Entailment, with nods to Lewy and Smiley

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Last week, we talked a bit about the Anderson-Belnap logic of entailment, as discussed in Priest's *Introduction to Non-Classical Logic*. For a quite different approach to entailment, we'll look next week at Neil Tennant's account. Doing things rather out of order, this week I'd like to say something more basic about the problems to which both Anderson and Belnap, on the one hand, and Tennant on the other, are responding. This will give me the chance for a bit of nostalgic philosophical time-travelling, revisiting work by Casimir Lewy and Timothy Smiley from the Cambridge of fifty years ago.[1]

1 What is the topic?

Our topic then is *entailment*. We mean the converse of the relation of logical consequence. So, we say that the premisses $A_1, A_2, \ldots, A_n$ entail the conclusion C if and only if C is a logical consequence of $A_1, A_2, \ldots, A_n$.

And when is C a logical consequence of $A_1, A_2, \ldots, A_n$? In first year lectures, for example, when trying to introduce neophytes to the concept, you offer a whole bunch of hopefully equivalent informal characterizations. The idea you are trying to get across is *something* along the lines of the conclusion C's being *absolutely guaranteed* to be true if the premisses $A_1, A_2, \ldots, A_n$ are true; or the joint truth of $A_1, A_2, \ldots, A_n$ being an *absolutely conclusive*, indefeasible, reason for the truth of C; or the inference from $A_1, A_2, \ldots, A_n$ to C being *absolutely watertight*; and so on.

But do these, and similar informal gestures, really suffice to fix a determinate concept of entailment? If so, how are we to analyse it? That's our topic.

2 The Lewis argument

Consider the following four principles:

P1. *A entails $(A \vee B)$.* That looks incontrovertible – at least given that $\vee$ means truth-functional disjunction.[2] If A is true then that absolutely guarantees that at least one of A and B is true, and that's all it takes for $(A \vee B)$ is true.

P2. *$(A \vee B)$ and $\neg A$ together entail B.* That too looks incontrovertible – at least given a natural understanding of negation which forces A and $\neg A$ to have incompatible truth-values. For if by the first premiss at least one of A and B is true, but by the second premiss it can't be the first because $\neg A$ is true, then that forces B to be the one which is true.

[1] Thanks to Tim Button and Luca Incurvati in particular for comments after I gave for a version of these notes as a talk.

[2] Students do often balk at the inference from A to A *or* B, as they are inclined to hear the vernacular 'A or B' as 'If not A then B' with a non-material 'if'. But here '$\vee$' is stipulated to be truth-functional.

P3. *Entailment is transitive.* The fundamental principle is that if A entails B and B entails C, then we can cut out the middle man: A entails C. If the truth of A absolutely guarantees the truth of B which absolutely guarantees the truth of C, then the truth of A guarantees the truth C.

The fundamental principle naturally generalizes in obvious ways. In particular we'll want this simple generalization: if A_1 entails B, and A_2 together with B entails C, then we can again cut out the middle man and we have A_1 and A_2 entails C. For if we have A_1 and A_2 true together, then (since A_1 entails B) we are guaranteed to have A_1, A_2 and B all true together, and hence (since A_2 and B entail C) we are guaranteed to have C. Or to put that diagrammatically, if we have

$$\frac{A_1}{B}$$

with the horizontal line indicating an entailment, and

$$\frac{B \quad A_2}{C}$$

then we can paste together the valid inferences to get

$$\frac{\dfrac{A_1}{B} \quad A_2}{C}$$

which shows that A_1 and A_2 entails C. It is, after all, the very rationale of proof-construction that we can in this way paste together small obvious inference steps, so that if each small step is indeed a genuine entailment, the bigger unobvious step from initial premisses to final conclusion via intermediate conclusions is also a genuine entailment.

P4. *It isn't, in general, the case that A and $\neg A$ together entail C.* It seems just absurd to say, for example, that the joint truth of A and $\neg A$ could be an absolutely conclusive reason for the truth of some arbitrary conclusion C. Or that the joint truth of A and $\neg A$ absolutely guarantees the truth of C. There is, in general, no connection at all between the premisses and the conclusion here – and the idea of premisses guaranteeing a conclusion/giving conclusive reasons seems to be tied up with some idea of connectedness.

Of course, the intuition that it *isn't* the case that contradictory premisses entail anything-you-like is the reason why beginners balk at the usual introductory logic book suggestion about how to tidy up our initial gestures at the notion of entailment into an official story. We say "Premisses A_i entail conclusion C if there is no possible situation in which the A_i is true together and C is false". If we define 'strict implication' be saying that the A_i strictly imply C when $\Box(A_1 \wedge A_2 \wedge \ldots \wedge A_n \supset C)$, then the official story is that entailment is just strict implication, which implies that contradictory premisses indeed entail anything. That is surely unhappy.

So, we'd *like* an account of entailment which conforms to the intuitively appealing principles P1 to P4. But of course, infamously, as C. I. Lewis seems to have been the first to remark, these four principles about entailment are inconsistent. For just apply the simple generalization of transitivity in P3 to put together P1 *A entails $(A \vee C)$*, and P2 *$(A \vee C)$ and $\neg A$ entail C* and we get the unwanted result *A and $\neg A$ together entail C*. Putting that diagrammatically, we can paste together entailments to get

$$\dfrac{\dfrac{A}{(A \vee C)} \qquad \neg A}{C}$$

and that apparently conclusively warrants the very entailment that we rejected in P4. Trouble!

3 Options

We can't really reject P1, for we can take that principle as partially stipulative of what we mean by '$\vee$' (inclusive, truth-functional, disjunction). So something else has got to give. That evidently leaves three options:

O1. Reject the (unrestricted) disjunctive syllogism principle that $(A \vee B)$ and $\neg A$ always together entail B.

O2. Reject the (unrestricted) transitivity of entailment.

O3. Bite the bullet, and accept that a pair of premisses A, $\neg A$ do indeed entail any conclusion B at all.

So which of these options is the most attractive?

Before turning to details, it is worth remarking in general terms on how things might turn out.

S1. It could be that we've simply made a *mistake* in initially endorsing all four principles P1, P2, P3 and P4. One of the principles turns out, on a little more reflection, simply not to be true to our actual concept of our entailment. The name of the game is to come up with an analysis of entailment that respects the remaining principles which *are* true.

S2. Alternatively, we might end up holding that the contradiction between the four principles P1, P2, P3 and P4 in fact shows that the intuitive notion of entailment is not fully coherent. For suppose (a) you hold that what fixes the sense of a non-observational predicate like 'entails' is in part the principles governing its use that you find primitively compelling. (I use Peacocke's term, though without committing to the details of his picture. But the basic idea is attractive: some principles governing the use of a concept are not really up for grabs: anyone who fully grasps the concept must conform their use to the principles, and not in virtue of inferential knowledge gleaned from more basic deployments of the concept.) And suppose (b) that, taken separately, the principles P1, P2, P3 and P4 do remain compelling, even on reflection. Then this shows that the very notion of entailment is not in good order. In this case, the name of the game is not analysis but to *decide* what the best 'rational reconstruction' of the concept is.

In his *Meaning and Modality* (CUP, 1976), Casimir Lewy basically takes the second line. It isn't that our concept of entailment is in good order, and that – in endorsing all of P1 to P4 – we initially make a simple mistake about it, one to be corrected by dropping the offending principle: rather, our intuitive concept is revealed as problematic (he says 'inconsistent') and it needs revising and reconstructing.

Now, I'm not sure how far we'd want to press a *sharp* distinction here between these two outcomes. After all, to suppose that there is a sharp distinction between changing our doctrine (about some fixed concept) and changing the concept (as we rationally reconstruct a conceptual mess) is in effect to suppose that there is a sharp analytic/synthetic distinction. Perhaps you don't want to do that. But be that as it may: for we can certainly draw a *graded* distinction here. However we analyse it, there's a difference between (a) reasonably happily giving up an initially attractive principle about entailment when further reflection releases our grip on it because we come to see the principle as straightforwardly false, and (b) rather unhappily having to settle on a story about entailment whose appeal is just that it the best option on a cost-benefit analysis, without letting go of the thought that some discarded principle still seems to accord with what we originally meant. Lewy thinks that (b) is the best we can do. Is he right?

4 Reject disjunctive syllogism?

Let's first revisit the idea of rejecting (unrestricted) disjunctive syllogism. Did we dismiss that too quickly?

Well, Anderson and Belnap recommend rejection as the way to go. And last week we saw some semantical stories that would supposedly warrant rejection. To take the simplest, suppose we hold that there are *three* different statuses that a proposition can have with respect to truth and falsity: it can be *solely true*, *solely false*, or *both true and false*. Then if P is *both*, and Q *solely false*, then – assuming the familiar truth-table definitions of the connectives in terms of truth and falsity assignments – $\neg P$ will be true (because *both* true and false), $(P \vee Q)$ will also be true (because *both* true and false), but Q is plain false. So the inference $\neg P, (P \vee Q) \therefore Q$ leads from truths to a falsehood so is not a valid entailment.

It can then be added that, when restricted to cases where P is 'normal' (doesn't take *both* values), disjunctive syllogism is indeed reliable, and that's why we wrongly think it is *generally* reliable: but – the story goes – we see that it *isn't* valid when we allow for truth-value gluts.

But does this sort of semantic story make any sense? We can say the words, of course, but do we have a defensible conception of propositions and a defensible conception of truth and falsehood that leaves it open that a proposition could be *both* true and false? Frankly, I very strongly doubt that there *is* such a conception. To be sure, dialetheists will disagree: and we can't even begin to debate the issue here. But most logicians do still think that dialetheism would be an *exorbitant* price to pay to block the Lewis argument. (And other semantical stories which are supposed to ground the rejection of disjunctive syllogism – like the Routley-star semantics for negation – seem equally unattractive.)

5 Lewy's paradox

However, we can say more about the option of rejecting P2. This *might* perhaps have something to be said for it if it blocks, in one fell swoop, *all* arguments with paradoxical conclusions about entailment. But it doesn't. We'll consider a type of argument from Lewy (rephrased in minor ways to avoid irrelevant distractions) which – like the Lewis argument – has an unwelcome conclusion but which doesn't involve disjunctive syllogism at all. Of course, it could be that different paradoxes of entailment need different solutions: but Lewy's work at least suggests that focussing on P2 isn't the way forward.

To set up Lewy's paradox – in the sense of a plausible argument to an implausible conclusion – we need another principle about entailment that again looks intuitively compelling.

P5. *Suppose A and C are contingent, and B is necessary: then if A and B together entail C, then A by itself entails C.* That is to say: in inferences about contingent matters, necessarily true premisses can be dropped.

Why is this so? And why in this principle do we specify that A and C are contingent?

Well take the second question first. Take the trivial entailment from the premisses that Caesar is dead and that $2 + 2 = 4$ to the conclusion $2 + 2 = 4$. If we can drop necessarily true premisses from entailments willy-nilly, then it would follow that the single premiss Caesar is dead entails that $2 + 2 = 4$ which is counterintuitive (what has Caesar to do with the truths of arithmetic?). But if A and C are contingent, we don't get into that sort of apparent fallacy of irrelevance. Since it is contingent, C isn't self-guaranteeing (so to speak). And it can only be guaranteed by premisses including contingencies. Suppose then that A and B together *do* absolutely guarantee the truth of C, and B is necessary. Then just given A – which could have been otherwise, so needs to be given! – you will always in fact have B as well since it is necessary, and so you will have A and B together which guarantee C. So the truth of A indeed relevantly guarantees the truth of C.

With that principle to hand, take the following three propositions:

(1) In Rome, the sounds "Due più due fa quattro" are in fact standardly used to express the proposition that two and two is four.

(2) Two and two *is* four.

(3) In Rome, the sounds "Due più due fa quattro" are in fact standardly used to express a true proposition.

And now consider the following argument.

L1. (1) and (2) entail (3). [Trivial and surely uncontentious!]

L2. But (1) and (3) are contingent, and (2) is necessary.

L3. Hence (1) entails (3) [From L1 by principle P5, given L2.]

L4. But (1) and (3) entails (2) [Triviality].

L5. Hence (1) and (1) entails (2) [By the transitivity of entailment: put $A_1 = A_2 =$ (1) in P3]

L6. But that's just absurd: contingent facts about what happens in Rome (even repeated!) do not entail arithmetical facts!

(You can obviously change the example of a necessary proposition here if for some reason (2) doesn't appeal, and make compensating adjustments elsewhere.[3] Choose

[3]In fact, Lewy's initially preferred argument involves propositions more along the lines of

(1′) In Rome, the sounds "Due più due fa quattro" are in fact standardly used to express the proposition that two and two is four.

(2′) It is necessarily true that two and two is four.

(3′) In Rome, the sounds "Due più due fa quattro" are in fact standardly used to express a necessarily true proposition.

and he appeals to the S4 principle to claim that (2′) is itself necessary: and then he derives the (greater?) absurdity that contingent facts about what happens in Rome apparently entails facts about the modal status of arithmetic.

some tautology if you must, if you want to keep the necessities logically necessary in the narrowest sense. And in 'proposition' talk sticks in your craw, then you can rephrase in obvious ways: it is not essential to the argument.)

So, faced with the Lewy argument we again have three live options.

O′1. Reject the principle P5 that necessary premisses can dropped when inferring about contingent matters

O′2. Reject the (unrestricted) transitivity of entailment.

O′3. Bite the bullet, and accept that facts about speech habits in Rome can entail facts about numbers.

This time, obviously, worrying about disjunctive syllogism is no help at all! And you might well think that bullet-biting is even less attractive in this case than in responding to the Lewis argument. (For you might muse: "Something *peculiar* is going on when we consider inferring from explicitly inconsistent premisses: maybe whatever we say about such a case will have to seem weird! But in Lewy's inference, there are no inconsistencies around – yet we end up with an intolerable conclusion.").

So should we modify O′2, and further restrict suppression of necessary propositions in valid entailments? Perhaps: but it is difficult to see how to do this in a principled way – yet surely we *can* often suppress logical necessities in inferences. (After all, consider the use of axiomatic systems of logic, where we show that the contingent A entails the contingent C by invoking a bunch of logical axioms en route in the derivation of C from A – we don't hesitate to move from the result that A + some logical axioms entails C to concluding that A entails C.) And in any case, L3 *does* looks intuitively true anyway (whether or not we base it on O′2).

To avoid the unwelcome conclusions of Lewy's argument and of the Lewis argument, then, maybe the thing to do is to restrict transitivity? But how might that work?

6 Smiley on entailment

The most attractive and principled way I know to restrict the transitivity of entailment is that developed by Neil Tennant. His account is our topic for next week. But it should be helpful if we first explore more of the background.

The Lewis argument purports to show that contradictory premisses entail anything at all. The Lewy argument can be naturally generalized (it wasn't to the purpose that we chose Rome and the proposition that two and two is four – in particular, any necessary proposition would have done): and the argument purports to show that a certain kind of necessary proposition can be entailed by apparently irrelevant premisses. One way to block the arguments is by brute force:

> I propose so to use the word 'entails' that no necessary statement and no negation of a necessary statement can significantly be said to entail or be entailed by any statement. (P. F. Strawson, 'Necessary Propositions and Entailment-statements', *Mind* (1948), at p. 186.)

But *that*'s hopeless: it would force us to say that the notion of entailment can't be applied in talking about mathematical proofs when we infer necessities from necessities. It would also block us from using reductio arguments (for these aim to show that unobviously inconsistent premisses are indeed inconsistent, precisely by deriving contradictions from them).

However, there is a neighbouring idea which is more plausible. Perhaps we should only say something along these lines:

> A entails C, if and only if, by means of logic, it is possible to come to know the truth of $A \supset C$ without coming to know the falsehood of A or the truth of C. (G. H. von Wright, *Logical Studies* (1957), p. 181.)

The idea is, for a trivial example, we can see that $P \wedge Q$ entails P by just doing the truth-table, and this entailment still holds in a mathematical context as doing the truth-table doesn't involve fixing the truth-values of P and Q. And generalizing, for similar, other familiar entailments similarly carry over to mathematics. So we avoid the problem that besets Strawson's crude proposal. However, as given, the von Wright definition doesn't do the intended job of paradox avoidance.

For consider, assuming a classical two-valued framework, we can establish the following two are tautologous forms by use of truth-tables

1. $(C \wedge A) \supset (C \wedge (A \vee B))$
2. $(\neg A \wedge (A \vee B)) \supset B$

So now, having by 'logical means' established these general principles without establishing anything about any particular propositions, we are entitled to their substitution instances

3. $(\neg P \wedge P) \supset (\neg P \wedge (P \vee Q))$
4. $(\neg P \wedge (P \vee Q)) \supset Q$

and putting those together, arguing with $\supset$ in the usual way which can be generally justified by truth-tables, we get

5. $(\neg P \wedge P) \supset Q$

by logical means, without en route establish the falsehood of $\neg P \wedge P$ or the truth of Q. So by von Wright's definition, it follows that

6. $\neg P \wedge P$ entails Q

which is just the sort of conclusion we want to avoid!

The trouble here, of course, is the sloppiness of the idea of it's being possible to come to know the truth of $A \supset B$ by means of logic. Can we improve things while still capturing something like the intended idea? Well, consider the following:

> $A_1, A_2, \ldots A_n$ entails C if and only if the implication $A_1 \wedge A_2 \wedge \ldots \wedge A_n \supset C$ is a substitution instance of a tautology $A'_1 \wedge A'_2 \wedge \ldots \wedge A'_n \supset C'$, such that neither C' nor $\neg(A'_1 \wedge A'_2 \wedge \ldots \wedge A'_n)$ is a tautology. (T. J. Smiley, 'Entailment and Deducibility', *Proc. Aristot. Soc*, 1958, at p. 240, with minor notational changes.)

Here we are to read 'tautology' as truth-functional tautology of the propositional calculus, as determinable by truth-tables. So what's going on here is that Smiley has sharply restricted what counts as establishing something 'by means of logic' for the purposes of establishing entailments (we'll consider a generalization in just a moment). And there's a natural movitation for a story of this shape. As we noted before when discussing P4, we naturally might think that an entailment between A and C should go with some kind of *connection* between the two, presumably a connection of meanings. So the idea is that A entails C when we could establish $A \supset C$ by logical means relying on some interaction between A and C (some connection that can be revealed when seeing them as instances of some necessity $A' \supset C'$, where we don't rely on either the contradictoriness of A' or necessity of C').

On *this* definition of entailment, the use of ordinary propositional reasoning in mathematics is again legitimized – as we are normally using substitution instances of tautological implications which indeed still hold even when the negated antecedent and the consequent are not tautologies. And the principles P1 and P2 of course hold. So does P4: for as Smiley writes

> A and $\neg A$ does not entail just any C, because there is in general no way of deriving $(A \wedge \neg A) \supset C$ from an implication which is itself tautologous but whose antecedent is not self-contradictory. (Smiley, p. 240, again with minor changes.)

Which means that P3 must fail – in other words, *Smiley entailment is not unrestrictedly transitive.*

As Lewy notes, Smiley's definition invites an obvious generalization:

> $A_1, A_2, \ldots A_n$ entails C if and only if the implication $A_1 \wedge A_2 \wedge \ldots \wedge A_n \supset C$ is a substitution instance of a logical necessity $A'_1 \wedge A'_2 \wedge \ldots \wedge A'_n \supset C$, such that neither C' nor $\neg(A'_1 \wedge A'_2 \wedge \ldots \wedge A'_n)$ is a logical necessity.

Again, generalized Smiley entailment is not unrestrictedly transitive. And Lewy's paradoxical argument fails for generalized Smiley entailment: L3 and L4 are both acceptable generalized Smiley entailments. But we can't infer from that L5, which *isn't* a generalized Smiley entailment – there's no way of deriving L5 as an instance of a necessary implication which isn't established by establishing the necessity of its consequent.

In sum, Smiley's story gives us an account of entailment that avoids the Lewis and Lewy paradoxes. And we've noted that there's quite a natural motivation for the story in the idea that there ought to be a meaning connection in genuine entailments (and since meaning-connection isn't transitive, neither is intuitive entailment).

So can we be content with this story? There are two sorts of difficulty, one 'local' to the details of Smiley's non-transitive notion, the other a 'global' problem for any account that rejects transitivity.

First, for local difficulties, note e.g.

1. On Smiley's story, it is false that (a) P entails $Q \vee \neg Q$. But (b) P *does* entail $P \wedge (Q \vee \neg Q)$, for $P \supset P \wedge (Q \vee \neg Q)$ is itself a tautology of the form $A \supset B$ with neither $\neg A$ nor B a tautology. But as Lewy remarks, it seems pretty counterintuitive to give different verdicts on (a) and (b).

2. As Smiley himself remarks,

    > Whereas it is true that $A \equiv B$ entails $A \equiv A$ if this is a mere abbreviation for $(A \supset B) \wedge (B \supset A)$ entails $(A \supset A) \wedge (A \supset A)$, because it then falls under the general principle $((A \supset B) \wedge (B \supset C)) \supset ((A \supset C) \wedge (A \supset C))$, it is not true that $A \equiv B$ entails $A \equiv A$ when '$\equiv$' is taken as a primitive term, because there is then no acceptable general principle available, as may be checked. This example shows incidentally that repetitions of premisses cannot always be ignored, for $A \equiv B$, $A \equiv B$ entails $A \equiv A$

even with '≡' taken as a primitive term, in virtue of the general principle $(A \equiv B \wedge C \equiv B) \supset A \equiv C$. (Smiley, pp. 241–242.)

But again, the two thoughts that the entailments of equivalence statements differ depending on whether equivalence is primitive or defined, and that the repetition of premisses can matter are again both highly counterintuitive.

So it seems that in fitting the notion of entailment to our intuitions so as to block the Lewis and Lewy arguments, Smiley has just shifted the bulge in the carpet to other, rather less immediately obvious, corners of the logical room. We still haven't got – even in its application to reasoning with propositional connectives – an everywhere-smooth fit with intuition (apart from intuitions about transitivity).

As for the global issue, the obvious worry is this: as Smiley himself puts it,

> The whole point of logic as an instrument, and the way in which it brings us new knowledge, lies in the contrast between the transitivity of 'entails' and the non-transitivity of 'obviously entails', and all this is lost if transitivity cannot be relied on. (Smiley, p. 242.)

If we now deny that entailment is transitive, aren't we in danger of confusing the relation of entailment that we care about, that holds even between the ends of long chains of deduction, with some other notion like 'obviously entails'?

And Lewy is even more trenchant in rejecting Smiley's account as a correct analysis of the intuitive notion of entailment:

> I believe that it is *more* counter-intuitive to reject unrestricted transitivity of entailment than to identify entailment with strict implication. For the former course dissociates the notion of "P entails Q" from that of "There is a valid deductive proof of Q from P". Obviously, if there is a valid deductive proof of Q from P, and a valid deductive proof of R from Q, then there is a valid deductive proof of R from P. *This* is undeniable: proof *is* transitive. (Lewy, p. 126)

If that's right, then a non-transitive notion of entailment can't reconstruct our intuitive notion.

But are Lewy and Smiley going too fast here? Perhaps so. For enter Neil Tennant. For what Lewy says is undeniable Tennant argues that we can and should deny: and he argues that all is certainly not lost if we restrict transitivity. The story, then, continues.