SANTA'S SINGLETON

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'After going round in circles with the metaphysics of concepts, it's time for a change. So lately, I've been wondering a bit about the metaphysics of functions instead.'

Really? I didn't know you were interested in the philosophy of biology.

'You misunderstand! Take expressions like 'father of', 'the mass in kilos of', 'successor of', and so on. They express functions in the logician's sense. I want to know what kind of thing they are. Not that we should really say 'thing', perhaps, but you know what I mean!'

Oh, I see. So where have you got to in thinking about them?

'Well, just at the moment, I'm wondering about what happens when you apply a function to Santa.[1]'

Santa? The jolly cove with a big beard, booming laugh, red coat and reindeer, comes down chimneys at Christmas?'

'Yes, that's the one.'

But he doesn't really exist!

'Of course not! I may have some profligate existential beliefs – all kinds of whatnots in Plato's heaven are fine by me (at least on weekdays, though I confess some Sunday doubts). But I'm entirely sound on Santa. There is, indeed, no such person (not even on Sundays).'

But if Santa doesn't exist, he isn't there to really have properties or to be fed as argument to a function. So the question doesn't arise: you can't apply a function to him.

'So, generalizing: for functions, no input means no output?'

That's it.

'You allow that there are functions which can take something as input but deliver nothing as output (like the reciprocal function which isn't defined for zero)?'

Indeed so: mathematicians talk about partial functions all the time.

'But you are saying that the situation is asymmetric and we can't have what we might call *co-partial* functions, functions which may take nothing as input but still deliver something as output?'

Seems right to me. If we fail to feed a genuine input to a function, the function has nothing to work on, its action isn't triggered, so we won't get an output.

'That sounds rather uncomfortably metaphorical, talking of triggering a function into getting to work!'

Fair point. But the metaphors hit off a metaphysical truth, don't they? Santa, Vulcan and the largest prime don't exist. If we try to offer Santa to the function *father of* we must fail, because he's not there to be offered; so we can get no real

[1]Throughout, 'function' means a one-or-more-place function. For present purposes, 'zero-place functions' are not properly so called.

person as output from that function. If we try to offer Vulcan to the function *the mass in kilos of* again nothing happens, and we get no number as output. If we try to offer the largest prime to the computable *factorial* function, again we don't get to the starting line for a possible computation. And the point surely generalises.

'Generalises more widely, perhaps. But does the point apply right across the board? I'm beginning to think that maybe it doesn't.'

Really? I'm intrigued. You'd better explain.

'OK, I will. But can we first agree that the issue does matter, at least a little bit? If we do allow that there are co-partial functions, we'll have to complicate our logic to deal with them, and no doubt complicate our metaphysics too.'

That sounds right. So I agree this is worth thinking about. At least a little bit. Tell me, then: why do you suppose that there might indeed be functions which can take nothing as input but still deliver an output?

'Well, I've been reading Alex Oliver and Timothy Smiley. With a trivial change of example, they write: "Consider the set-theoretic functor '{ }'. If there is no such person as Santa there is indeed such a set as {Santa}: it is the null set. Or consider the functor 'the cardinality of ... ' as in '0 is the cardinality of the even primes greater than 2'." (2006a, p. 323) In each case, the relevant function maps nothing to something, since Santa and the even primes greater than two don't exist, but the null set and zero do. Such examples of co-partial functions, Oliver and Smiley claim, have "always been to hand", and so a competent logic (and metaphysics) of functions will need to accommodate them.'

Let's come back in a moment to Santa's singleton (if we can call it that). I certainly don't find the 'cardinality' example very persuasive.

'Why not? I rather liked it!'

Well, take Oliver and Smiley's claim (A) 'the cardinality of the even primes greater than two is zero'.

I suppose we might rephrase this as (A′) 'the number of the even primes greater than two is zero'. And we might very well confuse this with the claim (B) 'the number of even primes greater than two is zero' (which you get from (A′) by dropping a definite article). Now (B) is certainly true, and it indeed features a function, the one expressed by 'the number of'. But *here* we have a second-level function which takes a concept (in Frege's sense) and yields a number. And its argument in (B) is the concept *even prime greater than two*, which is a perfectly good concept – a something, not a nothing (if we can put it that way, allowing ourselves Frege's grain of salt).

Secondly, compare (A) with the claim (C) 'the cardinality of the set of even primes greater than two is zero'. Again, this is uncontroversially true. And it features a function expressed by 'the cardinality of', but this time we are dealing with a first-level function from sets to numbers. Its argument is the set $\{x \mid x \text{ is an even prime greater than two}\}$, in other words the empty set. Which – at least on standard theories – is also a something, not a nothing.

Neither way, then, do we get a function mapping nothing to something ...

'Hold on: you are rather blatantly changing the subject! Oliver and Smiley aren't talking about either of the functions that you mention in (B) and (C). Their original claim (A) uses a functor 'The cardinality of' which neither takes a concept expression (as in your first replacement), nor a singular term specifically for a set (as in your second replacement), but rather takes a plural term which may refer to many, one, or zero things.'

> Well, consider a plural term like 'the cards': what about the Frege point that these could be one deck, four suits, 52 individual cards so there's no such thing as the cardinality of the cards?

'I thought you might say that. Frege is right that you can't just point at some things and ask 'how many?'. But terms, singular or plural, don't merely point: they come with associated criteria of identity (and hence criteria for counting). For example, the term 'the cards' picks things out – surprise, surprise – as *cards*: and the cardinality question for the cards as such has a determinate answer.'

> I'm not entirely confident that singular and plural terms *do* always come with associated principles of counting attached. But let's have that discussion another day: I'll grant you the assumption for now.

'Right. Then we can say – speaking with Oliver and Smiley – that the cardinality of the prime numbers is $\aleph_0$; the cardinality of the square roots of two is two; the cardinality of the positive square roots of two is one. And we can say (C) the cardinality of the even primes greater than two is zero. As they want, in the case of the last truth, the featured *the cardinality of* function maps nothing to something.

You seemed very eager to overlook the function with plural arguments which Oliver and Smiley actually use in their example, and to talk instead about two other functions. That makes me suspect that you are one of those singularists who pretends to unmask plural talk as really being singular talk in disguise (talk about a set, or talk about a concept). But singularism is ill-motivated and can't be consistently maintained across the board. Do you want me to remind you of Oliver and Smiley's forceful arguments elsewhere for that (e.g. in their 2001)? For a start, there's their surely convincing ... '

> Sorry for interrupting, but there's no need to continue. I'm *not* a singularist, and I certainly have no wish to ignore or parse away plurals. Honest! And *you* interrupted before I had the chance to finish my line of thought.

'Oh? Then it's your turn to explain.'

> As I said, there are two close neighbours to Oliver and Smiley's claim (A) that the cardinality of the even primes greater than two is zero. Those neighbouring claims – (B) the number of even primes greater than two is zero, (C) the cardinality of the set of even primes greater than two is zero – are both uncontroversially true, but neither involves a function mapping nothing to something. Agreed so far?

'I agree about your substitute claims (B) and (C). But that doesn't settle how things stand with Oliver and Smiley's original (A).'

> Indeed. My point *exactly.* So – just as you want – let's very clearly set aside those common-or-garden neighbouring truths which are not to be confused with Oliver and Smiley's claim. Let's read their (A) as we are supposed to do. But then why should we *now* be confident of the truth of this much less familiar claim, once we've sharply distinguished it from the more usual (B) and (C)? Is there really, already to hand, a decisive pre-theoretic verdict that (A), so interpreted, is true?
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> I don't see that there is. And certainly, there is no telling pre-theoretic verdict that plainly trumps the telling consideration the other way – I mean the consideration that in (A) the featured plural function is given no argument, so no output is triggered.

So at the very best we have a stand-off here, and not a decisive argument that (A) is both plainly true and witnesses the existence of a co-partial function.

'OK, you think that we will only be inclined to confidently endorse Oliver and Smiley's (A) if we elide it with one or other of the evident truths (B) or (C) which don't involve co-partial functions?'

That's it, in a nutshell.

'Well, be that as it may: you can't make a similar complaint of potential confusion about their other example which I mentioned, for that uses an entirely determinate and very well-understood set-theoretic notation.'

Let's see: remind me how the argument goes.

'They start by asking us to consider the set-theoretic functor "{ }" . . . '

Ah yes. Can we just pause over that?

'Heavens, are you going to complain already?'

Well, I do want to emphasise that in standard modern usage, the set-theoretic curly brackets are in fact canonically introduced as part of a variable-binding functor that takes a predicate φ and produces a singular term, thus: $\{\xi \mid \varphi(\xi)\}$, for some variable ξ. So it isn't the brackets by themselves that are a functor, it is the brackets-plus-variable.

'But we *do* use the notation "$\{a\}$" for the singleton of a.'

Of course! I'm not denying that. But '$\{a\}$' is standardly introduced as syntactic sugar for '$\{x \mid x = a\}$' (likewise, '$\{a, b\}$' is syntactic sugar for '$\{x \mid x = a \vee x = b\}$', and so it goes).

'Syntactic sugar?!'

Ooops! That's just CompSci speak for a neat abbreviatory device that makes things easier to read and so gives us notation that is a bit 'sweeter' for us humans to use. Indeed, you could say that the expression 'syntactic sugar' is syntactic sugar for 'a neat abbreviatory device etc.'.

'OK, I get the idea! But even if you are right about how '$\{a\}$' officially unpacks, it is still the case – as Oliver and Smiley go on to insist – that there is such a set as {Santa}.'

So Santa has a singleton!

'Of course not: there's no set whose one and only member is Santa.'

I agree with *that*: but it *is* entirely standard to say that $\{a\}$ is the singleton of a, and in *this* sense you do think there is a singleton of Santa.

'Well, if you must put it that way then, yes, I do. But let's not get hung up on terminology. The point is that, by your own lights, you ought to agree that "{Santa}" denotes the empty set. For you treat "{Santa}" as syntactic sugar for "$\{x \mid x = \text{Santa}\}$". Now, we must be working in a free logic here, since we are allowing empty names. So how are we going to treat the identity in our free logic? One obvious option is to construe identity *strongly*, so that "$a = b$" is false if one or both of the terms flanking the identity is empty. With identity construed strongly, "$x = \text{Santa}$" is then false, whatever "x" denotes when treated as a parameter, so it is equivalent to e.g. "$x \neq x$". Hence, on the obvious semantics, "$\{x \mid x = \text{Santa}\}$" will denote the same set as "$\{x \mid x \neq x\}$", i.e. the empty set.'

> Which is all fine by me. But then the compositional semantic story you've just been gesturing towards sees the term '$\{x \mid x = \text{Santa}\}$' as resulting from the application of the second-level functor '$\{x \mid \ldots x \ldots\}$' to the predicate '$\xi =$ Santa'. Now, turning from the level of language to the level of what it expresses, the predicate '$\xi =$ Santa' (given that 'Santa' has no denotation) expresses the same property as "$\xi \neq \xi$", i.e. the null property which is had by nothing. And so the functor here expresses a function which takes us from the null property to something, not from nothing to something. This case therefore certainly doesn't witness a co-partial function.

'I'm not sure I like the idea of the null property.'

> Well, if you want to start getting picky about properties and select some kosher ones from the gruesome gerrymandered one, that's another game we could play. Though I wonder: why buy null sets and balk at null properties? And if you *are* going to get picky about properties, you'd better get picky about functions as well. And do you then have to hand a story about why, for example, your favoured 'set of' function would count as kosher while the null property 'being identical to the set of itself' doesn't?

'Probably not! OK, we won't go down there today!'

> Anyway, my point was a general one. Even in the case of '$\{a\}$', the set-theoretic brackets are not operating as a functor that takes a term. The brackets don't express a first-level function. Rather, when the wraps are off, the functor here is a second-level variable-binding one, which is operating on the predicate '$\xi = a$'.

'Can we backtrack? Maybe I was too concessive in allowing you to privilege what you called the modern standard usage of the set-theoretic curly-brackets as (part of) a second-level functor. After all, that usage is probably shaped by decades of singularist prejudice. Let's go back to the founding father, Cantor. *He* introduced the curly-bracket notation in a different way. Right at the beginning of his epoch-making 1895 paper, he famously writes that "By an 'aggregate' we are to understand any collection into a whole M of definite and separate objects m .... These objects are called the 'elements' of M. In signs we express this thus: $M = \{m\}$." Here, then, the obvious reading makes "m" a plural variable (he talks of 'separate objects m'), and then "$\{\ \ \}$" expresses a first-level (though plural) function (see Oliver and Smiley, 2006b, p. 124).'

> Actually, that interpretation doesn't look compulsory to me, given that Cantor in fact gives us very little to go on, and he mostly uses the likes of 'm' as what are very clearly singular variables. Here's another way of understanding his practice. Like other mathematicians of his time (and later), Cantor's variables are strongly typed: that is to say, a variable m is understood as ranging over objects of some given type $\mathcal{M}$. And then we read '$\{m\}$' as meaning 'the aggregate of elements of associated type $\mathcal{M}$'. For example, κ serves for Cantor as a typed variable over cardinals; and '$\{\kappa\}$' denotes the set whose members are cardinals. In '$\{m\}$' and '$\{\kappa\}$', then, the variables aren't serving as object-denoting expressions, but are there to indirectly indicate a type; and so the set brackets aren't serving as first-level functor (singular or plural).

'I'm not sure why you should think yours is a *better* reading. But anyway, the first-level functor reading makes good sense of Cantor's notation doesn't it? So why not let's go with it? And then we have {Santa} is the empty set, with the functor taking an empty term and the result denoting the null set. So that does give us – as claimed – an example of a co-partial function.'

> That's still far too quick! Suppose we do read Cantor's notation as you want. Then this isn't the modern reading according to which, as we've agreed, '{Santa}' can be taken as denoting the empty set (but where there's no reason to suppose there's co-partial function involved). All bets are now off. We need to ask what '{Santa}' denotes when read Cantor-style, as involving a plural functor. Well, shall we take our lead from Cantor himself?

'That sounds a good move – even if I suspect your motives!'

> Well, Cantor himself didn't believe in the empty set. You yourself quoted him as talking of sets as collections; and that way of talking doesn't comfortably accommodate the empty set. My collection of priceless Ming vases is no collection at all. And Cantor takes this line consistently. To quote, ironically enough, Oliver and Smiley again, when Cantor is 'describing a putative point-set that turns out not to contain any points, he says that strictly speaking it does not exist. . . . In the same vein, he does not say that two point-sets with no point in common have an empty intersection; rather they have no intersection.' (2006b, p. 127) And so on through many examples. So, for Cantor, '{Santa}' couldn't denote the empty set as according to him there is no such thing: the expression must stand for nothing. So we'd certainly be going right against Cantor himself if we appealed to his putative way of understanding the set brackets in '{Santa}' as illustrating a co-partial function in action, mapping nothing to something.

'If you say so. But why couldn't we marry (what we are supposing is) Cantor's old way of understanding the 'set-of' brackets as a first-level plural functor with our modern acceptance of the empty set?'

> Maybe we could. You'd need to tell me more. But be clear that you'd have the weight neither of Cantor's authority nor of modern mathematics behind you. So this would be news, not a rehearsal of what we all know, *not* a reminder of what has 'always been to hand' (as Oliver and Smiley put it). We can therefore say at least this: it now seems a long way from just being obvious that we can simultaneously say that '{Santa}' denotes the empty set *and* that this exemplifies the operation of a co-partial function. I'd say that at this stage everything is still to argue for, and the principle that a function with an empty input yields no output remains attractively on the agenda.

'Oliver and Smiley do think they have another quick example of a co-partial function, but given what you've said so far I predict you won't like it.'

> Try me!

'Well, they suggest we should "consider mereological addition and ask whether the result of adding nothing to Russell is nothing or Russell".' (2006a, p. 323)

> Ah, mereology! The last refuge of the metaphysical scoundrel . . .

'Now your prejudices are showing!'

> Sorry. I'll try to behave myself. But I can't see that there is a serious argument here. Suppose I add nothing to Russell in the sense of just not going in for adding (compare: I added nothing to the debate, because I didn't utter a word). Then to be sure, we still have Russell – I haven't annihilated him. But no function is called, so a fortiori no co-partial function is called.

Suppose however you ask me to add nothing to Russell in the sense of forming the mereological sum of Russell with (say) Santa. Then I say: it can't be done – Santa isn't there to have a function applied to him and so we get no output. (Compare: you ask me to add 10 to the largest prime number. Again, I say there is nothing for me to do, because there is no such thing as the largest prime.) So this time a function is called, but not enough arguments are offered it and we get no output. So still we haven't not an illustration of a co-partial function in play.

'OK. Perhaps Oliver and Smiley's supposed examples don't conclusively show that co-partial functions have 'always been to hand'. But on reflection do we actually have to rely on prior examples at all? Can't we just *stipulate* that a certain function takes a given value when its argument is empty?'

Well, if partial functions are such strange beasts that we have to rely on unnatural stipulations to cook up them up, then we arguably don't need to accommodate them in a logical framework aimed at regimenting ordinary mathematical reasoning (which is what Oliver and Smiley are after). But let that pass. How is the stipulation story supposed to work?

'Well, here's a simple example. Let's not worry about pre-existing usage: suppose we now just stipulate that if 'a' denotes a, then '$[a]$' denotes a's singleton, while if 'a' is empty, then '$[a]$' denotes the empty set. What's wrong with that?'

Nothing at all. Be my guest: we are in control of our notation, so stipulate away!

'So after all our fussing, you are conceding just like that? You are allowing there's no problem in stipulating co-partial functions into being?'

Slow down! I didn't say *that*. We've agreed on a notation: but we haven't got an argument yet that '[ ]', as you've just introduced it, serves across the board to express a particular function. Have we got a functor here which uniformly expresses a single function (which has to be co-partial)? Or have we just overloaded notation so we have a mash-up of two different semantic roles?

'But what's the argument against semantic uniformity here? We're entirely happy to write, for example:

$$\text{if } x < 0, f(x) = 0 \text{ and if } x \geq 0, f(x) = x$$

and suppose that we've stipulated a perfectly good function. There's nothing wrong with such definitions by cases.

Again, I quite agree. Suppose we have two functions defined over two non-overlapping domains. Then we do typically allow taking the union of the domains and then pasting the functions together to give us a new function defined over the big domain. No problem. But that just isn't what's going on here.

We have a genuine function, taking us from objects (though you didn't say which) to their singletons; but what are we supposed to be pasting onto this? You say that when 'a' is empty, then '$[a]$' denotes the empty set. But that specification doesn't make the brackets here express a function. To evaluate '$[a]$' in this case we simply don't get round to having to call any function (let alone trying to present the referent of 'a' to it).

So think again about your stipulation of how to use the square brackets. To evaluate '$[a]$' we try to evaluate 'a'. If we succeed, then we call the singleton function to get the value of '$[a]$'; if we fail, we assign '$[a]$' the empty set as default value. Since the second alternative calls no function, we have no good reason to say that '[ ]' always expresses a function. And so, a fortiori, no good reason to say that it uniformly expresses a co-partial function.

I conjecture it will be similar for other attempts to stipulate copartial functions into being! Can you refute the conjecture?

'Ah well! Maybe I shouldn't have got so worried about co-partial functions after all. And, on reflection, I suppose this negative outcome is actually rather cheering.'

Good, I was only trying to be helpful!

'Somehow I doubt it. But as we agreed at the outset, spelling out the metaphysics and logic of functions *is* now going to be that bit easier if we *don't* have to accommodate co-partial functions.'

A little easier, perhaps, but still troublesome. For a start, what do you think about the function analogue of Frege's concept horse problem?

'Ah, now it is funny you should mention that . . . '

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