9 Elementary proof theory

The story of proof theory starts with David Hilbert and what has come to be known as 'Hilbert's Programme', which inspired the profoundly original work of Gerhard Gentzen in the 1930s.

Two themes from Gentzen are within easy reach for beginners in mathematical logic: (A) the idea of normalization for natural deduction proofs, (B) the move from natural deduction to sequent calculi, and cut-elimination results for these calculi. But the most interesting later developments in proof theory – in particular, in so-called ordinal proof theory – quickly become mathematically rather sophisticated. Still, at this stage it is at least worth making a first pass at (C) Gentzen's proof of the consistency of arithmetic using a cut-elimination proof which invokes induction over some small countable ordinals. So these three themes from elementary proof theory will be the focus of this chapter.

9.1 Preamble: a very little about Hilbert's Programme

Set theory, for example, is about – or at least, is *supposed* to be about – an extraordinarily rich domain of (mostly) infinite objects. How can we know that such a theory really does make good sense? Indeed, how can we know that it even gets to the starting line of being internally consistent?

David Hilbert had a wonderful insight. While the *topic* of a mathematical theory T such as set theory might be wildly infinitary, the *theory T itself* is built from thoroughly finite objects – namely sentences, and the finite arrays of sentences that are proofs. So perhaps we can use some very tame assumptions (assumptions that don't tangle with the infinite) to reason about T when it is thought of as a suite of finite objects. And in particular, perhaps we can use tame assumptions to prove T's consistency, without needing to worry about T's purported infinitary subject matter.

To make any progress with this idea, we'll need to fully pin down T's basic assumptions and to regiment the principles of reasoning that T can deploy – we'll need, in other words, to have a nice axiomatic formalization of T on the table. This formalization of the theory T (whether it's about sets, widgets, or whatnots) then gives us some definite, mathematically precise, *new* objects to reason about (beyond the sets, widgets, or whatnots), namely the T-wffs and T-proofs that make up the theory. And now, as Hilbert saw, we can set off to mathematically investigate *these*, developing a *Beweistheorie* (a theory about proofs).

We'll return in §9.3 to say something more about the resulting Programme of aiming to use entirely 'safe', merely finitary, reasoning about a theory T in order to prove its consistency (though you should already know that Gödel's Second Incompleteness Theorem is going to cause some trouble). But, for the moment, the point we want is simply this: the Programme presupposes that we can indeed regiment the theory that concerns us into a tidily disciplined formal shape – and in particular, we can regiment its required principles of reasoning into a formal deductive logic. Hence the central importance for Hilbert and his associates of constructing suitable formal systems for logic.

9.2 Deductive systems, normal forms, and cuts: a short overview

(a) The logical systems developed by Hilbert and Bernays[1] were axiomatic in style, and at some remove from the forms of deduction used in practice in mathematical proofs. It was Bernays' student Gerhard Gentzen who first introduced a style of deductive system which explicitly aimed to come, as he put it, "as close as possible to actual reasoning." The result was Gentzen's natural deduction calculi for intuitionistic and classical predicate logic.

Now, these calculi – which I'll take to be familiar from work on earlier topics in this Guide – have some lovely features: and as advertised, they do allow us to formally track natural lines of reasoning. But they also still allow us to construct some perversely unnatural proofs! For example, consider the following two derivations to show that from $\mathsf{P} \land \mathsf{Q}$ we can infer $\mathsf{P} \lor \mathsf{Q}$:

$$\text{(i)}\quad \dfrac{\dfrac{\mathsf{P} \land \mathsf{Q}}{\mathsf{P}}}{\mathsf{P} \lor \mathsf{Q}} \qquad\qquad \text{(ii)}\quad \dfrac{\dfrac{\dfrac{\mathsf{P} \land \mathsf{Q}}{\mathsf{P}}}{\mathsf{P} \lor (\mathsf{R} \land \mathsf{Q})} \qquad \dfrac{[\mathsf{P}]^{(1)}}{\mathsf{P} \lor \mathsf{Q}} \qquad \dfrac{\dfrac{[\mathsf{R} \land \mathsf{Q}]^{(1)}}{\mathsf{Q}}}{\mathsf{P} \lor \mathsf{Q}}}{\mathsf{P} \lor \mathsf{Q}}\,{\scriptstyle (1)}$$

(i) is an entirely natural mini-proof. But (ii) takes us on a pointless detour: on the leftmost branch, the 'wrong' disjunction is introduced which involves the quite irrelevant R, before we use a disjunction-elimination inference to get the proof back on track.

The detour in (ii) is not just inelegant; there is also a sense in which it makes the proof non-explanatory. After all, if a premiss A logically entails a conclusion C, this – we suppose – results from the conceptual content of A and C. So we want a proof to explain how the contents of A and C generate the entailment. A derivation like (ii), which introduces irrelevant content that is quite unrelated to either the premiss or conclusion, can't do that.

So, generalizing on the example of (ii), let's now define a detour as consisting in the use of the introduction rule for a logical operator (a connective or a

[1]Paul Bernays was nominally Hilbert's assistant, but in fact was an absolutely key figure in his own right.

quantifier) followed by the application of the corresponding elimination rule to this introduced operator. Then, as just noted, it is not merely to avoid inelegancies that we will want detour-free proofs.

Now, simple detours in a Gentzen-style natural deduction proof can easily be removed. For example, a detour which involves introducing a conditional (by conditional proof) and then eliminating it (by modus ponens), as on the left, can be simply smoothed away or *reduced*, as on the right:

$$\dfrac{\begin{matrix}\vdots\\ A\end{matrix} \qquad \dfrac{\begin{matrix}[A]^{(1)}\\ \wr\\ B\end{matrix}}{A \to B}\,(1)}{B} \qquad \rightsquigarrow \qquad \begin{matrix}\vdots\\ A\\ \wr\\ B\end{matrix}$$

For another example, going back to the case of introducing and then eliminating a disjunction, a proof of the shape on the left can be reduced to a proof with the shape on the right:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix}}{A \vee B} \qquad \begin{matrix}[A]^{(1)}\\ \wr\\ C\end{matrix} \qquad \begin{matrix}[B]^{(1)}\\ \wr\wr\\ C\end{matrix}}{C}\,(1) \qquad \rightsquigarrow \qquad \begin{matrix}\vdots\\ A\\ \wr\\ C\end{matrix}$$

And similarly for other simple detours involving other connectives and the quantifiers. However, what about the case where a detour gets entangled with the application of other rules in more complicated ways? Can detours *always* be removed?

Gentzen was able to show that – at least for his system of intuitionistic logic – if a conclusion can be derived from premisses at all, then there will indeed be a *normal*, i.e. detour-free, proof of the conclusion from the premisses. And he did this by giving a *normalization* procedure – i.e. instructions for systematically removing detours until we are left with a normal proof. The resulting detour-free proofs will then have particularly nice features such as the so-called subformula property: every formula that occurs in a proof will either be a subformula of one of the premisses or a subformula of the conclusion (as usual, counting instances of quantified wffs as subformulas of them). There won't be irrelevancies as in our silly proof (ii) above.

And now note that, as a corollary, we can immediately conclude that intuitionistic logic is consistent: we can't have a proof with the subformula property from no premisses to $\bot$. Which raises a hopeful prospect: can other normalization proofs be used to establish the sort of consistency results that Hilbert wanted?

(b) But now the story gets complicated. For a start, Gentzen himself couldn't find a normalization proof for his natural deduction system of classical logic (you can see why there might be a problem – a classical proof might, at least on the face of it, need to rely on an instance of excluded middle which isn't a subformula of either the premisses or the conclusion). In order to get a classical system for which he *could* prove an appropriate normalization theorem, Gentzen therefore introduced his sequent calculi, about which more in moment. And his normalization proof for intuitionistic logic then remained unpublished for seventy years. In the meantime, the proof was independently rediscovered by Dag Prawitz in his thesis, published as *Natural Deduction* (1965), which also presents a normalization proof for Gentzen's classical natural deduction system without $\lor$ and $\exists$ (which is of course equivalent to the complete system).

Since Prawitz's work brought Gentzen-style natural deduction back to centre stage, there has been a whole cottage industry of tinkering with the inference rules, and tinkering with the definition of a normal proof, in order to produce classical natural deduction systems with nice proof-theoretic features. But I rather think that the typical beginner in mathematical logic won't find the details of *these* further developments particularly exciting. However, it *is* well worth looking at the opening four chapters of Prawitz's wonderful short book, and perhaps just noting a few more ideas. This will be enough on our theme (A), natural deduction and normalization.

(c) How do we read off what depends on what in a natural deduction proof? By looking at the geometry of the proof, and its annotations.

For example, consider this derivation of $\mathsf{P} \to (\mathsf{Q} \to \mathsf{R})$ from $(\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}$:

$$\dfrac{\dfrac{\dfrac{(\mathsf{P} \land \mathsf{Q}) \to \mathsf{R} \qquad \dfrac{[\mathsf{P}]^{(2)} \qquad [\mathsf{Q}]^{(1)}}{\mathsf{P} \land \mathsf{Q}}}{\mathsf{R}}}{\mathsf{Q} \to \mathsf{R}}\,(1)}{\mathsf{P} \to (\mathsf{Q} \to \mathsf{R})}\,(2)$$

Then, reading upwards from R, we see that this wff depends on all three of $(\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}$, P, and Q as assumptions (for neither of the last two have yet been discharged); while $\mathsf{Q} \to \mathsf{R}$ on the next line depends only on $(\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}$ and P.

That's clear enough. But we could alternatively record dependencies quite explicitly, line by line. To do this, we will make use of so-called *sequents*. We'll write a sequent in the form $\Gamma \Rightarrow A$, and read this as saying that A is deducible from the finitely many (perhaps zero) wffs Γ.[2] Since an (undischarged) assumption depends just on itself, we can then explicitly record the deducibilities revealed in our natural deduction proof like this:

$$\dfrac{\dfrac{\dfrac{(\mathsf{P} \land \mathsf{Q}) \to \mathsf{R} \Rightarrow (\mathsf{P} \land \mathsf{Q}) \to \mathsf{R} \qquad \dfrac{\mathsf{P} \Rightarrow \mathsf{P} \qquad \mathsf{Q} \Rightarrow \mathsf{Q}}{\mathsf{P}, \mathsf{Q} \Rightarrow \mathsf{P} \land \mathsf{Q}}}{(\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}, \mathsf{P}, \mathsf{Q} \Rightarrow \mathsf{R}}}{(\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}, \mathsf{P} \Rightarrow \mathsf{Q} \to \mathsf{R}}}{(\mathsf{P} \land \mathsf{Q}) \to \mathsf{R} \Rightarrow \mathsf{P} \to (\mathsf{Q} \to \mathsf{R})}$$

[2] For present purposes, we can officially think of Γ as given as a set – though in the end we might prefer to treat Γ as a multi-set where repetitions matter: Gentzen himself treated Γ as an ordered sequence.

And now, following Gentzen, instead of thinking of this tree of sequents as in effect just a running commentary on an underlying natural deduction proof, we can treat it as *itself* a new sort of proof in its own right – a proof relating whole sequents rather than individual wffs.

At the tips of branches of this sequent proof about deducibilities we have 'axioms' of the form $A \Rightarrow A$ (since trivially, A is deducible, given A!). And then the proof is extended downwards by the application of two sorts of rules, rules governing specific logical operators, and general structural rules.

For the logical rules, we could replace the familiar natural deduction rules for wffs (on the left) with corresponding rules for deriving sequents (on the right):

$$\frac{A \qquad B}{A \wedge B} \quad \rightsquigarrow \quad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \wedge B}$$

$$\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B} \quad \rightsquigarrow \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

There should be nothing mysterious here. After all, the terse schematic presentation of the natural-deduction introduction rule for $\wedge$ is to be read as saying that if we have A (deduced perhaps from some other assumptions) and have B (again perhaps deduced from some other assumptions), we can infer $A \wedge B$ (with those earlier assumptions all remaining in play). And that's what the suggested sequent calculus rule says too. Likewise, the natural-deduction introduction rule for $\to$ is to be read as saying that if we derive B from the assumption A (and perhaps from some other assumptions), then we can drop that assumption A and infer $A \to B$ (with those other assumptions kept in play); and that's what the sequent calculus rule says too. There will be similar rules for other connectives and for quantifiers.

As for structural rules, we will mention here two candidates. The first is traditionally called *thinning* or *weakening* (neither of which is perhaps a very helpful label). The simple idea is that, if a wff is deducible from some assumptions, it remains deducible if we add in a further unnecessary assumption. So

$$\frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C}$$

Our second structural rule for sequent proofs corresponds to the structural fact that we can chain natural deduction proofs together into longer proofs. Thus, schematically, in natural deduction,

$$\text{We can splice a proof} \quad \begin{matrix}\Gamma\\ \vdots\\ A\end{matrix} \quad \text{with a proof} \quad \begin{matrix}\underbrace{A \qquad \Delta}\\ \vdots\\ B\end{matrix} \quad \text{to get} \quad \begin{matrix}\Gamma\\ \vdots\\ \underbrace{A \qquad \Delta}\\ \vdots\\ B\end{matrix}\ .$$

In sequent calculus terms this corresponds to the following *cut* rule:

$$\frac{\Gamma \ \Rightarrow\ A \qquad \Delta, A \ \Rightarrow\ B}{\Gamma, \Delta \ \Rightarrow\ B}$$

This intuitively sound rule allows us to cut out the middle man A.

So far, then, so good – though of course, we've left lots of detail to be filled out. And there is as yet nothing really novel involved in reworking natural deduction into sequent style like this. Now, however, Gentzen introduces two *very* striking new ideas.

(d) To introduce the first idea, let's think again about the *elimination* rules for conjunction. As a first shot, we might expect to transform the pair of natural-deduction rules into a corresponding pair of sequent-calculus rules like this:

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B} \quad \rightsquigarrow \quad \frac{\Gamma \ \Rightarrow\ A \wedge B}{\Gamma \ \Rightarrow\ A} \qquad \frac{\Gamma \ \Rightarrow\ A \wedge B}{\Gamma \ \Rightarrow\ B}$$

What could be more obvious? But we could alternatively adopt the following single sequent-calculus rule:

$$\frac{\Gamma, A, B \ \Rightarrow\ C}{\Gamma, A \wedge B \ \Rightarrow\ C}$$

This is obviously valid – if C can be derived from some assumptions Γ plus A and B, it can obviously be derived from Γ plus the conjunction of A and B. And we can use *this* rule introducing $\wedge$ on the *left* of the sequent sign instead of the expected pair of rules eliminating $\wedge$ to the *right* of the sequent sign. For note, given the new rule, we can restore the first of the elimination rules as a derived rule, because we can always give a derivation of this shape:

$$\dfrac{\Gamma \ \Rightarrow\ A \wedge B \qquad \dfrac{\dfrac{A \ \Rightarrow\ A}{A, B \ \Rightarrow\ A}\ \text{(Weakening)}}{A \wedge B \ \Rightarrow\ A}\ \text{(New rule for } \wedge)}{\Gamma \ \Rightarrow\ A}\ \text{(Cut)}$$

Similarly, of course, for the companion elimination rule.

And now the point generalizes. As Gentzen saw, in a sequent calculus for intuitionistic logic, we can get *all* the rules for handling connectives and quantifiers to *introduce* a logical operator – either on the right of the sequent sign (corresponding to a natural deduction introduction rule) or on the left of the sequent sign (corresponding to a natural deduction elimination rule).

(e) We can go further. Still working with a sequent calculus for $\Rightarrow$ read as intuitionistic deducibility, we can in fact *eliminate* the cut rule. Anything provable using cut can be proved without it.

This might initially seem pretty surprising. After all, didn't we just have to appeal to the cut rule to show that – using our new introduction-on-the-left rule for $\wedge$ – we can still argue from (1) $\Gamma \ \Rightarrow\ A \wedge B$ to (2) $\Gamma \ \Rightarrow\ A$? How can we possibly do without cut in this case?

Well, consider how we might actually have arrived at (1). Perhaps it was by the rule for introducing occurrences of $\wedge$ on the right of a sequent. So perhaps, to expose more of the proof from (1) to (2), it has the shape of the first proof below (supposing Γ to result from putting together Γ' and Γ''):

$$\dfrac{\dfrac{\Gamma' \Rightarrow A \qquad \Gamma'' \Rightarrow B}{\Gamma \Rightarrow A \land B} \qquad \dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}}{A \land B \Rightarrow A}}{\Gamma \Rightarrow A}\ (\text{Cut}) \qquad \rightsquigarrow \qquad \dfrac{\Gamma' \Rightarrow A}{\Gamma \Rightarrow A}\ (\text{Weakenings})$$

But if we *already* have $\Gamma' \Rightarrow A$, as in the first proof, then we don't need to go round the houses on that detour, introducing an occurrence of $\land$ to get the formula $A \land B$, and then cutting out that same formula: we can just get from $\Gamma' \Rightarrow A$ to $\Gamma \Rightarrow A$ by some weakenings (by adding in the wffs from Γ''). Here, then, eliminating the cut is just like normalizing (part of) a natural deduction proof.,

OK: that only shows that in just *one* rather special sort of case, we can eliminate a cut. Still, it's a hopeful start! And in fact, we can *always* eventually eliminate cuts from an intuitionistic sequent calculus proof.

But the process can be intricate. For example, take a slight variant of our previous example and suppose we want to eliminate the following cut (remember, combining Γ and Γ gives us Γ!):

$$\dfrac{\dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \land B} \qquad \dfrac{\Delta, A, B \Rightarrow C}{\Delta, A \land B \Rightarrow C}}{\Gamma, \Delta \Rightarrow C}\ (\text{Cut})$$

Then we can replace this proof-segment with the following:

$$\dfrac{\Gamma \Rightarrow A \qquad \dfrac{\Gamma \Rightarrow B \qquad \Delta, A, B \Rightarrow C}{\Gamma, \Delta, A \Rightarrow C}\ (\text{Cut})}{\Gamma, \Delta \Rightarrow C}\ (\text{Cut})$$

Again, as in normalizing a natural deduction proof, we have removed a detour – this time a detour through introducing-$\land$-on-the-right and introducing-$\land$-on-the-left. So we have now lost the cut on the more complex formula $A \land B$, albeit replacing it with two new cuts. But still, the new cuts are on the simpler formulas A and B, and we have also pushed one of the cuts higher up the proof. And that's typical: looking at the range of possible situations where we can apply the cut rule – a decidedly tedious hack though all the cases – we find we can indeed keep reducing the complexity of formulas in cuts and/or pushing cuts up the proof until all the cuts are completely eliminated.

(f) So we arrive at this result. In a sequent-calculus setting, we can use a *cut-free* deductive system for intuitionistic logic where all the rules for the connectives and quantifiers *introduce* logical operators, either to the left or to the right of the sequent sign. Analogously to a normalized natural-deduction proof, there are no detours. As we go down a branch of the proof, the sequents at each stage are steadily more complex (we can make the relevant notion of complexity precise in pretty obvious ways).

This proof-analysis immediately delivers some very nice results.

(i) The subformula property: every formula occurring the derivation of a sequent $\Gamma \Rightarrow C$ is a subformula of either one of formulas Γ or of C. (By inspection of the rules!)

(ii) There evidently can be no cut-free, ever-more-complex, derivation that ends with $\Rightarrow \bot$; in other words, absurdity isn't intuitionistically deducible from no premisses. Hence intuitionistic logic is internally consistent.

(iii) Equally evidently, the penultimate line of a cut-free, ever-more-complex, derivation of $\Rightarrow A \lor B$ has to be either $\Rightarrow A$ or $\Rightarrow B$, which establishes the disjunction property for intuitionistic logic – see §8.3 (a).

Note too that, at least for propositional logic, we can take any sequent and systematically try to work upwards from it to construct a cut-free proof with ever-simpler-sequents: the resulting success or failure then mechanically decides whether the sequent is intuitionistically valid.

(g) I said that Gentzen had two very striking new ideas in developing his sequent calculi beyond a mere re-write of a natural deduction system in which dependencies are made explicit. The first idea is to recast all the rules for logical operators as rules for *introducing* logical operators, now allowing introduction to the left as well as introduction to the right of the sequent sign, and to then show that we can get a cut-free proof (hence, a proof that malways goes from less complex to more complex sequents) for any intuitionistically correct sequent.

But this first idea doesn't by itself resolve the problem which Gentzen initially faced. Recall, he ran into trouble trying to find a normalization proof for *classical* natural deduction. And plainly, if we stick with a cut-free all-introduction-rules sequent calculus of the current style, we can't get a classical logical system at all. The point is trivial: one key additional classical principle we need to add to intuitionistic logic is the double negation rule. We need to be able to show, in other words, that from $\Gamma \Rightarrow \neg\neg A$ we can derive $\Gamma \Rightarrow A$. But obviously we can't do *that* in a system where we can only move from logically simpler to logically more complex sequents!

What to do? Well, at this point Gentzen's second (and quite original) idea comes into play. *We now liberalize the notion of a sequent.* Previously, we took a sequent $\Gamma \Rightarrow A$ to relate zero or more wffs on the left to a single wff on the right. Now we pluralize on both sides of the sequent sign, writing $\Gamma \Rightarrow \Delta$; and we read that as saying that at least one of Δ is deducible from the wffs Γ. If you like, you can regard Δ as delimiting the field within which the truth must lie if the premisses Γ are granted. (We'll continue, for our purposes, to treat Γ and Δ officially as sets, rather than multisets or lists: note that we will allow either or both to be empty.)

Keeping the idea that we want all our rules for the logical operators to be rules for *introducing* operators to the left or right of the sequent sign, how might these rules now go? There are various options, but the following can work nicely for conjunction and disjunction:

$$(\land\text{L})\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \land B \Rightarrow \Delta} \qquad (\land\text{R})\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \land B}$$

$$(\lor\text{L})\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \lor B \Rightarrow \Delta} \qquad (\lor\text{R})\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \lor B}$$

I won't give the rules for all the other logical operators here, but let's note the left and right rules for negation (these can either be built-in rules, if negation is treated as a primitive built-in connective, or derived rules, if negation is defined in terms of the conditional and absurdity):

$$(\neg L)\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma, \neg A \Rightarrow \Delta} \qquad (\neg R)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

These rules are evidently correct. For the first rule, suppose that given the assumptions Γ, then (at least) one of Δ and A follows: then given the same assumptions Γ but now also ruling out A, we can conclude that (at least) one of Δ is true. We can argue similarly for the second rule. But with these negation and disjunction rules in place we immediately have the following derivation:

$$\cfrac{\cfrac{A \Rightarrow A}{\Rightarrow A, \neg A}\ (\neg R)}{\Rightarrow A \vee \neg A}\ (\vee R)$$

Out pops the law of excluded middle! – so we know we are dealing with classical calculus.

(h) What about the structural rules for our classical sequent calculus which allows multiple alternative conclusions as well as multiple premisses? We can now allow weakening on both sides of a sequent. And we can generalize the cut rule to take this form:

$$\frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma', A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

(Think why this is a sound rule, given our interpretation of the sequents!). But then,as with our sequent calculus for intuitionistic logic, we can proceed to prove that we can eliminate cuts. If a sequent is derivable in our classical sequent calculus, it is derivable without using the cut rule!

As with intuitionist logic, this immediately gives us some nice results. Of course, we won't have the disjunction property (think excluded middle!). But we still have the subformula property in the form that if $\Gamma \Rightarrow \Delta$ is derivable, the every formula in the sequent proof is a subformula of one of Γ, Δ. And again, simply but crucially, $\Rightarrow \bot$ won't be derivable in the cut-free classical system, so it is consistent.

And that's perhaps enough by way of introduction to our theme (B), in which we begin to explore various elegant sequent calculi, prove cut-elimination theorems, and draw out their implications.

9.3 Proof theory and the consistency of arithmetic: a short overview

Now for our third theme (C), Gentzen's famed proof of the consistency of arithmetic (more precisely, the consistency of first-order Peano Arithmetic). Recall, Hilbert's Programme is the project of using tame proof-theoretic reasoning to prove the consistency of mathematical theories: PA gives us a first test case.

(a) You might very well wonder whether there can be *any* illuminating and informative ways of proving PA to be consistent. After all, proving consistency by appealing to a *stronger* theory like ZF set theory which in effect contains PA won't be a very helpful (for doubts about the consistency of PA will presumably just carry over to become doubts about the stronger theory). And you already know that Gödel's Second Incompleteness Theorem shows that it is impossible to prove PA's consistency by appealing to a *weaker* theory tame enough to be modelled inside PA (not even full PA can prove PA's consistency).

However, another possibility does remain open. It isn't ruled out that we can prove PA's consistency by appeal to an attractive theory which is weaker than PA in some respects but stronger in others. And *this* is what Gentzen aims to give us in his consistency proof for arithmetic.[3]

(b) Here then is an outline sketch of the key proof idea.

We start with a formulation of PA using for its logic a classical sequent calculus including the cut rule. (We will initially want the cut rule in making use of PA's axioms, and we can't assume straight off the bat that we can still eliminate cuts once we have more complex proofs appealing to non-logical axioms). Then,

> The 'correctness' of a proof depends on the correctness of certain other simpler proofs contained in it as special cases or constituent parts. This fact motivates the arrangement of proofs in linear *order* in such a way that those proofs on whose correctness the correctness of another proof depends precede the latter proof in the sequence. This arrangement of the proofs is brought about by correlating with each proof a certain transfinite ordinal number.

The idea, then, is that the various sequent proof-trees in this version of PA can be put into an ordering by a kind of dependency relation, with more complex proof trees (on a suitable measure of complexity) coming after simpler proofs. And this can be a well-ordering, so that the position along the ordering can indeed be tallied by an ordinal number.

But why is the relevant linear ordering of proofs said to be *transfinite* (in other words, why must it allow an item in the ordering to have an infinite number of predecessors)? Because

> [it] may happen that the correctness of a proof depends on the correctness of infinitely many simpler proofs. An example: Suppose that in the proof a proposition is proved for *all* natural numbers by complete induction. In that case the correctness of the proof obviously depends on the correctness of every single one of the infinitely many individual proofs obtained by specializing to a particular natural number. Here a natural number is insufficient as an ordinal number

[3] Gentzen in fact gives four different proofs, developed along somewhat different lines. But the master idea underlying the best known of the proofs is given in a wonderfully clear way in his wide-ranging lecture on 'The concept of infinity in mathematics' reprinted in his *Collected Papers*, from which the following quotations come.

> for the proof, since each natural number is preceded by only finitely many other numbers in the natural ordering. We therefore need the transfinite ordinal numbers in order to represent the natural ordering of the proofs according to their complexity.

Think of it this way: a proof by induction of the quantified $\forall x \varphi(x)$ leaps beyond all the proofs of $\varphi(0)$, $\varphi(1)$, $\varphi(2)$, ... And the result $\forall x \varphi(x)$ depends for its correctness on the correctness of the simpler results. So, in the sort of ordering of proofs which Gentzen has in mind, the proof by induction of $\forall x \varphi(x)$ must come infinitely far down the list, after all the proofs of the various $\varphi(n)$.

And now Gentzen's key step is to argue by an induction along this transfinite ordering of proofs. The very simplest proofs right at the beginning of the ordering transparently can't lead to contradiction. Then

> once the correctness [and specifically, freedom from contradicton] of all proofs preceding a particular proof in the sequence has been established, the proof in question is also correct precisely because the ordering was chosen in such a way that the correctness of a proof depends on the correctness of certain earlier proofs. From this we can now obviously infer the correctness of all proofs by means of a transfinite induction, and we have thus proved, in particular, the desired consistency.

Transfinite induction here is just the principle that, if we can show that a proof has a property P if all its predecessors in the relevant ordering have P, then *all* proofs in the ordering have property P.

(c) We can implement this same proof idea the other way around. We show that if any proof *does* lead to contradiction, then there must be an *earlier* proof in the linear ordering of proofs which also leads to contradiction – so we get an infinite sequences of proofs of contraction, ever earlier in the ordering. But then the ordinals which tally these proofs of contradiction would have to form an infinite descending sequence. And there can't be such a sequence of ordinals. Hence no proof leads to contradiction and PA is consistent.

(d) Two questions arising. First, *how* do we show that if a proof leads to a contradiction, then there must be an *earlier* proof in the linear ordering of proofs which leads to contradiction? By eliminating cuts using reduction procedures like those involved in the proof of cut-elimination for a pure logical sequent calculus – so here's the key point of contact with ideas we meet in tackling theme (B).

And second, what kind of transfinite ordering is involved here? Gentzen's ordering of possible proof-trees in his sequent calculus for PA turns out to have the order type of the ordinals less than ε_0 (these then are all the ordinals which are sums of powers of ω). So, what Gentzen's proof needs is the assumption that a relatively modest amount of transfinite induction – induction up to ε_0 – is legitimate.

Now, the PA proof-trees which we are ordering are themselves all finite objects; we can code them up using Gödel numbers in the familiar sort of way.

So in ordering the proofs, we are in effect thinking about a whacky ordering of (ordinary, finite) code numbers. And whether one number precedes another in the whacky ordering is nothing mysterious; a computation without open-ended searches can settle the matter.

So what resources does a Gentzen-style argument use, if we want to code it up and formalize it? The assignment of a place in the ordering to a proof can be handled by primitive recursive functions, and facts about the dependency relations between proofs at different points in the ordering can be handled by primitive recursive functions too. A theory in which we can run a formalized version of Gentzen's proof will therefore be one in which we can (a) handle primitive recursive functions *and* (b) handle transfinite induction up to ε_0, maybe via coding tricks. It turns out to be enough to have all p.r. functions 'built in' (in the way that addition and multiplication are built into PA) together with a formal version of transfinite induction just for simple quantifier-free wffs containing expressions for these p.r. functions. Such a theory is neither contained in PA (since it can prove PA's consistency by formalizing Gentzen's method, which PA can't), nor does it contain PA (since it needn't be able to prove instances of the ordinary Induction Schema for arbitrarily complex wffs).

So, in this sense, we can indeed prove the consistency of PA by using a theory which is weaker than PA in some respects while stronger in others.

(e) Of course, it is a very moot point whether – if you were *really* worried about the consistency of PA – a Gentzen-style proof when fully spelt out would help resolve your doubts. Are the resources it requires 'tame' enough to satisfy you?

Well, if you are globally worried about the use of induction in general, then appealing to an argument which deploys an induction principle won't help! But global worries about induction are difficult to motivate, and perhaps your worry is more specifically that induction over arbitrarily complex wffs might engender trouble. You note that PA's induction principle applies, inter alia, to wffs that themselves quantify over all numbers. And you might worry that if (like Frege) you understand the natural numbers to be what induction applies to, then there's a looming circularity here – numbers are understood as what induction applies to, but understanding some cases of induction involves understanding quantifying over numbers. If *that* is your worry, the fact that we can show that PA is consistent using an induction principle which is only applied to quantifier-free wffs (even though the induction runs over a novel ordering on the numbers) could soothe your worries.

Be that as it may: we can't pursue that kind of philosophical discussion and further here. The point remains that the Gentzen proof is a fascinating achievement, containing the seeds of wonderful modern work in proof theory. Perhaps we haven't quite executed Hilbert's Programme of proving consistency by appeal to tame proof-theoretic reasoning. But in the attempt, we have found how far along the ordinals we need to run our transfinite induction in order to prove the consistency of PA. And we can now set out to discover how much transfinite induction is required to prove the consistency of other theories. But the achievements of that kind of ordinal proof theory will have to be left for you (eventually) to explore . . .

9.4 Main recommendations on elementary proof theory

Let's start with a couple of very useful encyclopaedia entries by some notable proof theorists.

First, the following historical outline is particularly helpful for orientation:

1. Jan von Plato, 'The Development of Proof Theory', *Stanford Encyclopedia of Philosophy*. Available at tinyurl.com/sep-devproof.

And then look at the first half of the main entry on proof theory:

2. Michael Rathjen and Wilfrid Sieg, 'Proof Theory', §§1–3, *Stanford Encyclopedia of Philosophy*. Available at tinyurl.com/sep-prooftheory.

Skip over any passages that are initially unclear, and return to them when you've worked through some of the readings below.

In keeping with our overview in the previous section, I suggest that – in a first encounter with proof theory – you focus on (A) normalization for natural deduction and its implications; (B) the sequent calculus, cut-elimination and its implications; and (C) a Gentzen-style proof of the consistency of arithmetic. Now, there is book which aims to cover just these topics at the level we want:

3. Paolo Mancosu, Sergio Galvan and Richard Zach, *An Introduction to Proof Theory: Normalization, Cut-Elimination and Consistency Proofs* (OUP, 2021) – henceforth *IPL*.

However, as the authors say in their Preface, "in order to make the content accessible to readers without much mathematical background, we carry out the details of proofs in much more detail than is usually done." And this isn't as reader-friendly as they intend: expositions too often become wearyingly laborious. Also the authors stick very closely to Gentzen's own original papers, which isn't always the wisest choice. So, at least on topic areas (A) and (B), I will be highlighting some alternatives.

(A) You could find that the following *Handbook of the History of Logic* article gives some more helpful orientation:

4. F. J. Pelletier and Allen Hazen, 'Natural Deduction', §3. Available at tinyurl.com/pellhazen.

   It is §3.1 that is most immediately relevant. But do read the rest of §3. (And indeed, for your general logical education, why not read all this informative survey paper sometime?)

You could next tackle Chs 3 and 4 of *IPL*. But there's a lot to be said for just diving into the brisk opening chapters of a modern classic:

5. Dag Prawitz, *Natural Deduction: A Proof-Theoretic Study** (originally published 1965, reprinted by Dover Publications 2006), Chapters I to IV.

   Ch. I presents the now-standard Gentzen-style natural deduction systems for intuitionistic and classical logic. The short Ch. II explains the sense in which elimination rules are inverses to introduction rules. Then it notes some basic "reduction steps" for eliminating the sort of unnecessary detours which result from the application of an introduction rule being immediately followed by the application of the corresponding elimination rule. Ch. III shows that we can normalize proofs in a classical ND system – or at least, a cut down version without $\vee$ and $\exists$ built in as primitive – by systematically eliminating detours. Ch. IV extends the result to a full system of intuitionistic logic.

And that's perhaps about as much as you need on natural deduction. OK, you might be left wondering whether we can improve on Prawitz's Chapter III result and prove a similar normalization result for a full classical logic with the $\vee$ and $\exists$ rules restored. The answer is 'yes'. *IPL* §4.9 shows how it can be done for Gentzen's original natural deduction system. But it is more interesting to look at what happens if you revise Gentzen's original classical rules and use so-called 'general elimination rules'; this makes establishing normalization rather more straightforward. For something on this, see

6. Jan von Plato, *Elements of Logical Reasoning* (CUP, 2013). Chapters 3 to 6.

These very accessible chapters on intuitionistic and classical propositional logic also introduce the theme of proof-search.

Von Plato's book is, in fact, intended as a first introductory logic text, based on natural deduction: but it, very unusually, has a strongly proof-theoretic emphasis. And non-mathematicians, in particular, could find the whole book very helpful.

(B) Next, moving on to sequent calculi, you could start with Chs 5 and 6 of *IPL*. But the following is very accessibly written, ranges more widely, and is likely to prove quite a bit more enjoyable:

7. Sara Negri and Jan von Plato, *Structural Proof Theory* (CUP, 2001).

   The first four chapters gives us the basics. Ch. 1 helpfully bridges our topics, 'From Natural Deduction to Sequent Calculus'. Ch. 2 gives a sequent calculus for intuitionistic propositional logic and proves the admissibility of cut. Ch. 3 does the same for classical propositional logic. Ch. 4 adds the quantifiers.

You might well want to then read on to Ch. 5 which illuminatingly discusses some variant sequent calculi. Then you can jump to Ch. 8 which takes us ‘Back to Natural Deduction’. This relates the sequent calculus to natural deduction with general elimination rules, shows how to translate between the two styles of logic, and then derives a normalization theorem from the cut-elimination theorem: again this is very instructive.

Negri and von Plato note that, as we ‘permute cuts upward’ in a derivation – in order to eventually arrive at a cut-free proof – the number of cuts remaining in a proof can increase exponentially as we go along (though the process eventually terminates). So a cut-free proof can be much bigger than its original version. Pelletier and Hazen (4) in their §3.8 make some interesting related comments about sizes of proofs. And you will certainly want to read this famous short paper:

8. George Boolos, ‘Don’t Eliminate Cut’, reprinted in his *Logic, Logic, and Logic* (Harvard UP, 1998)

And *now*, if you really want to know more (in particular about how Gentzen originally arrived at his cut-elimination proof) you can make use of the relevant *IPL* chapters, skipping over a lot of the tedious proof-details.

(C) Next, Gentzen’s on the consistency of arithmetic. Von Plato (1) and Rathjen and Sieg (2) both provide some context for Gentzen’s work. And here’s a contemporary mathematician’s perspective on why we might be interested in the proofs of the consistency of PA:

9. Timothy Y. Chow, ‘The Consistency of Arithmetic’, *The Mathematical Intelligencer* 41 (2019), 22–30. Available at tinyurl.com/chow-cons.

Now we have two options, as Rathjen and Sieg (2) makes clear. We can tackle something like one of Gentzen’s own consistency proofs for PA; but we then have to tangle with a *lot* of messy detail as we negotiate the complications caused by having to deal with the induction axioms. Or alternatively we can box more cleverly, and prove consistency for a theory PA_ω which swaps the induction axioms for an infinitary rule. The proof uses the same overall strategy, but this time its implementation is a lot less tangled (yet the proof still does the needed job, since PA_ω’s consistency implies PA’s consistency).

There are a number of versions of the second line of proof in the literature. There is quite a neat but rather terse version here, from which you should be able to get the general idea (it assumes you know a bit about ordinals):

10. Elliott Mendelson, *Introduction to Mathematical Logic*, ‘Appendix: A Consistency Proof for Formal Number Theory’ (1st edn., 1964; later dropped but restored in the 6th edn., 2015).

But let’s suppose that you do want something much closer to Gentzen’s original proof:

There is a rather austere presentation of a Gentzen-style proof in the classic textbook on proof theory by Takeuti which I will mention in the next section: this might suit the more mathematical reader. But the following is more accessible – though with a distracting amount of detail:

3. Mancosu, Galvan and Zach, *IPL*, Chapters 7–9.

Read Chapter 8 on ordinal notations first. Then the main line of proof is in Chapters 7 and 9. Now, after an initial dozen pages saying something about PA, these two chapters together span another *sixty-five* pages(!), and it is consequently easy to get lost/bogged down in the details. And it is not as if the discussion is padded out by e.g. a philosophical discussion about the warrant for accepting the required amount of ordinal induction; the length comes from hacking through more details than any sensible reader will want or need.

However, if you have already tackled a modest amount of other mathematical logic, you should by now have enough nous to be able to read these chapters pausing over the key ideas and explanations while initially skipping/skimming over much of the detail. You could then quite quickly and painlessly end up with a very good understanding of at least the general structure of Gentzen's proof and of what it is going to take to elaborate it. So I suggest first skimming through to get the headline ideas, and then do a second pass to get more feel for the shape of some of the details. You can then drill down further again to work through as much of the remaining nitty-gritty that you then feel that you really want/need (which probably won't be much!).

9.5 Some parallel/additional reading

Here I will mention (parts of) just two other books for now. Both start again from scratch, but then their mode of presentation is perhaps half a step up in mathematical sophistication from the readings in the last section;

11. Gaisi Takeuti, *Proof Theory** (North-Holland 1975, 2nd edn. 1987: reprinted Dover Publications 2013).

    This is a true classic – if only because for a while it was about the only available book on most of its topics. Later chapters won't really be accessible to beginners. But you can certainly tackle Ch. 1 on logic, §§1–7 (and perhaps the beginnings of §8, pp. 40–45, which is easier than it looks if you compare how you prove the completeness of a tree system of logic). Then tackle Ch. 2, §9 on Peano Arithmetic. You can skip the next section on the incompleteness theorem, and skim §11 on ordinals (which makes rather heavy weather of what's really needed, which is the claim that a decreasing series of ordinals less than ε_0 can only be finitely long: see p. 98 on). The core consistency proof is then given in §12; read up to at least p. 114. This isn't exactly plain sailing – but if you skip and skim over some of the more tedious proof-details you should pick up a good basic sense of what happens in the consistency proof.

12. A. S. Troelstra and H. Schwichtenberg's *Basic Proof Theory* (CUP 2nd ed. 2000).

    This a volume in the series 'Cambridge Tracts in Computer Science'. Now, one theme that runs through the book concerns the computer-science idea of formulas-as-types and invokes the lambda calculus: however, it is in fact quite possible to skip over those episodes if you aren't familiar with the idea. The book, as the title indicates, is intended as a first foray into proof theory, and it *is* reasonably approachable. However it does spend quite a bit of time looking at slightly different ways of doing natural deduction and slightly different ways of doing the sequent calculus, and the differences may matter more for computer scientists with implementation concerns than for others.

    You can, however, with a bit of skipping, at this stage very usefully read just Chs. 1–3, the first halves of Chs. 4 and 6, and then Ch. 10 on arithmetic again.

We will return to consider more advanced texts on proof theory in Part III of the Guide.