12 Going further

This has been a Guide to *beginning* mathematical logic. So far, then, the suggested readings on different areas have been at entry level, or only a step or so up from that. In this final chapter, by contrast, we take a look at some of the more advanced literature on a selection of topics, taking us another step or two further.

If you have been tackling enough of the introductory readings, you should in fact be able to now follow your interests wherever they lead without needing help from this chapter. For a start, you can explore the many mathematical logic entries in *The Stanford Encyclopedia of Philosophy*, which are mostly excellent and have large bibliographies. The long essays in the eighteen(!) volumes of *The Handbook of Philosophical Logic* are of varying quality, but there are some good ones on straight mathematical logic topics, again with large bibliographies. Internet sites like math.stackexchange.com can be searched for useful lists of recommended books. And then there is always Google!

However, those resources do cumulatively point to a rather overwhelming range of literature to pursue. So perhaps some readers will still appreciate a few more limited menus of suggestions (even if they are less systematic and more shaped by my personal interests than in the core Guide).

Of course, the 'vertical' divisions between entry-level coverage and the further explorations in this chapter are pretty arbitrary; and the 'horizontal' divisions into different subfields can in places also be quite blurred. But we do need to impose *some* organization! So this chapter is divided up as follows. First, there is a very brief foray into logic-relevant algebra:

12.1 A very little light algebra for logic?

There then follows a series of sections taking topics in the same order as earlier core chapters:

12.2 Higher-order logic, the lambda calculus, and type theory
12.3 More model theory
12.4 More on formal arithmetic and computability
12.5 More on mainstream set theory
12.6 Choice, and the choice of set theory
12.7 More proof theory.

We could continue; but this is more than enough to be going on with . . . !

12.1 A very little light algebra for logic?

Depending on what you have read on classical propositional logic, you may well have touched on the notion of a Boolean algebra. And depending on what you have read on intuitionistic logic, you may have also also encountered Heyting algebras (a.k.a. pseudo-Boolean algebras). It is worth getting to know a bit more about these algebras, both because of their relevance to classical and intuitionistic logic, but also because Boolean algebra features in independence arguments in set theory.

For a gentle and clear first introduction (aimed at those with little mathematical background), see

1. Barbara Hall Partee, Alice G. B. ter Meulen, and Robert Eugene Wall, *Mathematical Methods in Linguistics* (1990, Springer). The (short!) Chs. 9 and 10 introduce some basic concepts of algebra (you can omit §10.3); Ch. 11 is on lattices; Ch. 12 is then on Boolean and Heyting algebras, and briefly connects Kripke's relational semantics for intuitionistic logic to Heyting algebras.

Also very accessible, adding a little more on Heyting algebras:

2. Morten Heine Sørensen and Pawel Urzyczyn, *Lectures on the Curry-Howard Isomorphism* (Springer, 2006), Ch. II, 'Intuitionistic logic'.

Then, for rather more about Boolean algebras, you need *very* little background to start tackling the opening chapters of

3. Steven Givant and Paul Halmos, *Introduction to Boolean Algebras* (Springer, 2009). This is an update of a classic book by Halmos, and is very accessible; any logician will want eventually to know the elementary material in the first third of the book.

If you already know a smidgin of algebra and topology, however, then there is a faster-track introduction to Boolean algebras in

4. René Cori and Daniel Lascar, *Mathematical Logic, A Course with Exercises: Part I* (OUP, 2000), Chapter 2.

And for a higher-level treatment of intuitionistic logic and Heyting algebras, you could read Chapter 5 of the book by Dummett mentioned in §8.5, or work up to Chapter 7 on algebraic semantics in the book on modal logic by Chagrov and Zakharyaschev mentioned in §10.5.

Then, if you want to pursue more generally e.g. questions about when propositional logics do have nice algebraic counterparts (in the sort of way that classical and intuitionistic logic relate respectively to Boolean and Heyting Algebras), then you *might* get something out of Ramon Jansana's 'Algebraic propositional logic' in *The Stanford Enclyclopedia of Philosophy*, tinyurl.com/alg-logic. But this does strike me as too rushed to be particularly useful. So instead, you could make a start reading

5. Josep Maria Font, *Abstract Algebraic Logic: An Introductory Textbook* (College Publications, 2016). This is written in an expansive and accessible style, and well worth diving into.

12.2 Higher-order logic, the lambda calculus, and type theory

The logical grammar of first-order logic is very restricted. We assume a domain of objects that we can quantify over; we can have names for some of these objects; we can express properties and relations defined over those objects; and can express (total) functions from one or more objects as inputs to objects as outputs. In informal mathematics, by contrast, we quantify over properties, relations and functions too (as in second-order logic). And we also consider e.g. properties of relations (like being symmetric), relations between functions (like being asymptotically equal), functions from one function to another (e.g. differentiation), and more.

Now, as is familiar, we can trade in properties of relations, relations between functions, functions of functions, etc. for *sets*. So we can compensate for the expressive limitations of first-order logic by adopting enough set theory. Still, we might reasonably look for a more expressive logical framework in which we can talk directly about more types of things, and quantify over more types of things, without playing the set-theory card. And exploring such a higher-order logic might even offer the prospect of an alternative, non-set-theoretic, foundation for mathematics.

We looked at a small fragment of higher-order logic in Chapter 4 on second-order logic. But now we want to explore theories with a richer type-structure. Such a theory of types goes back at least until Bertrand Russell's 1908 paper 'Mathematical logic as based on the theory of types'. Its history since Russell has been rather chequered. But particularly in the hands of theoretical computer scientists, type theories have come back into considerable prominence. And in the recent guise of homotopy type theory, one particular version is advertised as a new foundation for mathematics. But where to start?

You could first take a quick look at

1. Jouko Väänänen, 'Second-order and higher-order logic', *The Stanford Encyclopedia of Philosophy*, tinyurl.com/sep-vaan.
2. Thierry Coquand, 'Type theory', *The Stanford Encyclopedia of Philosophy*, tinyurl.com/sep-type.

But the first of these mostly revisits second-order logic at a probably quite unnecessarily sophisticated level for now, so don't get bogged down. The second gives us pointers forward, but is perhaps also rather too rushed.

Still, as you'll see from Coquand, basic topics to pursue include Simple Type Theory and the lambda calculus. For a clear and gentle introduction to the latter, see the first seven chapters of the following welcome short book which doesn't assume much mathematical background:

3. Chris Hankin, *An Introduction to Lambda Calculus for Computer Scientists** (College Publications 2004).

Next, as a spur to keep going, perhaps read this advocacy:

4. William M. Farmer, 'The seven virtues of simple type theory', *Journal of Applied Logic* 6 (2008) 267–286. Available at tinyurl.com/farm-STT.

And then for a bit more on Simple Type Theory/Church's Type Theory, though once more this is less than ideal, you could look at

5. Christoph Benzmüller and Peter Andrews, 'Church's type theory', *The Stanford Encyclopedia of Philosophy*, tinyurl.com/sep-CTT.

But then where to go next will depend on your interests and on how much more you want to know. The book we want, *Type Theories for Logicians, A Gentle Introduction*, has yet to be written. So you will have to make do with the following initial suggestions (in order of publication date):

6. Henk P. Barendregt, *The Lambda Calculus: Its Syntax and Semantics** (Originally 1980, reprinted by College Publications 2012). This is the weighty standard text: but the opening chapters are fairly accessible.
7. Peter Andrews, *An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof* (Academic Press, 1986). Chapter 5, under 50 pages, is a classic introduction to a version of Church's type theory developed by Andrews. It is often recommended, and worth battling through; but it *is* a rather terse bit of old-school exposition.
8. J. Roger Hindley, *Basic Simple Type Theory* (CUP, 1997). Again, this short book is a classic, but again it is pretty terse. Perhaps, in the end, mostly for those whose main interest is in computer science applications of type theory in the design of higher-level programming languages like ML.
9. Benjamin C. Pierce, *Types and Programming Languages* (MIT Press, 2002). An often-recommended text for computer scientists, and readable by others if you skip over some parts about implementation in ML. The first dozen or so shortish chapters are relatively discursive and accessible.
10. J. Roger Hindley and Jonathan P. Seldin, *Lambda-Calculus and Combinators: An Introduction* (CUP 2008). Attractively and clearly written, aiming to avoid excess technicalities. More of the feel of a modern maths book. Recommended.
11. Rob Nederpelt and Hedman Geuvers, *Type Theory and Formal Proof: An Introduction* (CUP 2014). Focuses, the authors say, "on the use of types and lambda terms for the complete formalisation of mathematics", so should be of particular interest to mathematical logicians. Also attractively and clearly written (as these things go!).

Then, pointing in a different direction, you might also want to follow up

12. Peter Dybjer and Erik Palmgren, 'Intuitionistic type theory', *The Stanford Encyclopedia of Philosophy*, tinyurl.com/sep-ITT.

And finally, I suppose I should finish by mentioning again one particular new incarnation of type theory:

13. The Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics* (2013), tinyurl.com/HOTT-book.

I leave it to you to make what you will of that program!

12.3 More model theory

(a) If you want to explore beyond the entry-level material of Chapter 5 on model theory, why not start with a quick warm-up, with some reminders of headlines and some very useful pointers to the road ahead:

1. Wilfrid Hodges and Thomas Scanlon, 'First-order model theory', *The Stanford Encyclopedia of Philosophy*, tinyurl.com/sep-fo-model.

Now, we noted before in §§3.6(c) and 5.3 that the wide-ranging mathematical logic texts by Hedman and Hinman cover a substantial amount of model theory. But why not look at two classic stand-alone treatments of the area which really choose themselves? In order of both first publication and eventual difficulty:

2. C. Chang and H. J. Keisler, *Model Theory** (originally North Holland 1973: the third edition has been inexpensively republished by Dover Books in 2012). This is the Old Testament, the first systematic text on model theory. Over 550 pages long, it proceeds at an engagingly leisurely pace. It is particularly lucid and is extremely nicely constructed with different chapters on different methods of model-building. A really fine achievement that still remains a good route in to the serious study of model theory.
3. Wilfrid Hodges, *A Shorter Model Theory* (CUP, 1997). The New Testament is Hodges's encyclopedic *Model Theory* (CUP 1993). This shorter version is half the size but still really full of good things. It does get tougher as the book progresses, but the earlier chapters of this modern classic, written with this author's characteristic lucidity, should certainly be readily manageable.

My suggestion would be to read the first three long chapters of Chang and Keisler, and then perhaps pause to make a start on

4. J. L. Bell and A. B. Slomson, *Models and Ultraproducts** (North-Holland 1969; Dover reprint 2006). Very elegantly put together: as the title suggests, the book focuses particularly on the ultra-product construction.

At this point read the first five chapters for a particularly clear introduction.

You could then return to Ch. 4 of C&K to look at (some of) their treatment of the ultra-product construction, before perhaps putting the rest of their book on hold and turning to Hodges.

(b) A level up again, here are two further books that should definitely be mentioned. The first has been around long enough to have become regarded as a modern standard text. The second is a bit more recent but also comes widely recommended. Their coverage is significantly different – so I suppose that those wanting to get really seriously into model theory should take a look at both:

5. David Marker, *Model Theory: An Introduction* (Springer 2002). Despite its title, this book would surely be hard going if you haven't already tackled some model theory (at least read Manzano or Kirby first). But despite being sometimes a rather bumpy ride, this highly regarded text will teach you a great deal. Later chapters, however, probably go far over the horizon for all except those most enthusiastic readers of this Guide who are beginning to think about specializing in model theory – it isn't published in the series 'Graduate Texts in Mathematics' for nothing!
6. Katrin Tent and Martin Ziegler, *A Course in Model Theory* (CUP, 2012). From the blurb: "This concise introduction to model theory begins with standard notions and takes the reader through to more advanced topics such as stability .... The authors introduce the classic results, as well as more recent developments in this vibrant area of mathematical logic. Concrete mathematical examples are included throughout to make the concepts easier to follow." Again, although it starts from the beginning, it could be a challenge to readers without some mathematical sophistication and some prior exposure to the elements of model theory – though I, for one, find it more approachable than Marker's book.

(c) So much for my principal suggestions. Now for an assortment of additional/alternative texts. Here are two more books which aim to give general introductions:

7. Philipp Rothmaler's *Introduction to Model Theory* (Taylor and Francis 2000) is, overall, comparable in level of difficulty with, say, the first half of Hodges. As the blurb puts it: "This text introduces the model theory of first-order logic, avoiding syntactical issues not too relevant to model theory. In this spirit, the compactness theorem is proved via the algebraically useful ultraproduct technique (rather than via the completeness theorem of first-order logic). This leads fairly quickly to algebraic applications, ..." Now, the opening chapters are indeed very clear: but oddly the introduction of the crucial ultraproduct construction in Ch. 4 is done very briskly (compared, say, with Bell and Slomson). And thereafter it seems to me that there is some unevenness in the accessibility

of the book. But others have recommended this text more warmly, so I mention it as a possibility worth checking out.

8. Bruno Poizat's *A Course in Model Theory* (English edition, Springer 2000) starts from scratch and the early chapters give an interesting and helpful account of the model-theoretic basics, and the later chapters form a rather comprehensive introduction to stability theory. This often-recommended book is written in a rather distinctive style, with rather more expansive class-room commentary than usual: so an unusually engaging read at this sort of level.

Another book which is often mentioned in the same breath as Poizat, Marker, and now Tent and Ziegler is *A Guide to Classical and Modern Model Theory*, by Annalisa Marcja and Carlo Toffalori (Kluwer, 2003) which also covers a lot: but I prefer the previously listed books.

The next two suggestions are of books which are helpful on particular aspects of model theory:

9. Kees Doets's short *Basic Model Theory** (CSLI 1996) highlights so-called Ehrenfeucht games. This is enjoyable and very instructive.
10. Chs. 2 and 3 of Alexander Prestel and Charles N. Delzell's *Mathematical Logic and Model Theory: A Brief Introduction* (Springer 1986, 2011) are brisk but clear, and can be recommended if you wanting a speedy review of model theoretic basics. The key feature of the book, however, is the sophisticated final chapter on serious applications to algebra, which might appeal to mathematicians with interests in that area.

Indeed, as we explore model theory, we quickly get entangled with algebraic questions. And as well as going (so to speak) in the direction from logic to algebra, we can make connections the other way about, starting from algebra. For something on this approach, see the following short, relatively accessible, and illuminating book:

11. Donald W. Barnes and John M. Mack, *An Algebraic Introduction to Mathematical Logic* (Springer, 1975).

(d) As an aside, let me also mention the sub-area of Finite Model Theory which arises particularly from consideration of problems in the theory of computation (where, of course, we are interested in *finite* structures – e.g. finite databases and finite computations over them). What happens, then, to model theory if we restrict our attention to finite models? Trakhtenbrot's theorem, for example, tells that the class of sentences true in any finite model is not recursively enumerable. So there is no deductive theory for capturing such finitely valid sentences (that's a surprise, given that there's a complete deductive system for the sentences which are valid in the usual broader sense!). It turns out, then, that the study of finite models is surprisingly rich and interesting. So why not dip into one or other of

12. Leonard Libkin, *Elements of Finite Model Theory* (Springer 2004).

13. Heinz-Dieter Ebbinghaus and Jörg Flum, *Finite Model Theory* (Springer 2nd edn. 1999).

Both are good, though I prefer Libkin.

(e) In §5.3 I warmly recommended that you read at least early chapters of *Philosophy and Model Theory* by Button and Walsh. Now you know more model theory, do revisit that book and read on!

Finally, I should mention John T. Baldwin's *Model Theory and the Philosophy of Mathematical Practice* (CUP, 2018). This presupposes a lot more background than Button and Walsh. Maybe some philosophers might be able to excavate more out of Baldwin's book than I did: but I find this book badly written and unnecessarily hard work.

12.4 More on formal arithmetic and computability

(a) The readings in §6.5 have introduced you to the canonical first-order theory of arithmetic, first-order Peano Arithmetic, as well as to some subsystems of PA (in particular, Robinson Arithmetic) and second-order extensions. So what to read next on formal arithmetics?

You will know by now that first-order PA has non-standard models: in fact, it even has uncountably many non-isomorphic models which can be built just out of natural numbers. It is worth pursuing this theme. For a taster, you could look at lecture notes by Jaap van Oosten, on 'Introduction to Peano Arithmetic: Gödel Incompleteness and Nonstandard Models', tinyurl.com/oosten-peano. But better to dive into

1. Richard Kaye's *Models of Peano Arithmetic* (Oxford Logic Guides, OUP, 1991), which tells us a great deal about non-standard models of PA. This reveals more about what PA can and can't prove, and will also introduce you to some non-Gödelian examples of incompleteness. This is a terrific book, and deservedly a modern classic.

As a sort of sequel, there is also another volume in the Oxford Logic Guides series for enthusiasts with more background in model theory, namely Roman Kossak and James Schmerl, *The Structure of Models of Peano Arithmetic*, OUP, 2006. But this is much tougher going. For a more accessible set of excellent lecture notes, see

2. Tin Lok Wong, 'Model theory of arithmetic', downloadable lecture by lecture from tinyurl.com/wong-model.

Next, going in a rather different direction, and explaining a lot about arithmetics weaker than full PA, here's another modern classic:

3. Petr Hájek and Pavel Pudlák, *Metamathematics of First-Order Arithmetic* (Springer 1993). This is pretty encyclopaedic, but at least the first

three chapters do remain surprisingly accessible for such a work. This is, eventually, a must-read if you have a serious interest in theories of arithmetic and incompleteness.

And what about going beyond first-order PA? We know that full second-order PA (where the second-order quantifiers are constrained to run over *all* possible sets of numbers) is unaxiomatizable, because the underlying second-order logic is unaxiomatiable. But there are axiomatizable subsystems of second order arithmetic. These are wonderfully investigated in another encyclopaedic modern classic:

4. Stephen Simpson, *Subsystems of Second-Order Logic* (Springer 1999; 2nd edn CUP 2009). The focus of this book is the project of 'reverse mathematics' (as it has become known): that is to say, the project of identifying the weakest theories of numbers-and-sets-of-numbers that are required for proving various characteristic theorems of classical mathematics.

   We know that we can reconstruct classical analysis in pure set theory, and rather more neatly in set theory with natural numbers as unanalysed 'urelemente'. But just *how much* set theory is needed to do the job, once we have the natural numbers? The answer is: stunningly little. The project of exploring what's needed is introduced very clearly and accessibly in the first chapter, which is a must-read for anyone interested in the foundations of mathematics. This introduction is freely available at the book's website tinyurl.com/2arith.

(b) Next, Gödelian incompleteness again. You could start with a short old *Handbook* article which is still well worth reading:

5. Craig Smoryński, 'The incompleteness theorems', in J. Barwise, editor, *Handbook of Mathematical Logic*, pp. 821–865 (North-Holland, 1977), which covers a lot very compactly. Available at tinyurl.com/smory.

Now, the further readings on incompleteness suggested in §6.6 finished by mentioning two wonderful books which could arguably have appeared on our main list of introductory readings. However – a judgement call – I think that the more abstract stories they tell can probably only be fully appreciated if you've first met the basics of computability theory and the incompleteness theorems in a more conventional treatment. But certainly, now is the time to read them, if you didn't tackle them before:

6. Raymond Smullyan, *Gödel's Incompleteness Theorems*, Oxford Logic Guides 19 (Clarendon Press, 1992). Proves beautiful, slightly abstract, versions of the incompleteness theorems. A modern classic.

7. Equally short and equally elegant is Melvin Fitting's, *Incompleteness in the Land of Sets** (College Publications, 2007). There is a simple correspondence between natural numbers and 'hereditarily finite sets' (i.e.

sets which have a finite number of members which in turn have a finite number of members which in turn ... where all downward membership chains bottom out with the empty set). Relying on this fact gives us another route in to proofs of Gödelian incompleteness, and other results of Church, Rosser and Tarski. Beautifully done.

After these, where should you go if you want to know more about matters more or less directly to do with the incompleteness theorems?

8. Raymond Smullyan's *Diagonalization and Self-Reference*, Oxford Logic Guides 27 (Clarendon Press 1994) is an investigation-in-depth around and about the idea of diagonalization that figures so prominently in proofs of limitative results like the unsolvability of the halting problem, the arithmetical undefinability of arithmetical truth, and the incompleteness of arithmetic. Read at least Part I.
9. Torkel Franzén, *Inexaustibility: A Non-exhaustive Treatment* (Association for Symbolic Logic/A. K. Peters, 2004). The first two-thirds of the book gives another take on logic, arithmetic, computability and incompleteness. The last third notes that Gödel's incompleteness results have a positive consequence: 'any system of axioms for mathematics that we recognize as correct can be properly extended by adding as a new axiom a formal statement expressing that the original system is consistent. This suggests that our mathematical knowledge is inexhaustible, an essentially philosophical topic to which this book is devoted.' Not always easy (you will need to know something about ordinals before you read this), but very illuminating.
10. Per Lindström, *Aspects of Incompleteness* (Association for Symbolic Logic/ A. K. Peters, 2nd edn., 2003). This rather terse book is probably for enthusiasts. It is not always reader-friendly in its choices of notation and the brevity of its arguments. However, the more mathematical reader will find that it again repays the effort.
11. Craig Smoryński, *Logical Number Theory I, An Introduction* (Springer, 1991). There are three long chapters. Ch. I discusses pairing functions and numerical codings, primitive recursion, the Ackermann function, computability, and more. Ch. II concentrates on 'Hilbert's tenth problem' – showing that we can't mechanically decide the solubility of certain equations. Ch. III considers Hilbert's Programme and contains proofs of more decidability and undecidability results, leading up to a version of Gödel's First Incompleteness Theorem. (The promised Vol. II which would have discussed the Second Incompleteness Theorem has never appeared.)

    The level of difficulty is rather varied, and there are a lot of historical disgressions and illuminating asides. So this is an idiosyncratic book; but is still an enjoyable and very instructive read.

And if you want the bumpier ride of a lecture course with problems assigned as you go along, this is notable:

12. Tin Lok Wong, 'The consistency of arithmetic', downloadable lecture by lecture from tinyurl.com/wong-consis.

(c) Now let's turn to books on computability. Among the Big Books on mathematical logic, the one with the most useful treatment is probably

13. Peter G. Hinman, *Fundamentals of Mathematical Logic* (A. K. Peters, 2005). Chs. 4 and 5 on recursive functions, incompleteness etc. strike me as the best written, most accessible (and hence most successful) chapters in this very substantial book. The chapters could well be read after my *IGT* as somewhat terse revision for mathematicians, and then as sharpening the story in various ways. Ch. 8 then takes up the story of recursion theory (the author's home territory).

However, good those these chapters are, I'd still recommend starting your more advanced work on computability with

14. Nigel Cutland, *Computability: An Introduction to Recursive Function Theory* (CUP 1980). This is a rightly much-reprinted classic and is beautifully lucid and well-organized. This *does* have the look-and-feel of a traditional maths text book of its time (so perhaps with fewer of the classroom asides we find in some modern, more discursive books). However, if you got through most of e.g. Boolos and Jeffrey without too much difficulty, you ought certainly to be able to tackle this as the next step. Very warmly recommended.

And of more recent books covering computability at this level, I also particularly like

15. S. Barry Cooper, *Computability Theory* (Chapman & Hall/CRC 2003). A very nicely done modern textbook. Read at least Part I of the book (about the same level of sophistication as Cutland, but with some extra topics), and then you can press on as far as your curiosity takes you, and get to excitements like the Friedberg-Muchnik theorem.

Of course, the inherited literature on computability is huge. But, being *very* selective, let me mention three classics from different generations:

16. Rósza Péter, *Recursive Functions* (originally published 1950: English translation Academic Press 1967). This is by one of those logicians who was 'there at the beginning'. It has that old-school slow-and-steady unflashy lucidity that makes it still a considerable pleasure to read. It remains very worth looking at.
17. Hartley Rogers, Jr., *Theory of Recursive Functions and Effective Computability* (McGraw-Hill 1967) is a heavy-weight state-of-the-art-then

classic, written at the end of the glory days of the initial development of the logical theory of computation. It quite speedily gets advanced. But the actin-packed opening chapters are excellent. At least take it out of the (e)library, read a few chapters, and admire!

18. Piergiorgio Odifreddi, *Classical Recursion Theory*, Vol. 1 (North Holland, 1989) is well-written and discursive, with numerous interesting asides. It's over 650 pages long, so it goes further and deeper than other books on the main list above (and then there is Vol. 2). But it certainly starts off quite gently paced and very accessible and can be warmly recommended for consolidating and then extending your knowledge.

(d) Classical computability theory abstracts away from considerations of practicality, efficiency, etc. Computer scientists are – surprise, surprise! – interested in the theory of feasible computation, and any logician should be interested in finding out at least a little about the topic of computational complexity. Here are three introductions to the topic, in order of increasing detail:

19. Herbert E. Enderton, *Computability Theory: An Introduction to Recursion Theory* (Associated Press, 2011). Chapter 7.
20. Shawn Hedman *A First Course in Logic* (OUP 2004): Ch. 7 on 'Computability and complexity' has a nice review of basic computability theory before some lucid sections discussing computational complexity.
21. Michael Sipser, *Introduction to the Theory of Computation* (Thomson, 2nd edn. 2006) is a standard and very well regarded text on computation aimed at computer scientists. It aims to be very accessible and to take its time giving clear explanations of key concepts and proof ideas. I think this is very successful as a general introduction and I could well have mentioned the book before. But I'm highlighting the book now because its last third is on computational complexity.

And for more expansive, stand-alone treatments, here are three more suggestions:

22. I don't mention many sets of lecture notes in this Guide, as they tend to be rather too terse for self-study. But Ashley Montanaro has an excellent and extensive lecture notes on *Computational Complexity*, lucid and detailed. Available at tinyurl.com/cocomp.
23. Oded Goldreich, *P, NP, and NP-Completeness* (CUP, 2010). Short, clear, and introductory stand-alone treatment.
24. You could also look at the opening chapters of the pretty encyclopaedic Sanjeev Arora and Boaz Barak *Computational Complexity: A Modern Approach* (CUP, 2009). The authors say that '[r]equiring essentially no background apart from mathematical maturity, the book can be used as a reference for self-study for anyone interested in complexity, including physicists, mathematicians, and other scientists, as well as a textbook for a variety of courses and seminars.' And at least it starts very readably! A late draft of the book can be freely downloaded from tinyurl.com/arora.

12.5 More on mainstream set theory

(a) Some of the readings on set theory suggested in Chapter 7 were beginning to get quite sophisticated: but still, we weren't tangling with more advanced topics like 'large cardinals' and 'forcing'. Now we move on.

And one option is immediately to go for broke and dive in to the modern bible, which is highly impressive not just for its size:

1. Thomas Jech, *Set Theory*, The Third Millennium Edition (Springer, 2003). The book is in three parts: the first, Jech says, every student should know; the second part every budding set-theorist should master; and the third consists of various results reflecting 'the state of the art of set theory at the turn of the new millennium'. Start at page 1 and keep going to page 705 – or until you feel glutted with set theory, whichever comes first!

This book is indeed a masterly achievement by a great expositor. And if you've happily read e.g. the introductory books by Enderton and then Moschovakis mentioned earlier in the Guide, then you should be able to cope pretty well with Part I of the book while it pushes on the story a little with some material on 'small large cardinals' and other topics. Part II of the book starts by telling you about independence proofs. The Axiom of Choice is consistent with ZF and the Continuum Hypothesis is consistent with ZFC, as proved by Gödel using the idea of 'constructible' sets. And the Axiom of Choice is independent of ZF, and the Continuum Hypothesis is independent with ZFC, as proved by Cohen using the much more tricky idea of 'forcing'. The rest of Part II tells you more about large cardinals, and about descriptive set theory. Part III is indeed for enthusiasts.

(b) Now, Jech's book is wonderful, but let's face it, the sheer size makes it a trifle daunting. It goes quite a bit further than many will need, and to get there it in places speeds along a bit faster than some will feel comfortable with. So what other options are there for if you want to take things more slowly?

Let's start with a book which I mentioned in passing in §7.6:

2. Azriel Levy, *Basic Set Theory** (Springer 1979, republished by Dover 2002). This is 'basic' in the sense of not dealing with topics like forcing. However it *is* a quite advanced-level treatment of the set-theoretic fundamentals at least in its mathematical style, and even the earlier parts are I think best tackled once you know some set theory (they could be very useful, though, as a rigorous treatment consolidating the basics – a reader comments that Levy's is his "go to" book when he needs to check set theoretical facts that don't involve forcing or large cardinals.). The last part of the book starts on some more advanced topics.

Levy's book ends with a discussion of some 'large cardinals'. However another much admired older book remains the recommended first treatment of this topic:

3. Frank R. Drake, *Set Theory: An Introduction to Large Cardinals* (North-Holland, 1974). This overlaps with Part I of Jech's bible, though at perhaps a gentler pace. But it also will tell you about Gödel's Constructible Universe and then some more about large cardinals. Very lucid.

For some other topics you could also look at the second volume of a book whose first instalment was a main recommendation in §7.2:

4. Winfried Just and Martin Weese, *Discovering Modern Set Theory II: Set-Theoretic Tools for Every Mathematician* (American Mathematical Society, 1997).

   This contains, as the authors put it, "short but rigorous introductions to various set-theoretic techniques that have found applications outside of set theory". Some interesting topics, and can be read independently of Vol. I.

(c) But now the crucial next step – that perhaps marks the point where set theory gets really challenging – is to get your head around Cohen's idea of forcing used in independence proofs. However, there is not getting away from it, this is tough. In the admirable

5. Timothy Y. Chow, 'A beginner's guide to forcing', tinyurl.com/chowf

Chow writes:

> All mathematicians are familiar with the concept of *an open research problem*. I propose the less familiar concept of *an open exposition problem*. Solving an open exposition problem means explaining a mathematical subject in a way that renders it totally perspicuous. Every step should be motivated and clear; ideally, students should feel that they could have arrived at the results themselves. The proofs should be 'natural' . . . [i.e., lack] any ad hoc constructions or brilliancies. I believe that it is an open exposition problem to explain forcing.

In short: if you find that expositions of forcing – including Chow's – tend to be hard going, then join the club.

Here though is a very widely used and much reprinted textbook, which nicely complements Drake's book and which has (inter alia) a relatively approachable introduction to forcing arguments:

6. Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs* (North Holland, 1980). If you have read (some of) the introductory set theory books mentioned in the Guide, you should actually find much of this text now pretty accessible, and can probably speed through some of the earlier chapters, slowing down later, until you get to the penultimate chapter on forcing which you'll need to take slowly and carefully. This is a rightly admired classic text.

Kunen has since published another, totally rewritten, version of this book as *Set Theory** (College Publications, 2011). This later book is quite significantly longer, covering an amount of more difficult material that has come to prominence since 1980. Not just because of the additional material, my current sense is that the earlier book may remain the somewhat gentler read.

Now, Kunen's classic text takes a 'straight down the middle' approach, starting with what is basically Cohen's original treatment of forcing, though he does relate this to some other approaches. Here are two of them:

7. Raymond Smullyan and Melvin Fitting, *Set Theory and the Continuum Problem* (OUP 1996, Dover Publications 2010). This medium-sized book is divided into three parts. Part I is a nice introduction to axiomatic set theory (in fact, officially in its NBG version – see §12.6). The shorter Part II concerns matters round and about Gödel's consistency proofs via the idea of constructible sets. Part III gives a different take on forcing. This is beautifully done, as you might expect from two writers with a quite enviable knack for wonderfully clear explanations and an eye for elegance.

8. Keith Devlin, *The Joy of Sets* (Springer 1979, 2nd edn. 1993) Ch. 6 introduces the idea of Boolean-Valued Models and their use in independence proofs. The basic idea is fairly easily grasped, but the details perhaps trickier.

   For more on this theme, see John L. Bell's classic *Set Theory: Boolean-Valued Models and Independence Proofs* (Oxford Logic Guides, OUP, 3rd edn. 2005). The relation between this approach and other approaches to forcing is discussed e.g. in Chow's paper and the last chapter of Smullyan and Fitting.

(d) Here is a selection of another four books with various virtues, in order of publication:

9. Akihiro Kanamori, *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings* (Springer, 1997, 2nd edn. 2003). This blockbuster is subtitled 'Large Cardinals in Set Theory from Their Beginnings', and is very clearly put together with a lot of helpful and illuminating historical asides. A classic.

10. Lorenz J. Halbeisen, *Combinatorial Set Theory, With a Gentle Introduction to Forcing* (Springer 2011). From the blurb "This book provides a self-contained introduction to modern set theory and also opens up some more advanced areas of current research in this field. The first part offers an overview of classical set theory wherein the focus lies on the axiom of choice and Ramsey theory. In the second part, the sophisticated technique of forcing, originally developed by Paul Cohen, is explained in great detail. With this technique, one can show that certain statements, like the continuum hypothesis, are neither provable nor disprovable from

the axioms of set theory. In the last part, some topics of classical set theory are revisited and further developed in the light of forcing."

True, this book gets quite hairy towards the end: but the earlier parts of the book should be much more accessible. This book has been strongly recommended for its expositional merits by more reliable judges than me; but I confess I didn't find it notably more successful than other accounts of forcing. A late draft of the book is available: tinyurl.com/halb-set.

11. Nik Weaver, *Forcing for Mathematicians* (World Scientific, 2014) is less than 150 pages (and the first applications of the forcing idea appear after just 40 pages: you don't have to read the whole book to get the basics). From the blurb: "Ever since Paul Cohen's spectacular use of the forcing concept to prove the independence of the continuum hypothesis from the standard axioms of set theory, forcing has been seen by the general mathematical community as a subject of great intrinsic interest but one that is technically so forbidding that it is only accessible to specialists ... This is the first book aimed at explaining forcing to general mathematicians. It simultaneously makes the subject broadly accessible by explaining it in a clear, simple manner, and surveys advanced applications of set theory to mainstream topics." This does strike me as a helpful attempt to solve Chow's basic exposition problem, to explain the Big Ideas very directly.

12. Ralf Schindler, *Set Theory: Exploring Independence and Truth* (Springer, 2014). The book's theme is "the interplay of large cardinals, inner models, forcing, and descriptive set theory". It doesn't presume you already know any set theory, though it does proceed at a cracking pace in a brisk style. But, if you already have some knowledge of set theory, this seems a clear and interesting exploration of some themes highly relevant to current research.

12.6 Choice, and the choice of set theory

But now let's leave the Higher Infinite and other excitements and get back down to earth, or at least to less exotic topics! And, to return to the beginning, we might wonder: is ZFC the 'right' set theory? Indeed, how do we choose which set theory to adopt?

(a) Let's start by thinking about the Axiom of Choice in particular. It is comforting to know from Gödel that AC is consistent with ZF (so adding it doesn't lead to contradiction). But we also know from Cohen's forcing argument that AC is independent with ZF (so accepting ZF doesn't commit you to accepting AC too). So why buy AC? Is it an optional extra?

Quite a few of the readings already mentioned will have touched on the question of AC's status and role. But for a useful overview/revision of some basics, see

1. John L. Bell, 'The axiom of choice', *The Stanford Encyclopedia of Philosophy*, tinyurl.com/sep-axch.

And for a short book also explaining some of the consequences of AC (and some of the results that you need AC to prove), see

2. Horst Herrlich, *Axiom of Choice* (Springer 2006), which has chapters really ather tantalisingly entitled 'Disasters without Choice', 'Disasters with Choice' and 'Disasters either way'.

Herrlich perhaps already tells you more than enough about the impact of AC: but there's also a famous book by H. Rubin and J.E. Rubin, *Equivalents of the Axiom of Choice* (North-Holland 1963; 2nd edn. 1985) worth browsing through: it gives over two hundred equivalents of AC!

Then next there is the nice short classic

3. Thomas Jech, *The Axiom of Choice** (North-Holland 1973, Dover Publications 2008). This proves the Gödel and Cohen consistency and independence results about AC (without bringing into play everything needed to prove the parallel results about the Continuum Hypothesis). In particular, there is a nice presentation of the so-called Fraenkel-Mostowski method of using 'permutation models'. Then later parts of the book tell us something about mathematics without choice, and about alternative axioms that are inconsistent with choice.

And for a more recent short book, taking you into new territories (e.g. making links with category theory), enthusiasts might enjoy

4. John L. Bell, *The Axiom of Choice** (College Publications, 2009).

(b) From earlier reading you should certainly have picked up the idea that, although ZFC is the canonical modern set theory, there are other theories on the market. I mention just a selection here (I'm not suggesting you need to follow up all these pointer – but it is worth stressing again that set theory is not quite the monolithic edifice that some presentations might suggest).

For a brisk overview, putting many of the various set theories we'll consider below into some sort of order, and mentioning yet further alternatives, see

5. M. Randall Holmes, 'Alternative axiomatic set theories', *The Stanford Encyclopedia of Philosophy*, tinyurl.com/alt-set.

At this stage, you might well find this a bit *too* brisk and allusive, but it is useful to give you a preliminary sense of the range of possibilities here. And I should mention that there is a longer version of this essay which you can return to later:

6. M. Randall Holmes, Thomas Forster and Thierry Libert. 'Alternative set theories'. In Dov Gabbay, Akihiro Kanamori, and John Woods, eds. *Handbook of the History of Logic, vol. 6, Sets and Extensions in the Twentieth Century*, pp. 559-632. (Elsevier/North-Holland 2012).

(c) It quickly becomes clear that some alternative set theories are more alternative than others! So let's start with the one which is the closest sibling to standard ZFC, namely NBG. You will have very probably come across mention of this already (e.g. even in the early pages of Enderton's set theory book).

We know that the universe of sets in ZFC is not itself a set. But we might think that this universe is a *sort* of big collection. Should we explicitly recognize, then, two sorts of collection, sets and (as they are called in the trade) proper classes which are too big to be sets? Some standard presentations of ZFC, such as Kunen's, do indeed introduce symbolism for classes, but then make it clear that class-talk is just a useful short-hand that can be translated away. NBG (named for von Neumann, Bernays, Gödel: some say VBG) takes classes a little more seriously. But things are a little delicate: it is a nice question just what NBG commits us to. An important technical feature is that its principle of class comprehension is 'predicative'; i.e. quantified variables in the defining formula for a class can't range over proper classes but range only over sets. Because of this we get a conservative extension of ZFC (nothing in the language of sets can be proved in NBG which can't already be proved in ZFC). For more, see:

7. Abraham Fraenkel, Yehoshua Bar-Hillel and Azriel Levy, *Foundations of Set-Theory* (North-Holland, 2nd edition 1973). Their Ch. II §7 remains a classic general discussion of the role of classes in set theory.

And also worth quickly consulting is

8. Michael Potter, *Set Theory and Its Philosophy* (OUP 2004) Appendix C is a brisker account of NBG and of other theories with classes as well as sets.

Then, if you want detailed presentations of set-theory via NBG, you can see either or both of

9. Elliott Mendelson, *Introduction to Mathematical Logic* (CRC, 4th edition 1997), Ch.4. is a classic and influential textbook presentation.
10. Raymond Smullyan and Melvin Fitting, *Set Theory and the Continuum Problem* (OUP 1996, Dover Publications 2010), Part I is another development of set theory in its NBG version.

(d) Recall, earlier in the Guide, we very warmly recommended Michael Potter's book which we just mentioned again. This presents a version of an axiomatization of set theory due to Dana Scott (hence 'Scott-Potter set theory', SP). This axiomatization is consciously guided by the conception of the set theoretic universe as built up in levels (the conception that, supposedly, also warrants the axioms of ZF). What Potter's book aims to reveal is that we can get a rich hierarchy of sets, more than enough for mathematical purposes, without committing ourselves to *all* of ZFC (whose extreme richness comes from the full Axiom of Replacement). If you haven't read Potter's book before, now is the time to look at it. Also, for a slightly simplified presentation of SP, see

11. Tim Button, 'Level Theory, Part I', *Bulletin of Symbolic Logic*, preprint available at tinyurl.com/level-th.

(e) We now turn to a somewhat more radical depart from standard ZF(C), namely ZFA (i.e. ZF − AF + AFA)

Here again is the now-familiar hierarchical conception of the set universe: We start with some non-sets (maybe zero of them in the case of pure set theory). We collect them into sets (as many different ways as we can). Now we collect what we've already formed into sets (as many as we can). Keep on going, as far as we can. On this 'bottom-up' picture AF, the Axiom of Foundation, is compelling (any downward chain linked by set-membership will bottom out, and won't go round in a circle).

But now here's another alternative conception of the set universe. Think of a set as a gadget that points you at some some things, its members. And those members, if sets, point to *their* members. And so on and so forth. On this 'top-down' picture, the Axiom of Foundation is not so compelling. As we follow the pointers, can't we for example come back to where we started? It is well known that in much of the usual development of ZFC the Axiom of Foundation AF does little work. So what about considering a theory of sets ZFA which drops AF and instead has an Anti-Foundation Axiom, AFA, which allows self-membered sets? To explore this idea, see

12. Start with Lawrence S. Moss, 'Non-wellfounded set theory', *The Stanford Encyclopedia of Philosophy*, tinyurl.com/sep-zfa.
13. Keith Devlin, *The Joy of Sets* (Springer, 2nd edn. 1993), Ch. 7. The last chapter of Devlin's book, added in the second edition of his book, starts with a very lucid introduction, and develops some of the theory.
14. Peter Aczel, *Non-well-founded Sets* (CSLI Lecture Notes 1988). This is a very readable short classic book, available at tinyurl.com/aczel.
15. Luca Incurvati, 'The graph conception of set' *Journal of Philosophical Logic* (2014) pp. 181-208, very illuminatingly explores the motivation for such set theories.

(f) Now for a much more radical departure from ZF.

Standard set theory lacks a universal set because, together with other standard assumptions, the idea that there is a set of all sets leads to contradiction. But by tinkering with those other assumptions, there are coherent theories with universal sets, of which Quine's 'New Foundations' is the probably the best known. For the headline news, see

16. T. F. Forster, 'Quine's New Foundations', *The Stanford Encyclopedia of Philosophy*, tinyurl.com/quine-nf.

For a full-blown but very readable presentation concentrating on NFU ('New Foundations' with urelements), and explaining motivations as well as technical details, see

17. M. Randall Holmes, *Elementary Set Theory with a Universal Set* (Cahiers du Centre de Logique No. 10, Louvain, 1998). Now freely available at tinyurl.com/holmesnf.

The following is rather tougher going, though with many interesting ideas:

18. T. F. Forster, *Set Theory with a Universal Set* Oxford Logic Guides 31 (Clarendon Press, 2nd edn. 1995).

(g) Famously, Zermelo constructed his theory of sets by gathering together some principles of set-theoretic reasoning that seemed actually to be used by working mathematicians (engaged in e.g. the rigorization of analysis or the development of point set topology), hoping to get a theory strong enough for mathematical use while weak enough to avoid paradox. The later Axiom of Replacement was added in much the same spirit. But does the result overshoot? We've already noted that SP is a weaker theory which may suffice. For a more radical approach, see this very engaging short piece:

19. Tom Leinster, 'Rethinking set theory'. Gives an advertising pitch for the merits of Lawvere's Elementary Theory of the Category of Sets (ETCS). tinyurl.com/leinst.

And for more on that, you could see e.g.

20. F. William Lawvere and Robert Rosebrugh, *Sets for Mathematicians* (CUP 2003) gives a presentation which in principle doesn't require that you have already done any category theory. But I suspect that it won't be an easy ride if you know no category theory (and philosophers will find it conceptually puzzling too – what *are* these 'abstract sets' that we are supposedly theorizing about?). In my judgement, to really appreciate what's going on, you will have to start engaging with more category theory. Which is a whole new ball game . . .

(h) I'll finish by briefly mentioning two other directions you could go in!

First, ZF/ZFC has a classical logic: what if we change the logic to intuitionistic logic? what if we have more general constructivist scruples? The place to start exploring is

21. Laura Crosilla, 'Set Theory: Constructive and Intuitionistic ZF', *The Stanford Encyclopedia of Philosophy*, tinyurl.com/crosilla.

Second, you'll recall from elementary model theory that Abraham Robinson developed a rigorous formal treatment that takes infinitesimals seriously. Later, a simpler and arguably more natural approach, based on so-called Internal Set Theory, was invented by Edward Nelson. He advertises it here:

22. Edward Nelson, 'Internal Set Theory: a new approach to nonstandard analysis', *Bulletin of The American Mathematical Society* 83 (1977), pp. 1165–1198. tinyurl.com/nelson-ist.

You can follow that up by looking at the approachable early chapters of Nader Vakin's *Real Analysis through Modern Infinitesimals* (CUP, 2011), a monograph developing Nelson's ideas.

12.7 More proof theory

(a) In §9.5, I mentioned three excellent books which are introductory in intent but which take us a step up from the basic recommendations on proof theory given earlier in Chapter 9, namely Takeuti's *Proof Theory*, Girard's *Proof Theory and Logical Complexity*, and Troelstra and Schwichtenberg's *Basic Proof Theory*. If you didn't take a look at them before, now is the time to do so!.

Also worth reading is the editor's own first contribution to

1. Samuel R. Buss, ed., *Handbook of Proof Theory* (North-Holland, 1998). Later chapters of this very substantial handbook do get pretty hard-core, though yhou might want to look at some of them later. But the 78 pp. opening chapter by Buss himself, a 'Introduction to Proof Theory', is readable, and freely downloadable from tinyurl.com/buss-intro.[1]

(b) And now the paths through proof theory fork. One path investigates what happens when we tinker with the structural rules shared by classical and intuitionistic logic.

Note for example the inference which takes us from the trivial $P \vdash P$ by weakening to $P, Q \vdash P$ and on, via conditional proof, to $P \vdash Q \to P$. If we want a conditional that conforms better to intuitive constraints of relevance, then we need to block that proof: is 'weakening' the culprit? The investigation of what happens if we vary rules such as weakening belongs to 'substructural logic', whose concerns are outlined in

2. Greg Restall, 'Substructural logics', *The Stanford Encyclopedia of Philosophy*, tinyurl.com/sep-subs

And the place to continue exploring these themes at length is the same author's

3. Greg Restall, *An Introduction to Substructural Logics* (Routledge, 2000), which will also teach you a more about proof theory generally in a very accessible way. Do try at least the first seven chapters.

(c) Another path forward picks up from Gentzen's proof of the consistency of arithmetic. Recall, that depends on transfinite induction along ordinals up to ε_0; and the fact that it requires just this much transfinite induction to prove the consistency of first-order PA is an important characterization of the strength of the theory.

The project of 'ordinal analysis' in proof theory aims to provide comparable characterizations of other theories in terms of the amount of transfinite induction

[1] Warning: there are, I am told, some confusing misprints in the cut-elimination proof.

that is needed to prove *their* consistency. Things do get quite hairy quite quickly, however. But you can start from two very useful sets of notes for mini courses:

4. Michael Rathjen, 'The realm of ordinal analysis' and 'Proof theory: from arithmetic to set theory', downloadable from tinyurl.com/rath-art and tinyurl.com/rath-ast.

(d) Finally, here are a couple more books of notable interest:

5. Wolfram Pohlers, *Proof Theory: The First Step into Impredicativity* (Springer 2009). This book officially has introductory ambitions, focusing on ordinal analysis. However, I would judge that it requires quite an amount of mathematical sophistication from its reader. From the blurb: "As a 'warm up' Gentzen's classical analysis of pure number theory is presented in a more modern terminology, followed by an explanation and proof of the famous result of Feferman and Schütte on the limits of predicativity." The first half of the book is probably manageable if (but only if) you already have done some of the other reading. But then the going indeed gets pretty tough.
6. H. Schwichtenberg and S. Wainer, *Proofs and Computations* (Association of Symbolic Logic/CUP 2012) "studies fundamental interactions between proof-theory and computability". The first four chapters, at any rate, will be of wide interest, giving another take on some basic material and should be manageable given enough background. However, to my surprise, I found the book to be not particularly well written and I wonder if it sometimes makes heavier weather of its material than seems really necessary. Still, worth getting to grips with.