Sets, relations, functions

Peter Smith

October 15, 2010

These notes amplify John L. Bell, David DeVidi and Graham Solomon, *Logical Options*, Broadview Press, 2001, §2.4, 'Set Theory': this book is a very useful source for material a step beyond *IFL*. All page references are to *LO*.

1 Set Theory and set theory

When we start doing serious logic, we very quickly find ourselves wanting to talk about *sets*. 'Domains of quantification' are sets, the 'extensions of predicates' are subsets of the domain of quantification. And so forth.

But note that this does *not* immediately entangle us with the heavy duty notion of set that it is the business of hard-core mathematical theories like Zermelo-Fraenkel Set Theory to regiment and explore.

Let me explain. A mathematical theory like ZF concerns the hierarchical universe you get if you take some things – or maybe just the empty set – and form all the possible sets of those. And then form all the possible sets whose members are drawn from the thing(s) you started off with and the sets you've just built; and then form all the possible sets whose members are drawn from what you've constructed so far; and then go up another level and form every possible collection out of what you have available now. Keep on going for ever . . . ; and when you have done all that, put together all the members from every set you have constructed, and now start set-building again, transfinitely. Keep on going again . . . . And that's just the beginning!

Evidently, we are going to end up with a *vast* hierarchical universe of sets, a universe rich enough to model more or less any kind of mathematical structure we might dream up – which is why we can in effect do more or less any mathematics inside ZF – or more accurately, within ZFC, which is ZF plus the Axiom of Choice – the canonical theory of this wildly proliferating universe. And equally evidently, when we interpret some version of QL, and talk about its 'domain of quantification' and the 'extensions of predicates' and so on we are – in the general case – *not* concerned with wild universes at all.

For example, the domain of quantification might be, very tamely, the set of people. The extension of the one-place predicate 'W' might be just the set of wise people (a subset of the domain). The extension of the two-place predicate 'L' might be just the set of pairs of people such that the first loves the second (a subset of the totality of arbitrary pairs from the domain) And so it goes. So all we need to worry about for interpreting a first-order language is normally some humdrum set of things (for the domain, e.g. the set of people, the set of natural numbers), subsets of domains (for extensions of monadic predicates), sets of pairs from the domain, and so on. There is no hierarchical *building up* from the initial domain to ever bigger sets; rather there is a *cutting down* of the domain to its subsets, or at any rate to sets of pairs culled from the domain, and so on.

OK, things get more complicated when we turn from thinking about this or that interpretation, and start generalizing over all possible interpretations – as when we say a quantificational argument is valid if there is no interpretation which makes the premisses true and conclusion false. Don't we *then* need to think about all possible domains, and hence all sets, including *huge* ones? Well actually, the answer is 'no': it turns out that if we can find an interpretation which makes the premisses of a first-order quantificational argument true and the conclusion false, then we can find a 'small' interpretation which does the job, whose domain is no bigger than the natural numbers. Moreover, that can be proved without heavy duty set-theoretic assumptions. Moral: you don't need to do any heavy lifting using serious set theory about *big* sets if you are interested just in elementary logic!

In short, hard-core set theory, a theory like ZF which tells us about the proliferating hierarchy of sets, sets of sets, sets of sets of sets, sets of sets of sets of sets, and so ever upwards, isn't our concern. So when relatively elementary logic books like *LO* have an introductory section 'Set Theory' with a capital 'S' and capital 'T', that's is a bit of a misnomer. It suggests that something rather heavy duty might be about to be needed. But not so. What we in fact typically get, as in *LO* is 'set theory' with a very small 's' and very small 't' – just some notation for how to refer to domains and their subsets, and notation for talking about pairs, and sets of pairs from a domain, and so on. Sure, there is a minimal amount of 'theory' involved in the presumption that given a domain, it has subsets, and that there will be pairs of elements from the domain, and sets of such pairs, and so on through what we need. But it is all anodyne and harmless stuff – humdrum school-room baby set theory, not the mathematician's Wonderful World of Set Theory.

2 Basic notions and notations

However, it certainly remains the case that you need to be able to understand some basic set concepts – e.g. the notions of union and intersection of sets, the Cartesian products of sets, and so on and so forth – quite apart from the particular use of these notions in talking about the semantics of first-order logical languages. So here's a quick stand-alone review of these basic gadgets, in company with *LO*.

These are things you ought to know about, yes. But not things you need obsess about. Don't worry too much about learning off by heart *all* the jargon and knowing how to use *all* the notational twiddly bits. You'll soon remember what the frequently-used stuff means: and you'll at least now know where to look up the meanings of the other devices if/when you encounter them again.

Anyway, here's an initial list of definitions. The page references to *LO* point to further explanations.

1. We use $a \in S$, $b \notin S$ to mean the element a is a member of the set S, and the element b isn't. (p. 73)

2. We can specify a set by listing its members explicitly, as in $\{2, 3, 5, 7, 11, 13, 17, 19\}$, or by definition, as in $\{x \mid x \text{ is a prime number less than } 20\}$, where $\{x \mid \varphi(x)\}$ is to be read 'the set of things x such that $\varphi(x)$. Use curly brackets and only curly brackets for (unordered) sets. (pp. 73–74)

3. Those two sets we just mentioned are in fact one and the very same set, and so indeed is e.g. $\{5, 7, 19, 7, 11, 13, 17, 19, 3, 5, 2\}$ – since sets are the same when they have the same members. That's the principle of extensionality. (p. 74)

4. A is a subset of B, $A \subseteq B$, when any member of A is a member of B. (p. 74) Note, $A \subseteq B$ is consistent with $A = B$. A is a *proper* subset of B, $A \subset B$, when any member of A is a member of B, and $A \neq B$, i.e. B also has members that A lacks.

5. The intersection of A and B, $A \cap B$ is the set of members common to A and B: $A \cap B = \{x \mid x \in A \wedge x \in B\}$. Dually, the union of A and B, $A \cup B$ is the set of the things which are in at least one of A and B: $A \cup B = \{x \mid x \in A \vee x \in B\}$. (p. 75) [What nice upshot do we get if we think of *propositions* as themselves sets, i.e. we identify the proposition A with the set of 'worlds' at which A is true?]

6. If A and B share no members, their intersection is the empty set $\emptyset$, the set with no members. 'The' empty set because it follows from our definitions that the empty set is unique (i.e. if A and B are empty sets, $a = b$). Also note that the $\emptyset$ is a subset of every set. (p. 74) [Check those claims! Are you a bit suspicious about the very idea of an empty set? Then you are in good company. After all, set-talk is introduced via metaphors about e.g. collecting together things; and a stamp collection with no stamps in it is surely *not* a collection at all. For much more of the same, see Alex Oliver and Timothy Smiley, 'What are sets and what are they for?', *Philosophical Perspectives* 20 (2006): 123–55. But in fact most mathematicians these days are entirely cheerful about the empty set; the usual line is that we shouldn't be *too* wedded to the introductory metaphors. But we can't go into this here, except for a remark in a moment.]

7. If a is an object then the set containing just a is its singleton set, $\{a\}$. The singleton of a is unique. And in general $a \neq \{a\}$, since a is a member of the singleton but not of itself. [Are you a bit suspicious about the very idea of a singleton set too? Then you are still in good company. After all, if there is just one stamp in my collection, surely my collection in the drawer is nothing other than that stamp. For more of the same, see Alex Oliver and Timothy Smiley again. But in fact most mathematicians these days are entirely cheerful about singletons too. Many would say something like this: don't think of a singleton as a sort of complex supposedly containing a inside it, but rather as – so to speak – a mathematical atom equipped with an 'arrow' pointing out to a. Likewise for other sets: they aren't complex structured objects 'containing' lots of members, but – so to speak – further unstructured atoms equipped with arrows pointing out to their 'members'. And the empty set is the limiting case of an atom like that, i.e. one which is equipped with no arrows out.]

But of course – leaving aside the square-bracketed riffs – you probably knew all *that* (the content of *LO*, §2.4.1) from school. So let's move on.

3 Pairs and Cartesian products

The set $\{a, b\}$ is the same set as $\{b, a\}$ (or indeed $\{a, b, a\}$, $\{a, b, b, a\}$ etc.). To repeat, what matters to set-identity is what the members are: and all those sets have the same members, i.e., a and b. But often we are interested not just pairs, but *ordered* pairs. Some notation and facts about these:

8. We use $\langle a, b\rangle$ for the ordered pair where, crucially, if $\langle a, b\rangle = \langle a', b'\rangle$, then $a = a'$ and $b = b'$. Some mathematicians use (a, b) as alternative notation – but that is to be deprecated as potentially too confusing. (p. 76)

9. Once we have the operation of forming ordered pairs available, we can form ordered n-tuples for any n. Thus the ordered triple $\langle a, b, c\rangle$ can be defined as the ordered pair $\langle a, b\rangle$ followed by c, i.e. as $\langle\langle a, b\rangle\ c\rangle$. And the ordered quadruple of a, b, c, d thus ordered is $\langle\langle\langle a, b\rangle\ c\rangle\ d\rangle$. And so on. (p. 77)

10. It would be perfectly in order to take the formation of ordered pairs to be a primitive operation, as primitive as forming unordered sets. But we don't need to. We can construct proxies for ordered pairs out of unordered sets as follows: now put
$$\langle a, b\rangle =_{\text{def}} \{\{a\}, \{a, b\}\}$$
Then this definition yields the essential property that if $\langle a, b\rangle = \langle a', b'\rangle$, then $a = a'$ and $b = b'$. (p. 77)

11. NB This definition gives us something which will do as well as an ordered pair, i.e. has the essential property just noted. But note that $\{\{a\}, \{a, b\}\}$ isn't really any more ordered than any other two element set. Nor indeed need it even be a pair. For if $a = b$, $\{a, b\} = \{a\}$, and the 'pair' set collapses into $\{\{a\}\}$. Is this a reason for rejecting the standard Wiener-Kuratowski 'definition' of an ordered pair? No! In a slogan, what we really care about is what an ordered pair *does* not what it *is* – anything that has the right essential structural property can serve as an 'ordered pair'.

12. Suppose A and B are sets. Then consider the collection of ordered pairs of the form $\langle a, b\rangle$ where a is a member of A and b is a member of B. This collection is called the Cartesian product of A with B. We can write
$$A \times B =_{\text{def}} \{\langle a, b\rangle \mid a \in A \wedge b \in B\}$$
Similarly
$$A \times B \times C =_{\text{def}} \{\langle a, b, c\rangle \mid a \in A \wedge b \in B \wedge c \in C\}$$
and so on. (p. 77, which gives an explanation of the origin of the terminology).

13. Particularly important are Cartesian products formed by taking ordered pairs, triples etc. from the same set A, i.e. the sets $A \times A$ (commonly written A^2), $A \times A \times A$ (commonly written A^3), and so on. (p. 77)

14. Thus note that if $A \subseteq D$ and $B \subseteq D$, then $A \times B \subseteq D^2$. [Make sure you understand why!]

4 Relations

LO says 'a binary relation is essentially just a subset of a Cartesian product of two sets' (p. 77). *No, it most certainly isn't!* What the authors should have written is: the *extension* of a binary relation is a subset of a Cartesian product of two sets. But it is Grave Sin against Great Uncle Frege to identify relations with their extensions.

Suppose we did think that relations themselves are subsets of Cartesian products, i.e. are sets, i.e. a type of object. So relations would be objects. And the relational proposition *Romeo loves Juliet* would apparently be equivalent to a list of names of objects, equivalent to e.g. *Romeo, the set of ordered pairs of lovers, Juliet.* But the latter isn't a proposition at all. There is no predication going on there! A relation is not an object (not even a fancy object like a subset of a Cartesian product). Rather, think of it as a *function*, i.e. a mapping of pairs of objects to truth-values (so the relation of loving is the function that maps the pair Romeo, Juliet to True, and maps the pair Elizabeth Bennett and Mr Collins to False). And functions aren't objects. Or so Great Uncle Frege taught us all.

Well, maybe Great Uncle Frege overdid it a bit: this isn't the place to discuss deep questions like the Unity of the Proposition and what it is about putting names and predicates together that gives us a genuine proposition rather than a mere list of names, and certainly not the place to discuss the Fregean doctrine that functions and relations are 'unsaturated' (somehow intrinsically gappy, with slots waiting to be filled). But there are serious issues here.

However, having harrumphed about the way that the authors of *LO* (like many authors of logic books) just ride rough-shod over these issues, we can systematically read most of their talk about relations the safe and uncontentious way, as in fact talk about the extensions of relations. Similarly, we'll read much of their talk about functions as talk about the extensions of functions. Then what they stay is standard and routine. Thus:

> (The extension of) a monadic property P is the set of things that have that property – i.e. the set $\{x \mid Px\}$.
>
> (The extension of) a dyadic relation R is the set of ordered pairs of items such that the first stands in R to the second – i.e. it is the set $\{\langle x, y\rangle \mid Rxy\}$.
> And so on.

Now, consider the relation Rxy which holds when x is real number, and y is a whole number greater than x. Then the extension of R is a set of pairs $\langle x, y\rangle$ where $x \in \mathbb{R}$ and $y \in \mathbb{Z}$ and $x < y$ (here, $\mathbb{R}$ is the standard symbol for the set of real numbers, and $\mathbb{Z}$ is the standard symbol for the set of integers, positive and negative). So the extension of R is – as *LO* says – a subset of the Cartesian product $\mathbb{R} \times \mathbb{Z}$.

If we start formalizing talk about the relation R in a first-order language, and want to talk about the reals $\mathbb{R}$ and the integers $\mathbb{Z}$ at the same time, then – since our logic is 'single sorted' with only one type of variable – we'll just have to bung them all into one big domain Δ – where $\mathbb{R} \subseteq \Delta$ and $\mathbb{Z} \subseteq \Delta$. The extension of R will still be a subset of $\mathbb{R} \times \mathbb{Z}$, and hence a subset of Δ^2 by the result (14) above. And every n-place relation defined over the all-in domain Δ, containing everything we want to be talking about in the context, will have as its extension a subset of the Cartesian product Δ^n.

5 Kinds of two-place relation

Now some definitions about kinds of relation.

15. Let R be a dyadic relation defined over a domain Δ:

    (a) R is *reflexive* if Raa for all $a \in \Delta$ [example: a is no taller than b, among people] ;

    (b) R is *symmetric* if $Rab \leftrightarrow Rba$ for all $a, b \in \Delta$ [example: a is a sibling of b];

    (c) R is *antisymmetric* if $Rab \wedge Rba \rightarrow a = b$ [example: $a \leq b$, $a, b \in \mathbb{N}$];

    (d) R is *transitive* if $Rab \wedge Rbc \rightarrow Rac$ for all $a, b, c \in \Delta$ [example: $a < b$].

    Think of it this way: a symmetric relation is one that always holds in both directions; an anti-symmetric relation is one that never holds in both directions except for the case when, so to speak, there *is* no special direction i.e. when the relation holds between a thing and itself. (p. 78)

16. A dyadic relation R which is reflexive, symmetric and transitive is an *equivalence* relation. An equivalence relation defined over the domain Δ carves it into equivalence classes – i.e. classes $A, B, C, \ldots$ such that every item in Δ is in one and only one of the classes; and all the members of one of the classes are R related to all the other members of the same class. Example, 'having the same surname as' divides

people into equivalence classes (counting compound surnames as a single name!); everyone is in one such class; and members of the same equivalence class all have the same surname as each other.

17. A dyadic relation R which is reflexive, antisymmetric and transitive is an *order* relation [example, $A \subseteq B$ is an order relation among sets]. If every item in the relevant domain is comparable by R, i.e. for every $a, b \in \Delta$, $Rab \vee Rba$, then R is a *total* order [examples: $A \subseteq B$ is not total, but $a \leq b$, $a, b \in \mathbb{N}$ is.]. (pp. 79–80)

18. A dyadic relation R (relating items in domain Δ to items in the codomain Γ where now Δ and Γ might be different) is *functional* if, for every $a \in \Delta$, there is one and only one $b \in \Gamma$ such that Rab. So a functional relation maps each item in Δ to one and only one target. So, corresponding to a functional relation R there is a function f such that $f(a) = b \leftrightarrow Rab$. a here is said to be the 'argument' given to the function, which then takes the 'value' $f(a)$. The extension of f and the extension of R here will be the same, the set of ordered pairs $\langle a, b\rangle$ such that $f(a) = b$ or equivalently Rab. It is a nice question whether we should think of the function f and the functional relation R as the same (so that the functional notation and the relational notation are just variant notations for the same thing).

6 Kinds of functions

For future reference, some more notation and jargon about functions. For simplicity, we'll focus here on one-place functions (it should be obvious how to generalize definitions to cover many-place functions which map more than one 'argument' to a 'value').

19. We write $f\colon \Delta \to \Gamma$, for a function which maps *every* element a of the *domain* Δ to exactly one corresponding value $f(a)$ in the *codomain* Γ. (f is said to be a function from Δ *into* Γ.) (p. 80)

NB we are here concerned with *total* functions, defined for every member of the domain. For wider mathematical purposes, the more general idea of a *partial* function becomes essential. This is a mapping f which is not necessarily defined for all elements of its domain (for an obvious example, consider the reciprocal function $1/x$ for rational numbers, which is not defined for $x = 0$). However, we won't say more about partial functions here.

20. The *range* of a function $f\colon \Delta \to \Gamma$ is $\{f(x) \mid x \in \Delta\}$, i.e. the set of elements in Γ that are values of f for arguments in Δ. So $\mathit{Range}(f) \subseteq \Gamma$.

21. A function $f\colon \Delta \to \Gamma$ is *surjective* iff $\mathit{Range}(f) = \Gamma$ – i.e. if for every $y \in \Gamma$ there is some $x \in \Delta$ such that $f(x) = y$. (If you prefer that in English, you can say that such a function is *onto*, since it maps Δ onto the whole of Γ.)

22. A function $f\colon \Delta \to \Gamma$ is *injective* iff f maps different elements of Δ to different elements of Γ – i.e. if $x \neq y$ then $f(x) \neq f(y)$. So f maps Δ to a sort of 'copy' inside Γ. (If you prefer that in English, you can say that such a function is *one-to-one*.)

23. A function $f\colon \Delta \to \Gamma$ is *bijective* if it is both surjective and injective. (In English again, f is then a *one-one correspondence* between Δ and Γ.)

(Cf. *LO* p. 81 which gives the 'English' jargon, but the Latinate jargon is the usage of preference these days.) Note that if there is a bijection between two sets then they are the same size. If there's a one-one correspondence between your knives and your forks, then you have the same number of knives and forks. And we can apply the same idea to infinite sets: if there is a bijection between A and B we'll again say they have the same size (or same 'cardinality', as the mathematicians say). And we can say that the size of A is smaller than the size of B if (i) there is a bijection between A and a proper subset of B, but there is no bijection between A and B (i.e. there is a one-one correspondence between A and a part of B but not with the whole of it).

7 Back to sets: powersets, very briefly

One more notion – though this takes us a step beyond what we need for most logical purposes.

Take the set $\{0, 1, 2\}$. How many subsets does it have? Eight!

$$\emptyset,\ \{0\},\ \{1\},\ \{2\},\ \{0, 1, \},\ \{0, 2\},\ \{1, 2\},\ \{0, 1, 2\}.$$

(Don't forget, the empty set is a subset of every set; and a set counts as a subset of itself!).

Here's a new definition:

24. The powerset of a set A, symbolized $\wp(A)$, is the set of all subsets of A.

We've just seen, then, that the powerset of the three-membered set $\{0, 1, 2\}$ has eight members. A little reflection shows that for any n, the powerset of an n-membered set has 2^n members.

So: for finite sets A, A is strictly smaller than $\wp(A)$. This result can be extended to infinite sets. But that's a story for another day!