Induction and Predicativity

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I am interested in the philosophical prospects of what is called 'predicativism given the natural numbers'. And today, in particular, I want to critically discuss *one* argument that has been offered to suggest that this kind of predicativism can't have a stable philosophical motivation.

Actually you don't really need to know about predicativism to find *some* stand-alone interest in the theme I will be discussing. But still, it's worth putting things into context. So I'm going to start by spending a bit of time introducing you to the background.

Let me start off by reminding you about the very idea of predicative versus impredicative definitions.

1 Impredicative definitions

1.1 Frege on *natural number*

Frege famously defines the property of being a natural number as follows: n is a natural number if and only if n has all the hereditary properties of zero – where a property is hereditary if, whenever something has it, so does its successor.

Frege has, of course already defined the successor relation without explicit reference to natural numbers, so his definition isn't explicitly circular. However, being a natural number is itself one of the hereditary properties of zero. So Frege *is* defining the property of being natural number is terms of a quantification over a totality including that very property.

Recall a Russellian term of art: *a definition is said to be* impredicative *if it defines an entity E by means of a quantification over a domain of entities which includes E itself.* In this sense, Frege's definition of the property *natural number* is plainly impredicative. And the impredicativity here turns out to be essential to the technical work that Frege wants his definition to do in the *Grundgesetze*.

1.2 Impredicativity and vicious circles

Now: Poincaré, and Russell following him, famously thought that impredicative definitions are as bad as more straightforwardly circular definitions. Such definitions, they suppose, offend against a principle banning viciously circular definitions. But are they right? Or are impredicative definitions harmless?

Well, Ramsey (and Gödel after him) famously noted that *some* impredicative definitions *are* surely quite unproblematic. Ramsey's example: picking out someone, by a Russellian definite description, as the tallest man in the room is picking him out by means of a quantification over the people in the room who include that very man, the tallest man. And where's the harm in that?

Surely, there's no harm at all. In this case, the men in the room are there anyway, independently of our picking any one of them out. So what's to stop us identifying one of them by appealing to his special status in the plurality of them? There is nothing logically or ontologically weird going on.

Likewise, it would seem, in other contexts where we take a realist stance, and where we suppose that – in some sense – reality already supplies us with a fixed totality of the entities to quantify over. If the entities in question are (as I put it before) 'there anyway', what harm can there be in picking out one of them by using a description that quantifies over some domain which includes that very thing?

Things are otherwise, however, if we are dealing with some domain with respect to which we take a less realist attitude. For example, there's a line of thought which runs through Poincaré, through the French analysts (especially Borel, Baire, and Lebesgue), and is particularly developed by Weyl in his *Das Kontinuum*: the thought is that mathematics should concern itself only with objects which can be defined. As the constructivist mathematician Errett Bishop later puts it

> A set [for example] is not an entity which has an ideal existence. A set exists only when it has been defined.

On this line of thought, defining a set is – so to speak – defining it into existence. And from this point of view, impredicative definitions involving set quantification will indeed be problematic.

For the definitist thought suggests a hierarchical picture. We define some things; we can then define more things in terms of those; and then define more things in terms of those; keep on going on. But what we can't do is define something into existence by impredicatively invoking a whole domain of things already including the very thing we are trying to define. That indeed would be going round in a vicious circle.

1.3 Realism, anti-realism, and impredicative definitions

So the initial headline thought is this. If you are full-bloodedly realist about some domain, think the entities in it are 'there anyway', then impredicative definitions over that domain can be just fine. If you are some stripe of anti-realist, you should probably see impredicative definitions as illegitimate. Thus, indeed, Frege vs Russell. Frege was a gung-ho realist about properties (strictly, for him, concepts), and hence could countenance impredicative definitions over a domain that reality had already populated for us. Early Russell was pretty much a definitionalist about properties (strictly, for him, propositional functions), and hence for him impredicative definitions for properties were viciously circular.

Though actually, I don't particularly want to stick with the headline thought in quite this form. For there's an evident snag in saying that realism justifies impredicative definitions, and that anti-realism justifies adherence to the vicious circle principle that bans such definitions. The snag is that, on further reflection, the realist idea of things being 'there anyway', and (say) the definitionalist's idea of something's 'only existing when it has been defined' are both pretty opaque. So rather than appeal to such ideas to ground accepting or rejecting predicative definitions over various domains, I'd prefer to put it the other way about. I'd rather say: accepting the legitimacy of impredicative definitions over a domain *constitutes* one kind of realism about that domain, constitutes a realist commitment over and above that taken on by someone who restricts herself to predicative definitions. But pursuing that thought further would have to be the topic for another day.

2 Predicativism given the natural numbers

So we now know what an *impredicative* definition is: to repeat, *a definition is said to be impredicative if it defines an entity E by means of a quantification over a domain of entities which includes E itself.* A *predicative* theory is one that does not deploy any impredicative theories. I'm now going to say something pretty brisk about one particular predicative theory – known as 'predicativism given the natural numbers'. This is, in headline terms, what you get when you add a predicative theory of sets of natural numbers on top of standard first-order Peano Arithmetic. But let me say just a bit more.

2.1 Implementing classical analysis in second-order arithmetic

As we all know, the familiar mathematics of classical analysis can be implemented in the universe of sets. But it is radical overkill to implement familiar mathematics in full-blown iterative hierarchy as described by ZFC. A very small fragment will in fact do the job. But just how small?

Well, all we actually need are the natural numbers and sets of natural numbers. True, we usually implement the integers, positive and negative, as (say) equivalence classes of ordered pairs of natural numbers – so for example, the number -7 is implemented as the set of all pairs of naturals $\langle m, n\rangle$ such that $n - m = 7$. On the usual treatment of ordered pairs, that implementation makes the integers sets of sets of sets of naturals. But *this* ascent three steps up the set-theoretic hierarchy is unnecessary if we have arithmetic to hand. For we can just use a pairing function to code up pairs of numbers by single numbers, which enables us to use naturals as codes for integers. We can use the same trick again to use naturals as codes for pairs of integer-codes, to give us a coding for the rational numbers. So you only really need to advance from the naturals to talking about *sets* of naturals when we come to implement the *real numbers* as infinite Cauchy sequences of rationals.

Now, even if you can code real numbers as just sets of numbers, when we come to talk about *functions* over the reals won't we need to talk about sets of argument-value pairs of reals? Won't we here need to go up the set theoretic hierarchy another step and allow sets containing sets of numbers? Well not so, if we only care about the familiar sorts of piecewise continuous functions we encounter in ordinary applicable classical mathematics. For the behaviour of a continuous function is dictated by its values at rational points, so we can implement such functions as sets of pairs of rationals. But pairs of rationals can be implemented by natural number codes, so continuous functions over the reals also again can be implemented at the level of sets of naturals. (Likewise for piecewise continuous functions with definable points of discontinuity).

In summary, the classical analysis of ordinary functions can be reconstructed in second-order arithmetic treated as the theory of natural numbers and sets of numbers. Which, of course, is why second-order arithmetic is sometimes just called 'analysis'.

2.2 Making do with even less

Ok, so far so good – and probably, so very familiar. But now we are in a minimalist frame of mind, asking just how little set theory we need for ordinary applicable mathematics, we can press on to ask: how much second-order arithmetic do we really need? Which sets of numbers do we *have* to countenance (if we want to reconstruct familiar mathematics)? And which sets of numbers *should* we countenance?

In one kind of standard presentation, a second-order arithmetic contains Robinson Arithmetic, i.e. the usual axioms for successor, addition and multiplication. Then there's an induction axiom, which tells us that if a set X contains 0, and contains the successor of n whenever it contains n, then it contains all numbers:

$$\textsf{Ind} \qquad \forall X((0 \in X \wedge \forall x(x \in X \rightarrow Sx \in X)) \rightarrow \forall x\, x \in X).$$

And then, crucially, there's a Comprehension Schema whose instances tell us what sets there are. The idea is that for any suitable open sentence $\varphi(\mathsf{x})$ with a free numerical variable, there's a set of the numbers satisfying that condition:

$$\textsf{Comp} \qquad \exists X \forall x(x \in X \leftrightarrow \varphi(x)).$$

So our question about which sets of numbers to countenance comes to this: which open sentences $\varphi(\mathsf{x})$ are we going to allow to instantiate the Comprehension Schema?

The generous line would be to allow that *any* open sentence of second-order arithmetic specifies a set. On this generous approach, then, we'll in particular allow instances of $\varphi(\mathsf{x})$ in Comp which embed universal set-quantifiers. This is to allow impredicative definitions of numerical sets, defining a set of numbers by means of a quantification over all sets of numbers including that one.

Well that's fine if we are realist about what sets of numbers there are. But should we be? Don't we swept along too readily by set-theorists' propaganda! There's a long tradition of hesitating over the distinctively set-theoretic idea that there is a plenum of all possible arbitrary infinite sets of numbers – sets which are supposedly perfectly determinate but are in the general case beyond any possibility of our specifying their members. For we can perhaps make sense of the membership of a set of numbers being determined by possession of some suitable characterizing property which gives a criterion for picking out the numbers. We can equally make sense of the membership being merely stipulated as an arbitrary list. However, the first idea gives us infinite sets but not arbitrary ones; and the second idea may give us arbitrary sets (whose members share nothing but the gerrymandered property of being on the membership list) but not infinite ones – unless we are prepared to conceive of a completed infinite series of arbitrary choices (and as Hermann Weyl puts it '[t]he notion that an infinite set is a "gathering" brought together by infinitely many individual arbitrary acts of selection . . . is nonsensical.'). So neither initial way of thinking of sets uncontentiously makes sense of the idea of arbitrary infinite sets of numbers.

Now, maybe there's a good riposte to this: so I'm not here dismissing the distinctively set-theoretic idea of arbitrary infinite sets of numbers out of hand. But the considerations just suggested indicate that grasping this idea, if we do, requires a conceptual move beyond the idea of the sort of definable collections of numbers that we might meet in ordinary mathematics. So we might well very wonder if we *need* to make such a conceptual move if we are in the business of reconstructing ordinary mathematics in second-order arithmetic.

So what happens if we try to do without the distinctively set-theoretic idea and stick to definable sets? The most obvious way to go is to first introduce the sets definable in purely arithmetic terms (without quantification over sets), and then allow in the sets definable in terms of those sets, and so on. And the natural way to do this is to simply require substitutions for $\varphi(\mathsf{x})$ in Comp not to have any embedded set-quantifiers. In other worlds we allow only predicative instances of the comprehension principle. If we do so, then we get the system of second-order arithmetic defined as follows:

> $\mathsf{ACA_0}$ is the second-order arithmetic whose axioms are those of Robinson Arithmetic Q, plus Ind, plus those instances of Comp where $\varphi(\mathsf{x})$ lacks bound set-variables.

Note, however, that banning *bound* set-variables in $\varphi(\mathsf{x})$ still allows *free* set-variables to occur as parameters. But that's what we want. If we think that a set X is respectable if it is arithmetically definable (without set quantification), then a set which is arithmetically definable in terms of being a member of X should be respectable too. In effect, $\mathsf{ACA_0}$'s restricted comprehension principle builds up the kosher sets in a natural way, by defining them 'from below'.

So $\mathsf{ACA_0}$ is a second-order arithmetic which doesn't commit us to countenancing arbitrary sets of numbers. And it is predicatively respectable: $\mathsf{ACA_0}$'s restricted comprehension principle is predicative, because it *doesn't* try to define any numerical set by a quantification over all numerical sets.

Two technical comments. Evidently, $\mathsf{ACA_0}$ includes standard first-order Peano Arithmetic (for all instances of first-order induction will come from a parameter-free instance of Comprehension plus the Induction Axiom). So, $\mathsf{ACA_0}$ is what you get when you take the standard first-order Peano Arithmetic of the natural numbers, and add a predicative theory of sets of numbers. It thus aims to regiment the commitments of a foundational position long since recommended by Hermann Weyl in his 1918 book *Das Kontinuum*, a position which was in turn influenced by Poincaré. This is the position dubbed 'predicativism given the natural numbers'.

And let's note that adding the predicative theory of sets doesn't – as it were – create any backwash disturbing the first-order theory: $\mathsf{ACA_0}$ is conservative over first-order Peano Arithmetic for first-order arithmetic sentences. That's really as you'd expect. For what $\mathsf{ACA_0}$ adds to first-order arithmetic is a way of talking about just those extensions of first-order arithmetic predicates to which we are already committed. And making explicit our commitments to logical extensions shouldn't give us new substantive results about numbers. (Enthusiasts will note the parallel with adding a predicative theory of classes to ZF to get von-Neumann/Bernays/Gödel set theory-with-classes, which is conservative over ZF for propositions purely about sets.)

2.3 What $\mathsf{ACA_0}$ can do for you

Now for the key question: how much familiar mathematics can we reconstruct in $\mathsf{ACA_0}$ (which is indeed a pretty weak theory)? I speculated that we ought to be able to get away without invoking the distinctively set-theoretic idea of arbitrary infinite sets of numbers. And if we recoil from that idea, the slide to a weak predicativist arithmetic seems pretty attractive. But have we overshot?

No. As Weyl foresaw and later work has confirmed *you can in fact reconstruct all familiar classical and complex analysis – and arguably* all *applicable mathematics – in* $\mathsf{ACA_0}$. Obviously we can't begin to show that here. Anyone who is interested can look at the constructions in that wonderful survey, Steven Simpson's great *Subsystems of Second Order Arithmetic* (of the which the first chapter is available online).

Suppose, then, just suppose you are gripped by the Quine/Putnam indispensability argument and hold what you should be realist about is that mathematics that is indispensable for science. Then *all* you are ultimately committed to – it seems – are the commitments of $\mathsf{ACA_0}$, that is to say the natural numbers and the extensions of arithmetic predicates (without set quantifiers). You can define the rest out of those minimal materials. Which is a rather stunning result.

3 Induction

But is everything quite as rosy as I've been painting it? Does ACA_0 in fact have a stable conceptual motivation? Is 'predicativism given the natural numbers' a coherent position? Feferman describes what he calls the 'the predicative conception' as holding that

> Only the natural numbers [are] regarded as 'given' to us. ...In contrast, sets are created by man to act as convenient abstractions ...from particular constructions or definitions.

Can we legitimately take such different attitudes to numbers and sets of numbers?

3.1 The challenge

Charles Parsons, in a paper entitled 'The Impredicativity of Induction', and again in his recent book *Mathematical Thought and Its Objects*, has argued that – as he puts it – the 'notion of natural number is *already* impredicative'. And this, he suggests, 'seriously weakens the case for the claim, deriving from Poincaré, that impredicativity is a sign of a vicious circle and altogether to be avoided'.

Now, in fact the enthusiast for ACA_0 doesn't have to say that impredicativity is always and altogether to be avoided. Her view might be that each time we allow a new class of impredicative definitions, we take on board new commitments, and the new commitments have to be separately warranted. So in a general spirit of not multiplying commitments beyond necessity, we should aim to avoid using impredicative constructions when we don't need them. And then what's interesting about ACA_0, the story goes, is that it reveals that to do 'ordinary' mathematics, we don't need to get impredicative about *sets*. That would *still* be an important and interesting result even if a full story about the notion of natural number which ACA_0 takes as given requires us to get impredicative about *numbers*.

Still, if Parsons is right and even grasping the notion of natural number requires grasping an impredicative definition of them, that at least might be thought to show that there can be nothing too alarming about impredicativity. And then I suppose the worry might be that the predicativist enthusiast for ACA_0 who tries to tell us scare stories about impredicative set theories might be over-selling his case. Well, is that right? What I want to do in the rest of this paper is to try to get clearer about some issues here by talking round and about Parsons's arguments.

3.2 Grasping the numbers by grasping the open-ended induction rule

Of course, if you think we are to be introduced to the natural numbers by the Frege-style definition I mentioned at the outset, then indeed our definition of the natural numbers will be impredicative. But, fascinating though the Fregean logicist project is, it would be hard to argue that our grasp of the numbers is in fact via the Fregean impredicative definition!

So what *does* it take to grasp the notion of natural number? Well, we start by learning some initial numbers and how to count with them. Then the penny drops that we can keep on going, adding one, and never running out numbers.

More carefully, then, we need to grasp that there's a first number (which we'll take to be zero rather than one, but nothing hangs on this). And we need to grasp that each number other than the first has another number as its successor, and different numbers have different successors. And we need to grasp too that zero and its successors are the only numbers.

This closure condition of course hooks up with the thought underlying the Fregean definition, for it ensures that this much must be true: if zero has a given property, and if the property is hereditary (is passed down from one number to the next), then any number at all has the property. That's because, by closure, zero and its successors are the only numbers that there are. So, grasping the natural numbers does go with acknowledging induction for any given property. But note, crucially, that *this* claim does *not* reinstate the content of Fregean definition. It is one thing to be committed to accept induction for *any* particular property of numbers which we get presented with; it is something further to suppose that there is clear sense to the idea of quantifying over *all* properties of numbers.

This induction inference, then, takes us from the premiss that zero satisfies some condition φ, and the premiss that if an arbitrary number a satisfies φ then so does Sa, to the conclusion that any given number t satisfies φ. In symbols, we can put it schematically as a rule like this:

$$\text{Rule} \qquad \frac{\varphi(0) \qquad Na \wedge \varphi(a) \to \varphi(Sa) \qquad Nt}{\varphi(t)}$$

where t is a term, and N expresses the property of being a number. And this rule we are taking to be open-ended in the sense that there's no limit on what can be put for φ, so long as it expresses some cogent condition. For how *can* there be a coherently motivated limit? If zero satisfies some condition, and that condition in inherited by successors, then zero and each of its successors satisfies the condition. So if t is a natural number – i.e. one among zero and its successors – it must satisfy the condition in question.

3.3 Schematic versus Second Order induction

It's worth pausing for two quick comments on the open-ended character of the induction rule. First, let's recall Georg Kreisel's often-quoted remark:

> A moment's reflection shows that the evidence [for] the first-order [induction] schema derives from the second-order [axiom].

This is right in one way and wrong in another. It is right that something more general does indeed lie behind our accepting (in particular) all the instances of the induction-schema in first-order Peano Arithmetic. We accept those instances because we accept the schematic induction rule applies, open-endedly, to any kosher condition on numbers, and we take it that the predicates of first-order arithmetic express such conditions. But Kreisel would be wrong to claim that a moment's reflection shows that we accept the schematic rule because we accept the second-order induction axiom. The second-order axiom supposedly quantifies over all properties sets of numbers; and it is, for a start, as we've already noted, far from clear we understand that quantification. So if anything, a moment's reflection suggests that it is the other way about: we accept the second-order induction axiom, if we do, because we accept the schematic open-ended induction rule, and then have the further (and contentious) thought that there is a determinate totality of instances which we can cogently logically sum together using a universal second-order quantification.

Second, note that the open-ended character of induction sustains our ordinary mathematical conviction that we've got hold of a unique number structure (up to isomorphism, of course). For suppose you try to challenge that conviction by offering me a description of some non-standard model of induction-free arithmetic. That will have to describe some junk that comes after zero and all the numbers you get by iterating the successor function. So, in describing the model, there will have to be a predicate (for short, 'not-junk') that will apply to zero and its successors, but not the end-added junk. Charitably assuming you are talking sense in presenting me with that model, I now have a new predicate, 'not-junk', to work with. So I'm committed, by open-ended induction, to the claim that any natural number isn't junk. So, by my lights, your non-standard model includes things that aren't numbers, the junk, and I beat off your challenge.

Of course, that kind of reasoning isn't going to satisfy certain kinds of philosophical sceptic (broadly, the Skolemite sceptic). But scepticism isn't our topic.

4 Induction and impredicativity?

So far, then, we've that grasp of the open-ended induction rule is involved in our grasp of the natural numbers, and we've just seen that it ensures that we are indeed – by ordinary mathematical standards – talking about a determinate structure. So let's say, in sum, that someone who has the idea of zero and successor, grasps the basic laws governing successor, accepts open-ended induction and – of course – knows how to use numbers to count has a basic structural notion of natural number. But is there more to the idea of natural numbers?

Here's Parsons:

> Because the number concept is characterized as one for which induction holds for any well-defined predicate or property, there is impredicativity if those involving quantification over numbers are included, as they evidently are.

And Parsons approvingly quotes Dummett, in his 'The Philosophical Significance of Gödel's Theorem', who writes:

> [T]he notion of 'natural number' . . . is impredicative. The totality of natural numbers is characterised as one for which induction is valid with respect to any well-defined property, . . . the impredicativity remains, since the definitions of the properties may contain quantifiers whose variables range over the totality characterised.

So the claim is this. First, accepting open-ended induction essentially involves preparedness, inter alia, to apply the induction rule to predicates that involve quantification over numbers. Second, it is impredicative to define the numbers in terms that involve the applicability of the induction rule to predicates including those which involve quantification over numbers.

I want to argue that this claim limps at both steps. First, why should it be thought that characterizing the numbers as a structure for which the open-ended induction rule applies to any well-defined predicate involves taking any stand at all on *which* predicates *are* well-defined? To be sure, the basic structural notion of number doesn't fix which properties of numbers are kosher: but why should it?

A finitist resists the idea that we can think of the natural numbers as a completed totality: so she will resist going further than schematic generalizations about numbers, and she doesn't think that predicates involving existential quantifications over the numbers make sense. A constructivist (though still resisting the idea of a completed infinity) allows some quantified predicates, when they express decidable properties of numbers.

And by contrast, a realist enthusiast for full first-order Peano Arithmetic will cheerfully allow arbitrarily complex first-order arithmetic predicates as making sense. But should we say that the finitist, constructivist and the realist have different notions of natural number? Shouldn't we rather say that they've the same notion – they've grasped the structure of the sequence of natural numbers – but they have different views about what we can sensibly say about it. The realist optimistically thinks that we can coherently deploy predicates such that it is quite beyond our capacities to determine whether they apply to a given number: he pretends we can survey the numbers all-at-once, as a completed infinity. The constructivist requires in-principle-decidability (even if that requires unbounded searches). The finitist is more restrictive again. But, we might very well think, whether optimists or pessimists about what we can sensibly say and know about them, all parties have latched on to the natural numbers, deploying the same thin structural concept that isn't freighted with contentious thicker implications about which predicates make sense.

However, let's wave this first point, as further discussion of it will probably be a sterile exercise. For pre-theoretically, talk about concepts probably just isn't determinate enough to fix whether the finitist, constructivist, and realist do or don't count as having the same concept of natural number. So let's now grant for the sake of argument that, at least once we move beyond finitism, our concept of natural number *will* be characterized, inter alia, by allowing induction involving predicates which quantify over all numbers. Will this make our characterization of the concept impredicative?

Well, recall once more that on the Russellian understanding of the idea, a definition of the set of natural numbers will count as 'impredicative' if it quantifies over some totality of sets including the set of natural numbers. Modulated into property talk, we'd have: a definition of the property of being a natural number will count as impredicative if it quantifies over some totality of properties including the property of being a natural number. Such was the case with the Fregean definition, we noted.

But now note that, even on the Parsons/Dummett story, there's no impredicativity in the *Russellian* sense in the characterization of the natural numbers via open-ended induction. Russellian impredicativity arises if a definition of the property of being a natural number quantifies over some totality of properties including that very property. The Parsons/Dummett idea is that a definition of the property of being a natural numbers (via the applicability of open-ended induction) indirectly quantifies over the natural numbers. So the difference is between defining a property by quantifying over a domain including *that very property*, and defining a property in a way that involves quantifying over a domain what includes *what falls under that property*. Ok, there's a rough analogy: but there's also a crucial distinction.

To point up the distinction, it might help for a moment to imagine that you are some kind of Bishop-style constructivist. Suppose you think that mathematical items are in some sense (however tenuous) 'constructed by us' and not determined to exist prior to our mathematical activity. Then, as we argued before, it is surely illegitimate to give a recipe for constructing a particular item which assumes we have *already* constructed a totality of items including the very one that we are now attempting to define. So any definition which is to play the role of a recipe-for-construction had better not be Russell-impredicative.

On the other hand, our constructivist need not balk at the idea that the numbers are what we can construct from zero by applying and re-applying the successor function (after all we have a canonical specification for each such number by a standard numeral). And, she adds, nothing else is a natural number. Accepting that closure clause comes to accepting the open-ended induction rule, which for the constructivist is applied to any constructively sensible predicate. And among the constructively sensible predicates, the story continues, some will involve quantification over the objects – zero and its successors – which we've *already* constructed. For example, there will be existential quantifications as we implement open-ended searches for a satisfier for some regular primitive recursive condition. So our constructivist arrives at a constructivist package conception of the natural numbers and their kosher properties. But there is, it seems, nothing here to offend constructive scruples in this: the basic objects with their successor properties are built first, and then further properties are defined over them. There's no smidgin of circularity here, despite the constructivist ending up with what Parsons and Dummett call, in *their* sense, an impredicative characterization of the numbers.

In short, a constructivist must balk at a Russell-impredicative definition of numbers, but need find no offence at all in a Parsons-impredicative characterization of the numbers. It is unhelpful, therefore, to call both notions varieties of 'impredicativity' (especially given the historical association of predicativism with broadly constructivist scruples).

4.1 So what have Dummett and Parsons got hold of?

So something has gone wrong. But of course, Dummett and Parsons are good philosophers. So I take it that they've latched on to *something*, but just – by my lights – rather unhappily misdescribed it. The question is what?

Well, to repeat, Dummett and Parsons focus on predicates of numbers which embed quantifiers. And I think that what really worries is them are the cases which go beyond anything constructively acceptable – as in standard first-order Peano Arithmetic, where we allow into the induction schema predicates that embed arbitrarily complex quantifications over the numbers. Such a predicate will in general express a condition which is satisfied by a given number n if something obtains of *all* numbers. Grasping such predicates involves thinking of the numbers all-at-once (as a completed totality), and directly determining whether n satisfies the predicate would in general involve a non-constructive supertask that only God could perform. The real concern here is not that deploying such a predicate in itself directly offends against any vicious circle principle (it doesn't), but that it involves a conception whose legitimacy goes beyond what is given to us by a basic level grasp of the numbers combined with anodyne constructions over the numbers.

And, without getting embroiled in textual exegesis, it is *this* point, I suggest, that the Dummett/Parsons notion of impredicativity seems really to be picking up on. Allowing the comprehensibility of unrestrictedly complex instances of the induction schema involves making sense of something that isn't given at the basic level of understanding the natural number structure. It involves, if you like, moving from thinking of the numbers as a potential infinity (as a finitist and constructivist might) to thinking of them as a completed actual infinitiy. And then it is very unclear what non-circular reason could be given to someone who doubts the coherence of making the move. Which I think is true, which but has nothing to do with Russell-impredicativity.

5 Back to ACA_0

Back to ACA_0, then. The situation is this. Someone who moves on from finitist or constructivist arithmetic to accepting full first-order Peano Arithmetic with its induction over arbitrarily complex predicates is making a significant conceptual move. There seems to be no non-circular way of arguing someone into making the move. Similarly, someone who moves on from Peano Arithmetic plus a predicative theory of sets of numbers like ACA_0 to accepting a richer, impredicative, second-order theory of arithmetic is making another singificant conceptual move. There seems to be no non-circular way of arguing someone into making this move either.

In talking of the first move – accepting induction over arbitrarily complex predicates – as involving going impredicative, Parsons makes it sound as if the first conceptual move and the second conceptual move are on a par. And then, since most of us readily make the first move, we might wonder why we should resist the second. But in fact the moves are quite different in character. The first move involves a willingness to be suppose we can – as it were – think about the domain of numbers all-at-once in a non-constructive way, and define properties using quantification over this fixed domain of objects we already know about (the numbers), properties such that only God could determine when the properties apply. The second move, going to impredicative second-order arithmetic, involves a willingness to define properties using quantification over a new domain we don't yet know about, a domain of properties which is now to include properties that aren't arithmetically definable. It's the difference between going fully realist talking about a subject matter we've already pinned down (the numbers), and taking on a new putative subject matter (moving beyond the arithmetically definable sets).

In sum, then, the moves to accepting full first-order Peano Arithmetic and to going beyond ACA_0 are different in character. We can make the first move without in any way diminishing the case for not making the second move. Predicativism given the natural numbers is a conceptually stable position.