# Cuts, consistency and axiomatized theories

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In the Wednesday Logic Reading Group, where we are working through Sara Negri and Jan von Plato's *Structural Proof Theory* – henceforth 'NvP' – I today introduced Chapter 6, 'Structural Proof Analysis of Axiomatic Theories'. In their commendable efforts to be brief, the authors are sometimes a bit brisk about motivation. So I thought it was worth trying to stand back a bit from the details of this action-packed chapter as far as I understood it in the few hours I had to prepare, and to try to give an overall sense of the project. These are the notes I wrote for myself. As often with such middle-of-term efforts dashed off in a couple of hours, I both would have liked to do better and do more justice to what we are reading, but I also just don't have time to do more now than make a few corrections to the first version. So the usual warning applies: *caveat lector*.

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## 1 Background

To keep these introductory remarks under control, let me focus on the *classical* multiple-conclusion sequent system for first-order logic without identity, G3c introduced by NvP at pp. 49 and 67 (if only because it's the classical case that most interests me).

G3c treats the antecedents and succedents of sequents as multisets, so immediately avoids the need for some of Gentzen's structural rules. And it avoids the need for weakening and contraction, inter alia, by adopting the generalized form $P, \Gamma \Rightarrow \Delta, P$ (with $P$ atomic) for logical axioms. Rules for the connectives come in matching left/right pairs. Thus the rules for conjunction are

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

G3c is a cut-free logic. We have seen, however, that the cut-rule that we know and love – the rule which encodes the transitivity of classical entailment – can be conservatively added to G3c (and doing so will in fact speed up proofs hyperexponentially, though NvP doesn't develop detailed estimates). Still, we might wonder whether there is something unnatural about *not* building such a fundamental feature of classical entailment into the basis of a theory of classical logic.

Let me see if I can give some shape to this inchoate worry which some might have about this by considering a different logic – different I think from any that we've met in NvP so far.

We'll again work in a multiple conclusion framework. Suppose, then, that we keep the same generalized logical axiom rule, but start by firstly replacing the rules for the conjunction with the following familiar looking package of additional axioms (where $A$, $B$ can be any wffs, of course):

$$A, B \Rightarrow A \wedge B, \quad A \wedge B \Rightarrow A, \quad A \wedge B \Rightarrow B.$$

Instances of these axioms can, like axioms of the basic axiom, appear at the top of any proof tree.

This new system we are building is to have cut as a basic structural rule (and contraction too). And with cut to work with, we can of course recover the previous rules for the conjunction as give by G3c:

$$\dfrac{\dfrac{}{A \wedge B \Rightarrow B} \qquad \dfrac{\dfrac{}{A \wedge B \Rightarrow A} \qquad A, B, \Gamma \Rightarrow \Delta}{A \wedge B, B, \Gamma \Rightarrow \Delta}\,Cut}{A \wedge B, \Gamma \Rightarrow \Delta}\,Cut$$

$$\dfrac{\Gamma \Rightarrow \Delta, B \qquad \dfrac{\Gamma \Rightarrow \Delta, A \qquad \dfrac{}{A, B \Rightarrow A \wedge B}}{\Gamma, B \Rightarrow \Delta, A \wedge B}\,Cut}{\Gamma \Rightarrow \Delta, A \wedge B}\,Cut$$

Now let's do (almost) the same for the other connectives. Instead of G3c's left and right disjunction rules, we'll use the obvious two $\vee$-introduction sequents as axioms, and then have a rule of proof to serve as $\vee$-elimination. Similarly, we will use modus ponens encoded as a sequent axiom, together with a conditional proof rule. We preserve the same ex-falso rule $L\bot$. Call this new system S. S will be evidently be equivalent as system of pure logic to G3c. That is to say, a sequent $\Gamma \Rightarrow \Delta$ will be provable in one system if and only it is provable in the other.

But now we'll note that adding a certain new pair of rules to G3c and to S has very different upshots. This will in fact nicely illustrate a claim made much earlier in NvP, namely that adding the same rule to differently formulated but equivalent systems can give augmented systems that behave quite differently.

So let's consider what happens when we add to G3c the following left and right rules:

$$\dfrac{\Gamma, B \Rightarrow \Delta}{\Gamma, A * B \Rightarrow \Delta} \qquad \dfrac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A * B, \Delta}$$

The resulting augmented calculus is still consistent, i.e. we still can't prove the sequent $\Rightarrow \bot$, for the same reason as G3c is consistent – for all axioms have wffs on the left, and no rules diminish the number of wffs on the left.

But our new rules are of course rules for the connective *tonk* in sequent calculus form. And adding those rules to a system which includes cut will, as you'd expect, quickly lead to trouble. Let $A$ be such that, already in S, we have $\Rightarrow A$. Then we can continue:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow A}}{\Rightarrow A * \bot} \qquad \dfrac{\dfrac{}{\bot \Rightarrow \bot}\,L\bot}{A * \bot \Rightarrow \bot}}{\Rightarrow \bot}\,Cut$$

So, the tonk rules added to S produce inconsistency, but not so when added to G3c.

Now, someone might say: the different behaviours of our two systems G3c and S arguably gives us some reason to suppose that S is the formulation more true to the shared classical motivation for the systems. For surely, it is agreed on all sides that adding tonk ought to be a disaster – intuitively, it is a run-about inference ticket allowing us to get from any $A$ to any $B$ in two steps. So if adding the sequent calculus version of the tonk rules to the sequent calculus G3c *doesn't* lead to disaster, then surely there is something central to classical logic that *isn't* being captured in this version of the calculus?

So what's going on? Well we can perhaps think of G3c as what you get if you take a multi-conclusion logic like S which has the natural cut rule and the most basic rules for the connectives, and then – so-to-speak – distribute the effects of cut into more complex rules for the various connectives. No surprise then that you can recover cut – *for arguments involving those connectives* – from G3c with its 'cut-distributed' rules. But there is no guarantee, of course, that when we add a new operator we will be able to recover cut for inferences involving the new operator. And that's what is illustrated by the tonk connective: add that, and we don't get cut for inferences involving that connective. Some might say: doesn't this just go to show that G3c misrepresents the character of classical entailment? By distributing the effect of cut into particular connective rules, a *global* property of classical entailment (its unrestricted transitivity applied to *any* propositions) is lost. But be that as it may. We have at least illustrated the point that equivalent logics may behave quite differently under augmentation by the same new rules.

## 2 Theories: the key question

Let's now turn from pure logic to logic-as-applied-to-a-regimented-theory. What morals can we draw from the investigations in previous chapters of cut-elimination and consistency proofs for pure logics which can be applied to regimented theories?

Well, suppose that $\Theta$ is a set of axioms. How should we regiment, in sequent style, the theory with those axioms and a classical logic? Here are some options:

1. Just say that $C$ is a $\Theta$-theorem if there is a proof in G3c (or equivalently S) of a sequent $\Gamma \Rightarrow C$, where $\Gamma \subseteq \Theta$.

   That's a natural definition, but we get no interesting payoffs from our preceeding discussions, at least not in the general case. Since G3c is cut-free we know the logic is consistent. But of course, nothing follows just from the fact that we are using a consistent logic about $\Theta$ being consistent.

2. Allow as new axioms every sequent $\Rightarrow A$, where $A \in \Theta$. Say that $C$ is a $\Theta$-theorem if there is a proof from these axioms of $\Rightarrow C$.

   But now it isn't enough to use G3c as that cut-free system wouldn't even allow us to get from axioms $\Rightarrow P$ and $\Rightarrow P \to Q$ to the conclusion $\Rightarrow Q$. We would need to use G3c + cut, or equivalently S.

   Since cut is now uneliminable, we of course can't use a cut-elimination proof of consistency. So again, this approach doesn't immediately promise to help us with getting interesting consistency proofs!

3. Somehow transmute the propositions in $\Theta$ into rules of inference $R$, and add these to G3c. Say that $C$ is a $\Theta$-theorem if there is a proof in G3c$'$ + $R$ of $\Rightarrow C$.

   The trouble is that this time a consistency proof for the new system of rules may be far *too* easy (we get as it were a false positive). For we now know from the case of tonk that we can add rules that do indeed still give us a syntactically consistent system, but only because of the essential absence of the cut rule, and hence a semantically artificial blocking of formal inconsistency.

So far so bad. But there is a chink of light. Perhaps in *some* cases we can follow the lead of the third route and recast a theory's axioms as rules $R$, and do this in such a way that (i) because G3c + $R$ is cut-free, and because of the form of the particular rules $R$, we indeed have an easy proof of G3c + $R$'s consistency, but also (ii) we can show that this syntactic consistency doesn't come at the expense of breaking the transitivity of classical entailment, i.e. we can prove that adding cut would be a conservative extension of this system.

Equivalently and possibly more naturally: perhaps in *some* cases we can recast a theory's axioms as rules $R$ such that (i) S + $R$ is such that we can prove the cut-elimination theorem for this system, and then (ii) we can use cut-elimination to quickly conclude that S + $R$ is consistent.

So what we want to get a handle on is the following question: is there a general story that we can tell about the shape of some rules such that (a) enough theories of some interest can be regimented into the form of such rules added to the rules for a classical sequent logic, and (b) we can prove a general cut-elimination theorem for a theory with non-logical rules of such a shape?

(If we can do that, and if there is a general sequents-as-axioms equivalent to rules of the required shape, then of course we could equally well do things in terms of Model 2 above with axioms restricted to sequents of the right shape. But, in the context, the rule version seems the more natural one to explore first.)

## 3 'Good' rules that allow cut-elimination

Let's say that a rule is 'good' (my term) if it has the following form:

$$\frac{Q_1, \Gamma \Rightarrow \Delta, \quad \ldots \quad Q_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}$$

where $n$ can be zero, but – just for the moment – we'll explicitly assume that $m > 0$. Here, the $P_i$ and $Q_j$ are atoms; but they *can* contain parameters. So although the top and bottom sequents – contexts $\Gamma$ and $\Delta$ apart – are quantifier-free, the rules can have the import of universally quantified axioms.

Note too that the form – single atoms $Q_j$ on the left of top sequents – looks restrictive, it isn't really. To work in terms of an example, note that the single rule with multiple $Q$'s on the right

$$\frac{Q_1, Q_2, \Gamma \Rightarrow \Delta, \quad R, \Gamma \Rightarrow \Delta}{P, \Gamma \Rightarrow \Delta}$$

is in fact elementarily equivalent to a pair of rules

$$\frac{Q_1, \Gamma \Rightarrow \Delta, \quad R, \Gamma \Rightarrow \Delta}{P, \Gamma \Rightarrow \Delta} \qquad \frac{Q_2, \Gamma \Rightarrow \Delta, \quad R, \Gamma \Rightarrow \Delta}{P, \Gamma \Rightarrow \Delta}$$

The two new rules are obtained from the original one by weakening the left premiss. In the other direction, consider the proof

$$\dfrac{\dfrac{\dfrac{Q_1, Q_2, \Gamma \Rightarrow \Delta \qquad \dfrac{R, \Gamma \Rightarrow \Delta}{R, Q_2, \Gamma \Rightarrow \Delta}}{P, Q_2, \Gamma \Rightarrow \Delta}}{Q_2, P, \Gamma \Rightarrow \Delta} \qquad \dfrac{R, \Gamma \Rightarrow \Delta}{R, P, \Gamma \Rightarrow \Delta}}{\dfrac{P, P, \Gamma \Rightarrow \Delta}{P, \Gamma \Rightarrow \Delta}}$$

Evidently if we add only good rules to NvP's classical sequent calculus $G3c$, we still cannot prove $\Rightarrow \bot$, for we don't ever have empty antecedents from logical axoims. Moreover, we can still prove, along much the same lines as for the pure logical system, i.e. by permuting cuts upwards, that cut can be conservatively added to $G3c$ plus good rules. Equivalently, good rules added to S give us a system for which cut-elimination can be proved, and hence is consistent. And note that the arguments about the cut rule – applied as they are to finite proof arrays – are nicely finitistic, and thoroughly constructive.

So far, so good. But it turns out that we can be a tad more liberal about what counts as a 'good' rule, and allow cases where $m$ – the number of $P_j$ on the left of the end-sequent of a rule – is zero. We just need to ensure that $\Rightarrow \bot$ still can't be arrived at (the proof of cut-elimination will still go through). But a proof of that sequent would have to end

$$\frac{Q_1 \Rightarrow \bot, \quad \ldots \quad Q_n \Rightarrow \bot}{\Rightarrow \bot}$$

And working upwards, we find that the non-logical axioms which generate such sequents $Q \Rightarrow \bot$ will have to be of the form $P_1, P_2, \ldots, P_n \Rightarrow \bot$. Ban such axioms as bad and consistency will be ensured.

## 4 Theories with good rules

NvP's result, as I've outlined it, is really very nice, as quite a few theories (theories that can be regimented in a quantifier-free form) can be regimented using good rules, and such theories are therefore provably consistent via a constructive proof-theoretic argument. Here are some examples:

1. The logic of identity. Use the evidently good rules

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \frac{P(b), a = b, P(a), \Gamma \Rightarrow \Delta}{a = b, P(a), \Gamma \Rightarrow \Delta}$$

   (To motivate these, reflect that the left-elimination of $a = a$ is like the right-introduction of $a = a$; and the identity $a = b$ makes one of $P(a)$ and $P(b)$ redundant.) This theory will be consistent. Work remains to be done, of course, to show that full Leibniz's Law holds for non-atomic contexts. But a proof by induction on the complexity of formulae does the trick.

2. The theory of lattices. First define a partial ordering $\preceq$, using the rules

$$\frac{a \preceq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \frac{a \preceq b, b \preceq c, a \preceq c, \Gamma \Rightarrow \Delta}{a \preceq b, b \preceq c, \Gamma \Rightarrow \Delta}$$

   And now add similar rules for meet, join, etc.

3. More excitingly, the laws of affine geometry can be given as good rules, and a consistency proof follows, as does a proof of the independence of the parallel postulate. But for details of *this*, I'll have to refer you to NvP.