not belonging to one of the matched pairs introduction/elimination rules. Now we see the significance of this. Its special status leaves room for an interpretation on which the remaining rules – the rules of IPL – hold good, but DN doesn't. Hence, as we wanted to show, DN is not derivable as a rule of intuitionistic propositional logic. Nor is LEM.

True, our version of the semantic argument as presented so far might seem all a bit too arm-waving for comfort; after all, the notion of warrant as we characterized it so can hardly be said to be ideally clear! But let's not fuss about details now. We'll soon meet a rigorous story partially inspired by this notion which gives us an entirely uncontroversial, technically kosher, proof that DN and its equivalents are, as claimed, independent of the rules of IPL.

Things *do* get controversial, though, when it is claimed that DN and LEM really don't apply in some particular domain of inquiry, because in this domain there can indeed be no more to correctness than having a warrant in the form of a direct informal proof. Now, so-called *intuitionists* do indeed hold that mathematics is a case in point. Mathematical truth, they say, doesn't consist in correspondence with facts about abstract objects laid out in some Platonic heaven (after all, there are familiar worries: what kind of objects could these ideal mathematical entities be? how could we possibly know about them?). Rather, the story goes, the mathematical world is in some sense our construction, and being mathematically correct can be no more than a matter of being assertible on the basis of a proof elaborating our constructions – meaning not a proof in this or that formal system but a chain of reasoning satisfying informal mathematical standards for being a direct proof.

Consider, for example, the following argument, intended to show that (C), there is a pair of irrational numbers a and b such that a^b is rational:

> Either (i) $\sqrt{2}^{\sqrt{2}}$ is rational, or (ii) it isn't. In case (i) we are done: we can simply put $a = b = \sqrt{2}$, and by assumption (C) holds. In case (ii) put $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$. Then a is irrational by assumption, b is irrational, while $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$ and hence is rational, so (C) again holds. Either way, (C).

It will be agreed on all sides that this argument isn't ideally satisfying. But the intuitionist goes further, and claims that this argument actually fails to establish (C), because we haven't yet constructed a specific a and b to warrant (C). The cited argument assumes that either (i) or (ii) holds, and – the intuitionist complains – we are not entitled to assume this when we are given no reason to suppose that one or other disjunct can be warranted by a construction.

(c) For an intuitionist, then, the appropriate logic is not full *classical* two-valued logic but rather our cut-down *intuitionistic* logic (hence the name!), because *this* is the right logic for correctness-as-informal-direct-provability.

Or so, roughly, goes the story. Plainly, we can't even begin to discuss here the highly contentious issues about the nature of truth and provability in mathematics which first led to the advocacy of intuitionistic logic (if you want to know a bit more, there are some initial references in the recommended reading). But no matter: there are plenty of other reasons too for being interested in intuitionistic logic, which keeps recurring in various contexts (e.g. in computer science and in category theory). And as we will see in the next chapter, the fact that its rules come in matched introduction/elimination pairs makes intuitionistic logic proof-theoretically particularly neat.

For now, though, let's just say a bit more about what can and can't be proved in IPL and its extension by the quantifier rules, and also introduce one of the more formal ways of semantically modelling it.

8.3 Overview: more proof theory, more semantics

(a) We use '$\vdash_{\text{C}}$' to symbolize classical derivability, and '$\vdash_{\text{I}}$' to symbolize derivability in intuitionistic logic. Then:

(i) The familiar classical laws governing just conjunctions and disjunctions stay the same: so, for example, we still have $A \wedge (B \wedge C) \vdash_{\text{I}} (A \wedge B) \wedge C$ and $A \vee (B \wedge C) \vdash_{\text{I}} (A \vee B) \wedge (A \vee C)$. However, although the conditional rules of inference are the same in classical and intuitionist logic, the laws governing the conditional are *not* the same. Classically, we have Peirce's Law, $(A \to B) \to A \vdash_{\text{C}} A$; but we do *not* have $(A \to B) \to A \vdash_{\text{I}} A$.

(ii) Classically, the binary connectives are interdefinable using negation. Not so in IPL. We do have for example $(A \vee B) \vdash_{\text{I}} \neg(\neg A \wedge \neg B)$. But the converse doesn't hold – a good rule of thumb is that IPL makes disjunctions harder to prove. However, $\neg(\neg A \wedge \neg B) \vdash_{\text{I}} \neg\neg(A \vee B)$.

Likewise, we do have $(\neg A \vee B) \vdash_{\text{I}} (A \to B)$. But the converse doesn't hold – though $(A \to B) \vdash_{\text{I}} \neg\neg(\neg A \vee B)$.

(iii) The connectives in IPL are not truth-functional. But their behaviour in a sense still tracks the classical truth-tables.

Take, for example, the classical table for the material conditional. We can read that as telling us that when A and B holds so does $A \to B$; when A holds and B doesn't (so $\neg B$ does), then $A \to B$ doesn't hold (so $\neg(A \to B)$ does); while when $\neg A$ holds, so does $(A \to B)$ (whether we also have B or $\neg B$).

Correspondingly, in intuitionistic logic, we still have $A, B \vdash_{\text{I}} (A \to B)$; $A, \neg B \vdash_{\text{I}} \neg(A \to B)$; and $\neg A \vdash_{\text{I}} (A \to B)$. The intuitionistic conditional therefore shares some of the same unwelcome(?) features as the classical material conditional.

(iv) Glivenko's theorem: if A is a propositional formula, $\vdash_C A$ just when $\vdash_I \neg\neg A$. Note, though, that this doesn't apply in general to quantified formulas.

(v) Moving to quantified intuitionistic logic, the disjunction property applies to propositional and quantified sentences alike: if $\Gamma \vdash_I (A \lor B)$ then either $\Gamma \vdash_I A$ or $\Gamma \vdash_I B$. Likewise, we only have $\Gamma \vdash_I \exists x Ax$ if we can provide a witness for the existentially quantified sentence, i.e. for some term t, $\Gamma \vdash_I At$.

(vi) Just as conjunction and disjunction are not intuitionistically interdefinable using negation, so too for the universal and existential quantifiers. Thus while $\exists xA \vdash_I \neg\forall x\neg A$, the converse doesn't hold – though, inserting a double negation, we do have $\neg\forall x\neg A \vdash_I \neg\neg\exists xA$. Likewise, $\forall xA \vdash_I \neg\exists x\neg A$. But again the converse doesn't hold – though $\neg\exists x\neg A \vdash_I \forall x\neg\neg A$.

(vii) A theme is emerging! While some classical results fail in intuitionistic logic, inserting some double negations *will* give corresponding intuitionistic results. This theme can be made more precise, in various ways. Consider, for example, the following translation scheme T for mapping classical to intuitionistic sentences – a *double-negation* translation:

   a) $A^T := \neg\neg A$, for atomic wffs A; $\bot^T := \bot$
   b) $(A \land B)^T := A^T \land B^T$
   c) $(A \lor B)^T := \neg\neg(A^T \lor B^T)$
   d) $(A \to B)^T := A^T \to B^T$
   e) $(\neg A)^T := \neg A^T$
   f) $(\forall xA)^T := \forall xA^T$
   g) $(\exists xA)^T := \neg\neg\exists xA^T$

   Suppose Γ^T comprises the double-negation translations of the sentences in the set Γ. Then we have the following key theorem due (independently) to Gödel and Gentzen:

   $$\Gamma \vdash_C A \text{ if and only if } \Gamma^T \vdash_I A^T.$$

(b) Two comments on the Gödel/Gentzen theorem. First, it shows that for every classical result, there is a corresponding intuitionistic one that you get by judiciously inserting double negation signs. So we can think of classical logic not so much as what you get by *adding* to intuitionist logic but rather as what you get by *ignoring a distinction* that the intuitionist thinks is of central importance, namely the distinction between A and $\neg\neg A$.

Second, note this particular consequence of the theorem: $\Gamma \vdash_C \bot$ if and only if $\Gamma^T \vdash_I \bot$. So if the classical theory Γ is inconsistent by classical standards, then its intuitionistic translation Γ^T is already inconsistent by intuitionistic standards. Roughly speaking, then, if we have worries about the consistency of a classical theory, retreating to an intuitionistic version isn't going to help. As you'll see from the readings, this observation had significant historical impact in debates in the foundations of mathematics.

(c) Let's now return to those earlier arm-waving semantic remarks in §8.2(a). They can be sharpened up in various ways, but here I'll just briefly consider (a version of) Saul Kripke's semantics for IPL. I'll leave it to you to find out how the story can be extended to cover quantified intuitionistic logic.

Take things in stages. First, imagine a mathematical enquirer, starting from a ground state of knowledge g; she then proceeds to expand her knowledge, through a sequence of possible further states K. Different routes forward can be possible, so we can think of these states as situated on a branching array of possibilities rooted at g (not strictly a 'tree' though, as we can allow branches to later rejoin, reflecting the fact that our enquirer can arrive at the same knowledge state by different routes). If she can get from state $k \in K$ to the later state $k' \in K$ by zero or more steps, then we'll write $k \leq k'$. So let's say, more abstractly,

> An *intuitionistic model structure* is a triple $(g, K, \leq)$, where K is a set, $\leq$ is a partial order defined over K, and g is its minimum (so $g \leq k$ for all $k \in K$).

As our enquirer investigates the truth of the various sentences of her propositional language, at any stage k a sentence A is either *established to be true* or *not [yet] established*. We can symbolize those alternatives by $k \Vdash A$ and $k \nVdash A$; it is usual, for reasons that needn't now detain us, to read '$\vDash$' as *forces*.

Now as far as *atomic* sentences are concerned, the only constraint on a forcing relation is this: once P is established in the knowledge state k, it stays established in any expansion on that state of knowledge, i.e. at any k' such that $k \leq k'$. Formally then, we require,

> For any atomic sentence P and $k \in K$, if $k \vDash P$, then $k' \vDash P$, for all $k' \in K$ such that $k \leq k'$.

And now, next stage, let's expand a forcing relation defined for a suite of atoms so that it now covers *all* wffs built up from those atoms by the connectives. So, for all $k, k' \in K$, and all relevant sentences A, B, we put

(i) $k \nVdash \bot$.
(ii) $k \Vdash A \wedge B$ iff $k \Vdash A$ and $k \Vdash B$.
(iii) $k \Vdash A \vee B$ iff $k \Vdash A$ or $k \Vdash B$.
(iv) $k \Vdash A \to B$ iff, for any k' such that $k \leq k'$, if $k' \Vdash A$ then $k' \Vdash B$.
(v) $k \Vdash \neg A$ iff, for any k' such that $k \leq k'$, $k' \nVdash A$.

It's a simple consequence of these definitions that for any A, whether atomic or molecular, once A is established it stays established. In other words,

($*$) If $k \vDash A$, then $k' \vDash A$, for all k' such that $k \leq k'$.

What motivates these clauses (i) to (v) in our characterisation of $\vDash$? (i) The absurd is never established as true, in any state of knowledge. And (ii) establishing a conjunction is equivalent to establishing each conjunct, on any sensible story. So we needn't pause over these first two.

But (iii) reveals our enquirer's intuitionist/constructivist commitments! – she is taking establishing a disjunction in an acceptably direct way to require establishing one of the disjuncts. For (iv) the thought is that establishing $A \to B$ is tantamount to giving you an inference-ticket: with the conditional established, should you (eventually) get to also establish A, then you will then be entitled to B too. Finally, (v) falls out from the definition of $\neg A$ as $A \to \bot$ and the evaluation rules for $\to$ and $\bot$. Or more directly, the idea is that to establish $\neg A$ is to rule out, once and for all, A turning out to be correct as we later expanded our knowledge.

With these pieces in place, we can – next stage! – define a formula of a propositional language to be intuitionistically valid in a natural way. Classically, such a formula is valid (is a tautology) if it is true however things turn out with the values of the relevant atoms. Now we say that a propositional formula A is intuitionistically valid on Kripke's semantics if it can be established in the ground state of knowledge, however things later turn out with respect to the truth of relevant atoms as our knowledge expands. Formally,

> A is intuitionistically valid iff $g \vDash A$, whatever the model structure $(g, K, \leq)$ and whatever forcing relation $\vDash$ is defined over the relevant atoms.[1]

And now for the big reveal! Kripke proved in 1965 the following soundness and completeness result:

> A formula is a theorem of IPL (can be derived from no premisses) if and only if it is intuitionistically valid.

Neither direction of the biconditional is particularly hard.

Expanding the idea of valuations over an intuitionistic model structure to accommodate quantified formulas and then proving soundness and completeness for quantified intuitionistic logic is, however, rather more involved.

(d) Let's finish by briefly showing that – given Kripke's soundness result that every IPL theorem is intuitionistically valid on his semantic story – it is immediate that the law of excluded middle fails for IPL.

It couldn't be easier. Consider a propositional language with just a single atom P; and take the model structure which has just two states g, k such that $g \leq k$. And now suppose that P is not yet established at g but *is* established at k, hence $g \nVdash \mathsf{P}$ while $k \Vdash \mathsf{P}$. By the rule for negation, $g \nVdash \neg\mathsf{P}$. So $g \nVdash (\mathsf{P} \lor \neg\mathsf{P})$. Hence $\mathsf{P} \lor \neg\mathsf{P}$ is not valid. Hence, by the soundness result, $\mathsf{P} \lor \neg\mathsf{P}$ can't be an IPL theorem.

[1] Fine print, just to link up with other presentations you will meet. First, given $(*)$, $g \vDash A$ holds iff $k \vDash A$ for all k. So we can redefine validity by saying A is valid just when $k \vDash A$ for all k. But then, second, we can in fact let g drop right out of the picture. For it is quite easy to see that it will make no difference whether we require the partial order $\leq$ to have a minimum or not: the same sentences will come out valid either way. Indeed, third, we don't even require the relation we symbolized $\leq$ to be a true partial order: again, if we allow any reflexive, transitive relation over K in its place, it will make no different to what comes out as valid.

8.4 Basic recommendations on intuitionistic logic

So much for some quick introductory remarks – enough, I hope, to spark interest in the topic! There is room, then, for a short introductory book which would develop these and related themes at the kind of accessible level we currently want. And Grigori Mints's *A Short Introduction to Intuitionistic Logic* (Springer, 2000) is indeed brief enough; however, it soon becomes entangled with more advanced topics in a way that will put it too quickly mystify beginners. So we will have to patch together readings from a few different sources.

We will cherry-pick from the following:

1. Joan Moschovakis, 'Intuitionistic Logic', in *The Stanford Encyclopaedia of Philosophy*, §§1–3, §4.1, §5.1. Available at tinyurl.com/sep-intuit.
2. Dirk van Dalen, *Logic and Structure* (Springer, 1980; 5th edition 2012), Chapter 5, 'Intuitionistic logic'.
3. A.S. Troelstra and Dirk van Dalen, *Constructivism in Mathematics, An Introduction: Vol. I* (North-Holland, 1988), Chapter 2, 'Logic', §1, §3 (up to Prop 3.8), §4?, §5, §6?.

You could read these in the order given, initially skimming/skipping over passages that aren't immediately clear.

Or perhaps better, start with (1)'s §1, 'Rejection of Tertium Non Datur', and then (2)'s §5.1, 'Constructive reasoning' which introduces the BHK interpretation of the logical operators.

Then look at a presentation of a *natural deduction* system for intuitionistic logic (as sketched in our overview): this is briskly covered in (2) in the first half of §5.2. But in fact the discussion in (3) – though this is not an introductory textbook – is notably more relaxed and clearer: see §1 of the chapter.

Next, read up on the *double-negation translation* between classical and intuitionistic logic. This is described in (1) §4.1, and explored a bit more in the second half of (2) §5.2. But again, a more relaxed presentation can be found in (3), §3 (up to Prop. 3.8).

Now you want to find out more about *Kripke semantics*, which is also covered in all three resources. (1) §5.1 gives the brisk headline news. (2) gives a compressed account in the first half of §5.3. But again (3) is best: Troelstra and Van Dalen give a much more expansive and helpful account in their Ch. 2 §5 – which sensibly treats propositional logic first before expanding the story to cover full quantified intutionistic logic.

I would suggest, though, leaving detailed soundness and completeness proofs for Kripke semantics – covered in (2) §5.3 or (3) §6 – for later (if indeed tackled at all, at this stage.)

For a few more facts about intuitionistic logic, such as the *disjunction property*, see also the first couple of pages of (2) §5.4 (the rest of that section is interesting but not really needed at this stage).

Return to (1) to look at §2.1 (an *axiomatic* version of intuitionistic logic), and the first half of §3 (on Heyting's intuitionistic arithmetic). Then finally, for more on Heyting Arithmetic and a spelt-out proof that it is consistent if and only if classical Peano Arithmetic is consistent, you could dip into

4. Paolo Mancosu, Sergio Galvan, and Richard Zach, *An Introduction to Proof Theory*, (OUP, 2021). §2.15 on 'Intuitionistic and classical arithmetic' can be read as an approachable stand-alone treatment.

8.5 Some parallel/additional reading

Kripke semantics for intuitionistic logic involves evaluating formulas not once and for all but at different points in a structure. We informally talked about these points as various 'states of knowledge'; in a different idiom we could have talked about various 'possible worlds'. Now, the use of this kind of 'possible world semantics' is characteristic of modal logics – the simplest modal logics being logics of necessity and possibility, with their semantics modelling the idea that being necessarily true is being true at all relevant possible worlds. So another way of approaching intuitionistic logic is by *first* discussing modal logics more generally, *before* looking at intuitionistic logic in particular. If you want to explore this route, you can jump to this Guide's Chapter 11. Or you can go straight to the rightly admired

5. Graham Priest, *An Introduction to Non-Classical Logic* (CUP, expanded 2nd edition 2008). Particularly, Chapters 6 and 20.

   In this excellent book, Priest treats a whole range of logics systematically by using a tableau/tree approach. His Chapter 1 provides a quick revision tutorial on tableaux for classical propositional logic. He then discusses propositional modal logics in Chapters 2 and 3, and then – for present purposes – you can skip forward to Chapter 6 on propositional intuitionistic logic. Priest's Chapter 12 revises tableaux for quantified logic. Next, read up on quantified modal logics in Chapters 14 and 15, and then you can skip forward to the discussion of quantified intuitionistic logic in in Chapter 20. (It's a minor pity that the tableaux aren't more prettily displayed: otherwise, this is all *very* well done.)

There is a somewhat different way using tableaux for intuitionistic logic (which doesn't rely on first treating modal logic), which is quite nicely explored

6. Harrie de Swart, *Philosophical and Mathematical Logic* (Springer, 2018), Chapter 18.

However, I prefer the treatment of the same tableau approach in an earlier excellent book:

7. Melvin Fitting, *Intuitionistic Logic, Model Theory, and Forcing* (North Holland, 1969), Part I.

   Ignore the scary title: it is only the beautifully clear but sophisticated first part of the book which concerns us now! It should particularly appeal to those who appreciate mathematical elegance.

Finally, for a bit more on natural deduction, the sequent calculus and semantics for intuitionistic logic, you should look at two chapters from a modern classic:

8. Michael Dummett, *Elements of Intuitionism* (OUP, 2nd ed. 2000), Chapters 4 and 5.

In fact, you could well want to read the opening two chapters and the final one as well!

8.6 A little more history, a little more philosophy

A number of the readings mentioned so far include brief remarks about the history of intuitionism (and constructivism more generally). For something more substantial, look at

9. A.S. Troelstra and Dirk van Dalen, *Constructivism in Mathematics, An Introduction: Vol. I* (North-Holland, 1988), Chapter 1,

which gives a brief characterization of various forms of constructivism (not all of them motivate the adoption of a non-classical logic like intuitionistic logic).

The early days of intuitionism were wild! To get a sense of *how* wild Brouwer's ideas were, you could take a look at

10. Mark van Atten, *On Brouwer* (Wadsworth, 2004), Chapters 1 and 2.

The same author has a *Stanford Encyclopedia* article on 'The Development of Intuitionistic Logic' at tinyurl.com/dev-intuit; but that's much more detailed than you are likely to want.

Turning to more philosophical discussions – and it is a bit difficult to separate thinking about intuitionism as a philosophy of mathematics from thinking about intuitionistic logic more specifically – one key article that you will want to read (which was hugely influential in reviving interest in a 'tamer' intuitionism among philosophers) is

11. Michael Dummett, 'The Philosophical Basis of Intuitionistic Logic' (originally 1973, reprinted in Dummett's *Truth and Other Engimas*).

Then, for more recent discussions, here's a trio of articles:

12. Carl Posy, 'Intuitionism and Philosophy'; D. C. McCarty, 'Intuitionism in Mathematics'; and Roy Cook, 'Intuitionism Reconsidered', all in S. Shapiro, ed., *The Oxford Handbook of the Philosophy of Mathematics and Logic* (OUP, 2005).