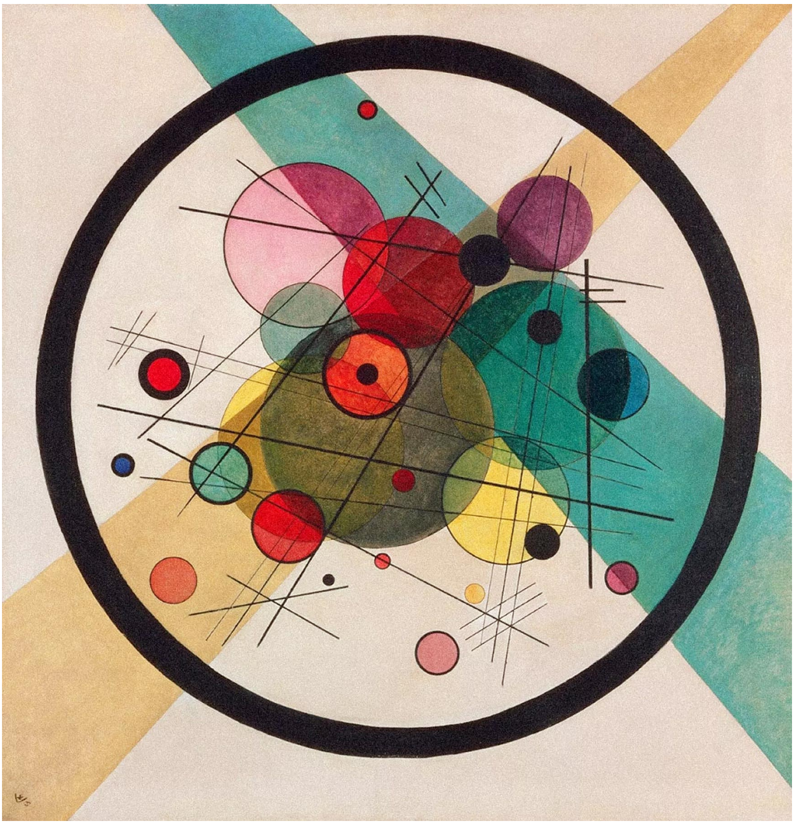
Logic: A Study Guide

Parts II and III, unrevised

Peter Smith



Chapters originally appearing in the TYL 2020 Guide

Pass it on, ... That's the game I want you to learn. Pass it on.

Alan Bennett, *The History Boys*

The Guide's layout is optimized for reading on-screen: try, for example, reading in Books on an iPad, or in double-page mode in Adobe Reader on a laptop.

URLs in blue are live links to external web-pages or PDF documents. Internal cross-references to sections etc. are also live. But to avoid the text being covered in ugly red rashes, internal cross-references are not coloured. If you are reading on a computer, your cursor should change as you hover over such live links: *try here*.

Against the entry for a book, one star means (relatively) cheap, two stars means (legally) free.

The latest version of this Guide can always be downloaded from logicmatters.net/tyl/

The cover image is Wassily Kandinsky, Circles in a Circle, 1923.

Contents

Part II

Other Logics

About Part II

Here's the menu for this Part, which is probably more of interest to philosophers rather than mathematicians:

10 We begin by looking at intuitionist logic – a key form of non-classical logic where we drop the classical principle of excluded middle.

11 We continue with modal logic – like second-order logic, an *extension* of classical logic – for two reasons. First, the basics of modal logic don't involve anything mathematically more sophisticated than the elementary first-order logic covered in Chiswell and Hodges (indeed to make a start on modal logic you don't even need as much as that). Second, and much more importantly, philosophers working in many areas surely *ought* to know a little modal logic.

12 Classical logic demands that all terms denote one and one thing – i.e. it doesn't countenance empty terms which denote nothing, or plural terms which may denote more than one thing. In this section, we look at logics which remain classical in spirit (retaining the usual sort of definition of logical consequence) but which do allow empty and/or plural terms.

13 Among variant logics which are non-classical in spirit, we have already mentioned intutionist logic. Here we consider some other deviations from the classical paradigm, starting with those which require that conclusions be related to their premisses by some connection of *relevance* (so the classical idea that a contradiction entails anything is dropped).

Chapter 10

Intuitionist logic

(a) Ask yourself: Why should we endorse the principle that $\varphi \lor \neg\varphi$ is always true no matter the domain? Even prescinding from issues of vagueness, does the world have to cooperate to determine any given proposition to be true or false? Could there perhaps be domains – mathematics, for example – where truth is in some good sense a matter of provability-in-principle, and falsehood a matter of refutability-in-principle? And if so, would every proposition from such a domain be either true or false, i.e. provable-in-principle or refutable-in-principle? Why so?

Perhaps then we *shouldn't* suppose that the principle that $\varphi \lor \neg\varphi$ always holds true no matter the domain, as a matter of logic. Maybe the law of excluded middle, even when it does hold for some domain, doesn't hold as a matter of pure *logic* but e.g. as a matter of metaphysics or of the specific nature of truth for that domain.

Thoughts like this give rise to one kind of challenge to classical two-valued logic, which of course does assume excluded middle across the board. For more first words on this 'intuitionist' challenge, see e.g. the brief remarks in Theodore Sider's *Logic for Philosophy* (OUP, 2010), §3.5, or in John L. Bell, David DeVidi and Graham Solomon's *Logical Options: An Introduction to Classical and Alternative Logics* (Broadview Press 2001), pp. 192–196.

Then here is an introductory treatment which gives both motivation and a first account of intuitionist logic in a natural deduction framework.

1. Richard Zach et al., *Open Logic Text*, §§49.1–49.4.

Now, in this natural deduction framework, the syntactic shape of intuitionist logic is straightforward. Assuming uncontroversial background, it is easily seen that having all instances of $\varphi \lor \neg\varphi$ as a theorem is equivalent to allowing all instances of the inference rule from $\neg\neg\varphi$, *infer* φ. So, putting it crudely, we just have to drop the usual double negation rule from the rules of your favourite proof system so as to block the derivation of excluded middle.

Things get more complicated when we turn to look at a revised *semantics* apt for a non-classical understanding of the logical operators. The now-standard version due to Kripke is a brand of 'possible-world semantics' of a kind that is also used in modal logic. Philosophers might prefer, therefore, to cover intuitionism after first looking at modal logic more generally. Still, you should have no great difficulty diving straight into

2. Dirk van Dalen, *Logic and Structure**, (Springer, 1980; 5th edition 2012), §§5.1–5.3.

This might usefully be read in tandem with the first 29 pages of

3. Nick Bezhanishvili and Dick de Jongh, *Intuitionistic Logic*.

If, however, you want to approach intuitionistic logic *after* looking at some modal logic, then you could read the appropriate chapters of the terrific

4. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, much expanded 2nd edition 2008). Chs. 6 and 20 on intuitionistic logic respectively flow on naturally from Priest's treatment in that book of modal logics, first propositional and then predicate.

Oddly, Priest's tableaux-based book seems to be one of the few introductory texts covering modal logic which take the natural sideways step of discussing intuitionistic logic too.

(b) One theme not highlighted in these initial readings is that intuitionistic logic, from a more proof-theoretic point of view, seemingly has a certain naturalness compared with classical logic. Suppose we think of the natural deduction introduction rule for a logical operator as fixing the meaning of the operator (rather than a prior semantics fixing what is the appropriate rule). Then the corresponding elimination rule surely ought to be in harmony with the introduction rule, in the sense of just 'undoing' its effect, i.e. giving us back from a wff φ with O as its main operator no more than what an application of the O-introduction rule to justify φ would have to be based on. For this idea of harmony see e.g. Neil Tennant's *Natural Logic*, §4.12. From this perspective the characteristically classical double negation rule is seemingly an outlier, not 'harmonious'. There's now a significant literature on this idea: but for some initial discussion, and pointers to other discussions, you could still usefully start with Peter Milne, 'Classical harmony: rules of inference and the meaning of the logical constants', *Synthese* vol. 100 (1994), pp. 49–94.

For an introduction to intuitionistic logic in a related spirit, see

5. Stephen Pollard, *A Mathematical Prelude to the Philosophy of Mathematics* (Springer, 2014), Ch. 7, 'Intuitionist Logic'.

(c) If you want to pursue things further, both of the following range widely and have a large number of further references:

6. Joan Moschovakis, 'Intuitionistic Logic', *The Stanford Encyclopedia of Philosophy*,

7. Dirk van Dalen, 'Intuitionistic Logic', in the *Handbook of Philosophical Logic*, Vol. 5, ed. by D. Gabbay and F. Guenthner, (Kluwer 2nd edition 2002). This, however, rachets up the level of difficulty, and has parts you will probably want to/need to skip.

Note, by the way, that what we've been talking about is intuitionist *logic* not intuitionist *mathematics*. For more on the relation, see both the SEP entry by Moschovakis and

8. Rosalie Iemhoff, 'Intuitionism in the Philosophy of Mathematics', *The Stanford Encyclopedia of Philosophy*.

And then the stand-out recommendation is

9. Michael Dummett, *Elements of Intuitionism*, Oxford Logic Guides 39 (OUP 2nd edn. 2000). A classic text. But (it has to be said) this is quite hard going in parts, and some of the formal aspects are perhaps tougher than they need be. Note though that thefinal chapter, 'Concluding philosophical remarks', is very well worth looking at, even if you bale out from reading all the formal work that precedes it.

But further investigation of intuitionistic mathematics really needs to be set in the context of a wider engagement with varieties of constructive mathematics.

Chapter 11

Modal logic

(a) Basic modal logic is the logic of the one-place propositional operators '$\Box$' and '$\Diamond$' (read these as 'it is necessarily true that' and 'it is possibly true that'); it adopts new principles like $\Box\varphi \to \varphi$ and $\varphi \to \Diamond\varphi$, and investigates more disputable principles like $\Diamond\varphi \to \Box\Diamond\varphi$. Different readings of the box and diamond generate different modal logics, though initially you can concentrate on just three main systems, known as T, S4 and S5.

There are some nice introductory texts written for philosophers, though I think the place to start is clear:

1. Rod Girle, *Modal Logics and Philosophy* (Acumen 2000; 2nd edn. 2009). Girle's logic courses in Auckland, his enthusiasm and abilities as a teacher, are justly famous. Part I of this book provides a particularly lucid introduction, which in 136 pages explains the basics, covering both trees and natural deduction for some propositional modal logics, and extending to the beginnings of quantified modal logic. Philosophers may well want to go on to read Part II of the book, on applications of modal logic.

Also introductory, though perhaps a little brisker than Girle at the outset, is

2. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, much expanded 2nd edition 2008): read Chs 2–4 for propositional modal logics, Chs 14–18 for quantified logics. This book – which is a terrific achievement and enviably clear and well-organized – systematically explores logics of a wide variety of kinds, using trees throughout in a way that

can be very illuminating indeed. Although it starts from scratch, however, it would be better to come to the book with a prior familiarity with non-modal logic via trees, as in my chapters available here. We will be mentioning Priest's book again in later sections for its excellent coverage of other non-classical themes.

If you do start with Priest's book, then at some point you will want to supplement it by looking at a treatment of natural deduction proof systems for modal logics. One option is to dip into Tony Roy's comprehensive 'Natural Derivations for Priest, *An Introduction to Non-Classical Logic*' which presents natural deduction systems corresponding to the propositional logics presented in tree form in the first edition of Priest (so the first half of the new edition). Another possible way in to ND modal systems would be via the opening chapters of

3. James Garson, *Modal Logic for Philosophers** (CUP, 2006; 2nd end. 2014). This again is certainly intended as a gentle introductory book: it deals with both ND and semantic tableaux (trees), and covers quantified modal logic. It is reasonably accessible, but not – I think – as attractive as Girle. But that's a judgement call.

(b) We now go a step up in sophistication (and the more mathematical might want to start here):

4. Melvin Fitting and Richard L. Mendelsohn, *First-Order Modal Logic* (Kluwer 1998). This book starts again from scratch, but then does go rather more snappily, with greater mathematical elegance (though it should certainly be accessible to anyone who is modestly on top of non-modal first-order logic). It still also includes a good amount of philosophically interesting material. Recommended.

A few years ago, I would have said that getting as far as Fitting and Mendelsohn will give most philosophers a good enough grounding in basic modal logic. But e.g. Timothy Williamson's book *Modal Logic as Metaphysics* (OUP, 2013) calls on rather more, including e.g. second-order modal logics. If you need to sharpen your knowledge of the technical background here, I guess there is nothing for it but to tackle

5. Nino B. Cocchiarella and Max A. Freund, *Modal Logic: An Introduction to its Syntax and Semantics* (OUP, 2008). The blurb announces that "a variety of modal logics at the sentential, first-order, and second-order levels

> are developed with clarity, precision and philosophical insight". However, when I looked at this book with an eye to using it for a graduate seminar, I confess I didn't find it very appealing: so I do suspect that many philosophical readers will indeed find the treatments in this book rather relentless. However, the promised wide coverage could make the book of particular interest to determined philosophers concerned with the kind of issues that Williamson discusses.

Finally, I should certainly draw your attention to the classic book by Boolos mentioned at the end of §14.4, where modal logic gets put to use in exploring results about provability in arithmetic, Gödel's Second Incompleteness Theorem, and more.

Postscript for philosophers Old hands learnt their modal logic from G. E. Hughes and M. J. Cresswell *An Introduction to Modal Logic* (Methuen, 1968). This was at the time of original publication a unique book, enormously helpfully bringing together a wealth of early work on modal logic in an approachable way. Nearly thirty years later, the authors wrote a heavily revised and updated version, *A New Introduction to Modal Logic* (Routledge, 1996). This newer version like the original one concentrates on *axiomatic* versions of modal logic, which doesn't make it always the most attractive introduction from a modern point of view. But it is still an admirable book at an introductory level (and going beyond), and a book that enthusiasts can still learn from.

I didn't recommend the first part of Theodore Sider's *Logic for Philosophy** (OUP, 2010). However, the second part of the book which is entirely devoted to modal logic (including quantified modal logic) and related topics like Kripke semantics for intuitionistic logic is significantly better. Compared with the early chapters with their inconsistent levels of coverage and sophistication, the discussion here develops more systematically and at a reasonably steady level of exposition. There is, however, a lot of (acknowledged) straight borrowing from Hughes and Cresswell, and – like those earlier authors – Sider also gives axiomatic systems. But if you just want a brisk and pretty clear explanation of Kripke semantics, and want to learn e.g. how to search systematically for countermodels, Sider's treatment in his Ch. 6 could well work as a basis. And then the later treatments of quantified modal logic in Chs 9 and 10 (and some of the conceptual issues they raise) are also brief, lucid and approachable.

Postscript for the more mathematical Here are a couple of good introductory modal logic books with a mathematical flavour:

6. Sally Popkorn, *First Steps in Modal Logic* (CUP, 1994). The author is, at least in this possible world, identical with the late mathematician Harold Simmons. This book, which entirely on propositional modal logics, is written for computer scientists. The Introduction rather boldly says 'There are few books on this subject and even fewer books worth looking at. None of these give an acceptable mathematically correct account of the subject. This book is a first attempt to fill that gap.' This considerably oversells the case: but the result is illuminating and readable.

7. Also just on propositional logic, I'd recommend Patrick Blackburn, Maarten de Ricke and Yde Venema's *Modal Logic* (CUP, 2001). This is one of the Cambridge Tracts in Theoretical Computer Science: but again don't let that provenance put you off – it is (relatively) accessibly and agreeably written, with a lot of signposting to the reader of possible routes through the book, and interesting historical notes. I think it works pretty well, and will certainly give you an idea about how non-philosophers approach modal logic.

Going in a different direction, if you are particularly interested in the relation between modal logic and intuitionistic logic (see §10), then you might want to look at

> Alexander Chagrov and Michael Zakharyaschev *Modal Logic* (OUP, 1997). This is a volume in the Oxford Logic Guides series and again concentrates on propositional modal logics. Written for the more mathematically minded reader, it tackles things in an unusual order, starting with an extended discussion of intuitionistic logic, and is pretty demanding. But enthusiasts should find this enlightening.

Finally, if you want to explore even more, there's the giant *Handbook of Modal Logic*, edited by van Bentham et al. (Elsevier, 2005). You can get an idea of what's in the volume by looking at this page of links to the opening pages of the various contributions.

Chapter 12

Free logic, plural logic

We next look at what happens if you stay first-order (keep your variables running over objects), stay classical in spirit (keep the same basic notion of logical consequence) but allow terms that fail to denote (free logic) or allow terms that refer to more than one thing (plural logic).

12.1 Free Logic

Classical logic assumes that any term denotes an object in the domain of quantification, and in particular assumes that all functions are total, i.e. defined for every argument – so an expression like '$f(c)$' always denotes. But mathematics cheerfully countenances partial functions, which may lack a value for some arguments. Should our logic accommodate this, by allowing terms to be free of existential commitment? In which case, what would such a 'free' logic look like?

For some background and motivation, see the gently paced and accessible

1. David Bostock, *Intermediate Logic* (OUP 1997), Ch. 8.

Then for more detail, we have a helpful overview article in the ever-useful Stanford Encyclopedia:

2. John Nolt, 'Free Logic', *The Stanford Encyclopedia of Philosophy.*

For formal treatments in, respectively, natural deduction and tableau settings, see:

3. Neil Tennant, *Natural Logic*** (Edinburgh UP 1978, 1990), §7.10.
4. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, 2nd edition 2008), Ch. 13.

If you want to explore further (going rather beyond the basics), you could make a start on

5. Ermanno Bencivenga, 'Free Logics', in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic, vol. III: Alternatives to Classical Logic* (Reidel, 1986). Reprinted in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, 2nd edition, vol. 5 (Kluwer, 2002).

Postscript Rolf Schock's *Logics without Existence Assumptions* (Almqvist & Wiskell, Stockholm 1968) is still well worth looking at on free logic after all this time. And for a collection of articles of interest to philosophers, around and about the topic of free logic, see Karel Lambert, *Free Logic: Selected Essays* (CUP 2003).

12.2 Plural logic

In ordinary mathematical English we cheerfully use plural denoting terms such as '2, 4, 6, and 8', 'the natural numbers', 'the real numbers between 0 and 1', 'the complex solutions of $z^2 + z + 1 = 0$', 'the points where line L intersects curve C', 'the sets that are not members of themselves', and the like. Such locutions are entirely familiar, and we use them all the time without any sense of strain or logical impropriety. We also often generalize by using plural quantifiers like 'any natural numbers' or 'some reals' together with linked plural pronouns such as 'they' and 'them'. For example, here is a version of the Least Number Principle: given any natural numbers, one of them must be the least. By contrast, there are some reals – e.g. those strictly between 0 and 1 – such that no one of *them* is the least.

Plural terms and plural quantifications appear all over the place in everyday mathematical argument. It is surely a project of interest to logicians to regiment and evaluate the informal modes of argument involving such constructions. This is the business of plural logic, a topic of much recent discussion.

For an introduction, see

1. Øystein Linnebo, 'Plural Quantification', *The Stanford Encyclopedia of*

Philosophy.

And do read at least these two of the key papers listed in Linnebo's expansive bibliography:

2. Alex Oliver and Timothy Smiley, 'Strategies for a Logic of Plurals', *Philosophical Quarterly* (2001) pp. 289–306.
3. George Boolos, 'To Be Is To Be a Value of a Variable (or to Be Some Values of Some Variables)', *Journal of Philosophy* (1984) pp. 430–50. Reprinted in Boolos, *Logic, Logic, and Logic* (Harvard University Press, 1998).

(Oliver and Smiley give reasons why there is indeed a real subject here: you can't readily eliminate all plural talk in favour e.g. of singular talk about sets. Boolos's classic will tell you something about the possible relation between plural logic and second-order logic.) Then, for much more about plurals, you can follow up more of Linnebo's bibliography, or could look at

4. Thomas McKay, *Plural Predication* (OUP 2006),

which is clear and approachable. However:

Real enthusiasts for plural logic will want to dive into the philosophically argumentative and formally rich

5. Alex Oliver and Timothy Smiley, *Plural Logic* (OUP 2013: revised and expanded second edition, 2016).

Chapter 13

Relevance logics (and wilder logics too)

(a) Classically, if $\varphi \vdash \psi$, then $\varphi, \chi \vdash \psi$ (read '$\vdash$' as 'entails': irrelevant premisses can be added without making a valid entailment invalid). And if $\varphi, \chi \vdash \psi$ then $\varphi \vdash \chi \to \psi$ (that's the Conditional Proof rule in action, a rule that seems to capture something essential to our understanding of the conditional). Presumably we have $P \vdash P$. So we have $P, Q \vdash P$. Whence $P \vdash Q \to P$. It seems then that classical logic's carefree attitude to questions of relevance in deduction and its dubious version of the conditional are tied closely together.

Classically, we also have $\varphi, \neg\varphi \vdash \psi$. But doesn't the inference from P and $\neg P$ to Q commit another fallacy of relevance? And again, if we allow it and also allow conditional proof, we will have $\neg P \vdash P \to Q$, another seemingly unhappy result about the conditional.

Can we do better? What does a more relevance-aware logic look like?

For useful introductory reading, see

1. Edwin Mares, 'Relevance Logic', *The Stanford Encyclopedia of Philosophy.*
2. Graham Priest, Koji Tanaka, Zach Weber, 'Paraconsistent logic', *The Stanford Encyclopedia of Philosophy.*

These two articles have many pointers for further reading across a range of topics, so I can be brief here. But I will mention two wide-ranging introductory texts:

3. Graham Priest, *An Introduction to Non-Classical Logic** (CUP, much expanded 2nd edition 2008). Look now at Chs. 7–10 for a treatment of propositional logics of various deviant kinds. Priest starts with relevance logic and goes on to also treat logics where there are truth-value gaps, and – more wildly – logics where a proposition can be both true and false (there's a truth-value glut), Then, if this excites you, carry on to look at Chs. 21–24 where the corresponding quantificational logics are presented. This book really is a wonderful resource.
4. J. C. Beall and Bas van Fraassen's *Possibilities and Paradox* (OUP 2003), also covers a range of logics. In particular, Part III of the book covers relevance logic and also non-standard logics involving truth-value gaps and truth-value gluts. (It is worth looking too at the earlier parts of the book on logical frameworks generally and on modal logic.)

The obvious next place to go is then the very lucid

5. Edwin Mares, *Relevant Logic: A Philosophical Interpretation* (CUP 2004). As the title suggests, this book has very extensive conceptual discussion alongside the more formal parts elaborating what might be called the mainstream tradition in relevance logics.

(b) Note, however, that although they get discussed in close proximity in the books by Priest and by Beall and van Fraasen, there's no tight connection between (i) the reasonable desire to have a more relevance-aware logic (e.g. without the principle that a contradiction implies everything) and (ii) the highly revisionary proposal that there can be propositions which are both true and false at the same time.

At the risk of corrupting the youth, if you are interested in exploring the latter immodest proposal further, then I can point you to

6. Graham Priest, 'Dialetheism', *The Stanford Encyclopedia of Philosophy.*

(c) There is however a minority tradition on relevance that I myself find extremely appealing, developed by Neil Tennant, initially in scattered papers.

Classically, we can unrestrictedly paste proofs together – so can e.g. paste together an uncontroversial proof that for the inference $P \therefore P \vee Q$ and a proof for the inference $\neg P, P \vee Q \therefore Q$ to give us a proof for the (dubious) inference $P, \neg P \therefore Q$. But maybe what is getting us into trouble is pasting together proofs with premisses which explicitly contradict each other. What if we restrict that? You will need to know a little proof theory to appreciate how Tennant handles this thought – though you can get a flavour of the approach from the early programmatic paper

6. Neil Tennant, 'Entailment and proofs', *Proceedings of the Aristotelian Society* LXXIX (1979) 167–189.

Tennant wrote a sequence of interesting follow-up papers over the next decades, but he has now brought everything neatly together in his argumentative and technically engaging

7. Neil Tennant, *Core Logic* (OUP, 2017).

Part III

Beyond the basics

About Part III

In this third part of the Guide there are suggestions for more advanced reading on the various areas of logic we have already touched on in Part II. As noted in Chapter 16, there are yet further areas of logic which the Guide does not (yet!) cover: e.g. type theory, the lambda calculus, infinitary logics. But sufficient unto the day!

Three points before we begin:

- Before tackling the more difficult material in the next two chapters, it could be very well worth first taking the time to look at one or two of the wider-ranging Big Books on mathematical logic which will help consolidate your grip on the basics at the level of Part I and/or push things on just a bit. See the slowly growing set of Book Notes for some guidance on what's available.

- I did try to be fairly systematic in Part I, aiming to cover the different core areas at comparable levels of detail, depth and difficulty. The coverage of various topics from here on is more varied: the recommendations can be many or few (or non-existent!) depending on my own personal interests and knowledge.

- I do, however, still aim to cluster suggestions within sections or subsections in rough order of difficulty. In this Part, boxes are used set off a number of acknowledged classics that perhaps any logician ought to read one day, whatever their speciality.

And a warning to those philosophers still reading: some of the material I point to is inevitably mathematically quite demanding!

Chapter 14

More advanced reading on some core topics

In this chapter, there are some suggestions for more advanced reading on a selection of topics in and around the core mathematical logic curriculum we looked at in Chs. 4 and 5 – other than set theory, which we return to at length in the next chapter.

14.1 Proof theory

Proof theory has been (and continues to be) something of a poor relation in the standard Mathematical Logic curriculum: the usual survey textbooks don't discuss it. Yet this is a fascinating area, of interest to philosophers, mathematicians, and computer scientists who all *ought* to be concerned with the notion of proof! So let's start to fill this gap next.

(a) I mentioned in §**??** the introductory book by Jan von Plato, *Elements of Logical Reasoning** (CUP, 2014), which approaches elementary logic with more of an eye on proof theory than is at all usual: you might want to take a look at that book if you didn't before. However, you should start serious work by reading the same author's extremely useful encyclopaedia entry:

1. Jan von Plato, 'The development of proof theory', *The Stanford Encyclopedia of Philosophy.*

This will give you orientation and introduce you to some main ideas: there is also an excellent bibliography which you can use to guide further exploration.

That biblio perhaps makes the rest of this section a bit redundant; but for what they are worth, here are my less informed suggestions. Everyone will agree that you should certainly read the little hundred-page classic

2. Dag Prawitz, *Natural Deduction: A Proof-Theoretic Study** (Originally published in 1965: reprinted Dover Publications 2006).

And if you want to follow up in more depth Prawitz's investigations of the proof theory of various systems of logic, the next place to look is surely

3. Sara Negri and Jan von Plato, *Structural Proof Theory* (CUP 2001). This is a modern text which is neither too terse, nor too laboured, and is generally very clear. When we read it in a graduate-level reading group, we did find we needed to pause sometimes to stand back and think a little about the motivations for various technical developments. So perhaps a few more 'classroom asides' in the text would have made a rather good text even better. But this is still *extremely* helpful.

Then in a more mathematical style, there is the editor's own first contribution to

4. Samuel R. Buss, ed., *Handbook of Proof Theory* (North-Holland, 1998). Later chapters of this very substantial handbook do get pretty hard-core; but the 78 pp. opening chapter by Buss himself, a 'Introduction to Proof Theory'**, is readable, and freely downloadable. (Student health warning: there are, I am told, some confusing misprints in the cut-elimination proof.)

(b) And now the path through proof theory forks. In one direction, the path cleaves to what we might call classical themes (I don't mean themes simply concerning classical logic, as intuitionistic logic was also treated as central from the start: I mean themes explicit in the early classic papers in proof theory, in particular in Gentzen's work). It is along this path that we find e.g. Gentzen's famous proof of the consistency of first-order Peano Arithmetic using proof-theoretic ideas. One obvious text on these themes remains

5. Gaisi Takeuti, *Proof Theory** (North-Holland 1975, 2nd edn. 1987: reprinted Dover Publications 2013). This is a true classic – if only because for a while it was about the only available book on most of its topics. Later chapters won't really be accessible to beginners. But you could/should try reading Ch. 1 on logic, §§1–7 (and perhaps the beginnings of §8, pp. 40–45, which is easier than it looks if you compare how

you prove the completeness of a tree system of logic). Then on Gentzen's proof, read Ch. 2, §§9–11 and §12 up to at least p. 114. This isn't exactly plain sailing – but if you skip and skim over some of the more tedious proof-details you can pick up a very good basic sense of what happens in the consistency proof.

Gentzen's proof of the consistency of depends on transfinite induction along ordinals up to ε_0; and the fact that it requires just so much transfinite induction to prove the consistency of first-order PA is an important characterization of the strength of the theory. The project of 'ordinal analysis' in proof theory aims to provide comparable characterizations of other theories in terms of the amount of transfinite induction that is needed to prove *their* consistency. Things do get quite hairy quite quickly, however.

6. For a glimpse ahead, you could look at (initial segments of) these useful notes for mini-courses by Michael Rathjen, on 'The Realm of Ordinal Analysis' and 'Proof Theory: From Arithmetic to Set Theory'.

Turning back from these complications, however, let's now glance down the other path from the fork, where we investigate not the proof theory of theories constructed in familiar logics but rather investigate non-standard logics themselves. Reflection on the structural rules of classical and intuitionistic proof systems naturally raises the question of what happens when we tinker with these rules. We noted before the inference which takes us from the trivial $P \vdash P$ by 'weakening' to $P, Q \vdash P$ and on, via 'conditional proof', to $P \vdash Q \rightarrow P$. If we want a conditional that conforms better to intuitive constraints of relevance, then we need to block that proof: is 'weakening' the culprit? The investigation of what happens if we tinker with standard structural rules such as weakening belongs to substructural logic, outlined in

7. Greg Restall, 'Substructural Logics', *The Stanford Encyclopedia of Philosophy*.

(which again has an admirable bibliography). And the place to continue exploring these themes at length is the same author's splendid

8. Greg Restall, *An Introduction to Substructural Logics* (Routledge, 2000), which will also teach you a lot more about proof theory generally in a very accessible way. Do read at least the first seven chapters.

(You could note again here the work on Neil Tennant mentioned at the very end of Chapter 13.)

(c) For the more mathematically minded, here are a few more books of considerable interest. I'll start with a couple that in fact aim to be accessible to beginners. They wouldn't be my recommendations of texts to start from, but they could be very useful if you already know a bit of proof theory.

9. Jean-Yves Girard, *Proof Theory and Logical Complexity. Vol. I* (Bibliopolis, 1987) is intended as an introduction]. With judicious skipping, which I'll signpost, this is readable and insightful, though some proofs are a bit arm-waving.

   So: skip the 'Foreword', but do pause to glance over 'Background and Notations' as Girard's symbolic choices need a little explanation. Then the long Ch. 1 is by way of an introduction, proving Gödel's two incompleteness theorems and explaining 'The Fall of Hilbert's Program': if you've read some of the recommendations in §**??** above, you can probably skim this fairly quickly, though noting Girard's highlighting of the notion of 1-consistency.

   Ch. 2 is on the sequent calculus, proving Gentzen's *Hauptsatz*, i.e. the crucial cut-elimination theorem, and then deriving some first consequences (you can probably initially omit the forty pages of annexes to this chapter). Then also omit Ch. 3 whose content isn't relied on later. But Ch. 4 on 'Applications of the *Hauptsatz*' is crucial (again, however, at a first pass you can skip almost 60 pages of annexes to the chapter). Take the story up again with the first two sections of Ch. 6, and then tackle the opening sections of Ch. 7. A rather bumpy ride but very illuminating.

   (Vol. II of this book was never published: though there are some draft materials here.)

10. A. S. Troelstra and H. Schwichtenberg's *Basic Proof Theory* (CUP 2nd ed. 2000) is a volume in the series Cambridge Tracts in Computer Science. Now, one theme that runs through the book indeed concerns the computer-science idea of formulas-as-types and invokes the lambda calculus: however, it is in fact possible to skip over those episodes in you aren't familiar with the idea. The book, as the title indicates, is intended as a first foray into proof theory, and it *is* reasonably approachable. However it is perhaps a little cluttered for my tastes because it spends quite a bit of time looking at slightly different ways of doing natural deduction and slightly different ways of doing the sequent calculus, and the differences may matter more for computer scientists with implementation concerns than for others. You could, however, with a bit of skipping, very usefully read just Chs. 1–3, the

first halves of Chs. 4 and 6, and then Ch. 10 on arithmetic again.

And now for three more advanced offerings, worth commenting on:

11. I have already mentioned the compendium edited by Samuel R. Buss, *Handbook of Proof Theory* (North-Holland, 1998), and the fact that you can download its substantial first chapter. You can also freely access Ch. 2 on 'First-Order Proof-Theory of Arithmetic'. Later chapters of the Handbook are of varying degrees of difficulty, and cover a range of topics (though there isn't much on ordinal analysis).

12. Wolfram Pohlers, *Proof Theory: The First Step into Impredicativity* (Springer 2009). This book has introductory ambitions, to say something about so-called ordinal analysis in proof theory as initiated by Gentzen. But in fact I would judge that it requires quite an amount of mathematical sophistication from its reader. From the blurb: "As a 'warm up' Gentzen's classical analysis of pure number theory is presented in a more modern terminology, followed by an explanation and proof of the famous result of Feferman and Schütte on the limits of predicativity." The first half of the book is probably manageable if (but only if) you already have done some of the other reading. But then the going indeed gets pretty tough.

13. H. Schwichtenberg and S. Wainer, *Proofs and Computations* (Association of Symbolic Logic/CUP 2012) "studies fundamental interactions between proof-theory and computability". The first four chapters, at any rate, will be of wide interest, giving another take on some basic material and should be manageable given enough background. Sadly, I found the book to be not particularly well written and it sometimes makes heavier weather of its material than seems really necessary. Still worth the effort though.

There is a recent more introductory text by Katalin Bimbó, *Proof Theory: Sequent Calculi and Related Formalisms* (CRC Press, 2014); but having looked at it, I'm not minded to recommend this.

14.2 Beyond the model-theoretic basics

(a) If you want to explore model theory beyond the entry-level material in §**??**, why not start with a quick warm-up, with some reminders of headlines and some useful pointers to the road ahead:

1. Wilfrid Hodges, 'First-order model theory', *The Stanford Encyclopedia of Philosophy.*

Now, we noted before that e.g. the wide-ranging texts by Hedman and Hinman eventually cover a substantial amount of model theory. But you will do even better with two classic stand-alone treatments of the area which really choose themselves. Both in order of first publication and of eventual difficulty we have:

2. C. Chang and H. J. Keisler, *Model Theory** (originally North Holland 1973: the third edition has been inexpensively republished by Dover Books in 2012). This is the Old Testament, the first systematic text on model theory. Over 550 pages long, it proceeds at an engagingly leisurely pace. It is particularly lucid and is extremely nicely constructed with different chapters on different methods of model-building. A really fine achievement that still remains a good route in to the serious study of model theory.
3. Wilfrid Hodges, *A Shorter Model Theory* (CUP, 1997). The New Testament is Hodges's encyclopedic original *Model Theory* (CUP 1993). This shorter version is half the size but still really full of good things. It does get tougher as the book progresses, but the earlier chapters of this modern classic, written with this author's characteristic lucidity, should certainly be readily manageable.

My suggestion would be to read the first three long chapters of Chang and Keisler, and then perhaps pause to make a start on

4. J. L. Bell and A. B. Slomson, *Models and Ultraproducts** (North-Holland 1969; Dover reprint 2006). Very elegantly put together: as the title suggests, the book focuses particularly on the ultra-product construction. At this point read the first five chapters for a particularly clear introduction.

You could then return to Ch. 4 of C&K to look at (some of) their treatment of the ultra-product construction, before perhaps putting the rest of their book on hold and turning to Hodges.

(b) A level up again, here are two more books. The first has been around long enough to have become regarded as a modern standard text. The second is more recent but also comes well recommended. Their coverage is significantly different – so those wanting to get seriously into model theory should probably take a look at both:

5. David Marker, *Model Theory: An Introduction* (Springer 2002). Despite its title, this book would surely be hard going if you haven't already tackled some model theory (at least read Manzano first). But despite being sometimes a rather bumpy ride, this highly regarded text will teach you a great deal. Later chapters, however, probably go far over the horizon for all except those most enthusiastic readers of this Guide who are beginning to think about specializing in model theory – it isn't published in the series 'Graduate Texts in Mathematics' for nothing!

6. Katrin Tent and Martin Ziegler, *A Course in Model Theory* (CUP, 2012). From the blurb: "This concise introduction to model theory begins with standard notions and takes the reader through to more advanced topics such as stability, simplicity and Hrushovski constructions. The authors introduce the classic results, as well as more recent developments in this vibrant area of mathematical logic. Concrete mathematical examples are included throughout to make the concepts easier to follow." Again, although it starts from the beginning, it could be a bit of challenge to readers without any prior exposure to the elements of model theory – though I, for one, find it more approachable than Marker's book.

(c) So much for my principal suggestions. Now for an assortment of additional/alternative texts. Here are two more books which aim to give general introductions:

7. Philipp Rothmaler's *Introduction to Model Theory* (Taylor and Francis 2000) is, overall, comparable in level of difficulty with, say, the first half of Hodges. As the blurb puts it: "This text introduces the model theory of first-order logic, avoiding syntactical issues not too relevant to model theory. In this spirit, the compactness theorem is proved via the algebraically useful ultraproduct technique (rather than via the completeness theorem of first-order logic). This leads fairly quickly to algebraic applications, ..." Now, the opening chapters are indeed very clear: but oddly the introduction of the crucial ultraproduct construction in Ch. 4 is done very briskly (compared, say, with Bell and Slomson). And thereafter it seems to me that there is some unevenness in the accessibility of the book. But others have recommended this text, so I mentioned it as a possibility worth checking out.

8. Bruno Poizat's *A Course in Model Theory* (English edition, Springer 2000) starts from scratch and the early chapters give an interesting and helpful

account of the model-theoretic basics, and the later chapters form a rather comprehensive introduction to stability theory. This often-recommended book is written in a rather distinctive style, with rather more expansive class-room commentary than usual: so an unusually engaging read at this sort of level.

Another book which is often mentioned in the same breath as Poizat, Marker, and now Tent and Ziegler as a modern introduction to model theory is *A Guide to Classical and Modern Model Theory*, by Annalisa Marcja and Carlo Toffalori (Kluwer, 2003) which also covers a lot: but I prefer the previously mentioned books.

(d) The next two suggestions are of books which are helpful on particular aspects of model theory:

9. Kees Doets's short *Basic Model Theory** (CSLI 1996) highlights so-called Ehrenfeucht games. This is enjoyable and very instructive.

10. Chs. 2 and 3 of Alexander Prestel and Charles N. Delzell's *Mathematical Logic and Model Theory: A Brief Introduction* (Springer 1986, 2011) are brisk but clear, and can be recommended if you wanting a speedy review of model theoretic basics. The key feature of the book, however, is the sophisticated final chapter on applications to algebra, which might appeal to mathematicians with special interests in that area. For a very little more on this book, see my Book Note.

Indeed, as we explore model theory, we quickly get entangled with algebraic questions. And as well as going (so to speak) in the direction from logic to algebra, we can make connections the other way about, starting from algebra. For something on this approach, see the following short, relatively accessible, and illuminating book:

11. Donald W. Barnes and John M. Mack, *An Algebraic Introduction to Mathematical Logic* (Springer, 1975).

(e) As an aside, let me also briefly allude to the sub-area of Finite Model Theory which arises particularly from consideration of problems in the theory of computation (where, of course, we are interested in *finite* structures – e.g. finite databases and finite computations over them). What happens, then, to model theory if we restrict our attention to finite models? Trakhtenbrot's theorem, for example, tells that the class of sentences true in any finite model is not recursively enumerable. So there is no deductive theory for capturing such finitely valid sentences (that's a surprise, given that there's a complete deductive system for the valid sentences!). It turns out, then, that the study of finite models is surprisingly rich and interesting (at least for enthusiasts!). So why not dip into one or other of

12. Leonard Libkin, *Elements of Finite Model Theory* (Springer 2004).
13. Heinz-Dieter Ebbinghaus and Jörg Flum, *Finite Model Theory* (Springer 2nd edn. 1999).

Either is a very good standard text to explore the area with, though I prefer Libkin's.

(f) Three afterthoughts. First, it is illuminating to read something about the history of model theory: there's a good, and characteristically lucid, unpublished piece by a now-familiar author here:

14. W. Hodges, 'Model Theory'.

Second, one thing you will have noticed if you tackle a few texts beyond the level of Manzano's is that the absolutely key compactness theorem (for example) can be proved in a variety of ways – indirectly via the completeness proof, via a more direct Henkin construction, via ultraproducts, etc. How do these proofs inter-relate? Do they generalize in different ways? Do they differ in explanatory power? For a quite excellent essay on this – on the borders of mathematics and philosophy (and illustrating that there is indeed very interesting work to be done in that border territory), see

15. Alexander Paseau, 'Proofs of the Compactness Theorem', *History and Philosophy of Logic* 31 (2001): 73–98.

Finally, I suppose that I should mention John T. Baldwin's *Model Theory and the Philosophy of Mathematical Practice* (CUP, 2018). This presupposes a lot more background than the excellent book by Button and Walsh mentioned in §**??**. A few philosophers might be able to excavate more out of this than I did: but – as far as I read into it – I found this book badly written and unnecessarily hard work.

14.3 Computability

In §**??** we took a first look at the related topics of computability, Gödelian incompleteness, and theories of arithmetic. In this and the next two main sections, we return to these topics, taking them separately (though this division is necessarily somewhat artificial).

14.3.1 Computable functions

(a) Among the Big Books on mathematical logic, the one with the most useful treatment of computability is probably

1. Peter G. Hinman, *Fundamentals of Mathematical Logic* (A. K. Peters, 2005). Chs. 4 and 5 on recursive functions, incompleteness etc. strike me as the best written, most accessible (and hence most successful) chapters in this very substantial book. The chapters could well be read after my *IGT* as somewhat terse revision for mathematicians, and then as sharpening the story in various ways. Ch. 8 then takes up the story of recursion theory (the author's home territory).

However, good those these chapters are, I'd still recommend starting your more advanced work on computability with

2. Nigel Cutland, *Computability: An Introduction to Recursive Function Theory* (CUP 1980). This is a rightly much-reprinted classic and is beautifully lucid and well-organized. This *does* have the look-and-feel of a traditional maths text book of its time (so with fewer of the classroom asides we find in some modern, more discursive books). However, if you got through most of e.g. Boolos, Burgess and Jeffrey without too much difficulty, you ought certainly to be able to tackle this as the next step. Very warmly recommended.

And of more recent books covering computability this level (i.e. a step up from the books mentioned in §**??**, I also particularly like

3. S. Barry Cooper, *Computability Theory* (Chapman & Hall/CRC 2003). This is a very nicely done modern textbook. Read at least Part I of the book (about the same level of sophistication as Cutland, but with some extra topics), and then you can press on as far as your curiosity takes you, and get to excitements like the Friedberg-Muchnik theorem.

(b) The inherited literature on computability is huge. But, being *very* selective, let me mention three classics from different generations:

4. Rósza Péter, *Recursive Functions* (originally published 1950: English translation Academic Press 1967). This is by one of those logicians who was 'there at the beginning'. It has that old-school slow-and-steady un-flashy lucidity that makes it still a considerable pleasure to read. It remains very worth looking at.

5. Hartley Rogers, Jr., *Theory of Recursive Functions and Effective Computability* (McGraw-Hill 1967) is a heavy-weight state-of-the-art-then classic, written at the end of the glory days of the initial development of the logical theory of computation. It quite speedily gets advanced. But the opening chapters are still excellent reading and are action-packed. At least take it out of the library, read a few chapters, and admire!

6. Piergiorgio Odifreddi, *Classical Recursion Theory*, Vol. 1 (North Holland, 1989) is well-written and discursive, with numerous interesting asides. It's over 650 pages long, so it goes further and deeper than other books on the main list above (and then there is Vol. 2). But it certainly starts off quite gently paced and very accessible and can be warmly recommended for consolidating and extending your knowledge.

(c) A number of books we've already mentioned say something about the fascinating historical development of the idea of computability: as we noted before, Richard Epstein offers a very helpful 28 page timeline on 'Computability and Undecidability' at the end of the 2nd edn. of Epstein/Carnielli (see §**??**). Cooper's short first chapter on 'Hilbert and the Origins of Computability Theory' also gives some of the headlines. Odifreddi too has many historical details. But here are two more good essays on the history:

7. Robert I. Soare, 'The History and Concept of Computability', in E. Griffor, ed., *Handbook of Computability Theory* (Elsevier 1999).

8. Robin Gandy, 'The Confluence of Ideas in 1936' in R. Herken, ed., *The Universal Turing Machine: A Half-century Survey* (OUP 1988). Seeks to explain why so many of the absolutely key notions all got formed in the mid-thirties.

14.3.2 Computational complexity

Computer scientists are – surprise, surprise! – interested in the theory of feasible computation, and any logician should be interested in finding out at least a little about the topic of computational complexity.

1. Shawn Hedman *A First Course in Logic* (OUP 2004): Ch. 7 on 'Computability and complexity' has a nice review of basic computability theory before some lucid sections discussing computational complexity.

2. Michael Sipser, *Introduction to the Theory of Computation* (Thomson, 2nd edn. 2006) is a standard and very well regarded text on computation aimed at computer scientists. It aims to be very accessible and to take its time giving clear explanations of key concepts and proof ideas. I think this is very successful as a general introduction and I could well have mentioned the book before. But I'm highlighting the book in this subsection because its last third is on computational complexity.

3. Ofed Goldreich, *P, NP, and NP-Completeness* (CUP, 2010). Short, clear, and introductory stand-alone treatment of computational complexity.

4. Ashley Montanaro, *Computational Complexity*. Excellent 2012 lecture notes, lucid and detailed and over 100 pages (also include a useful quick guide to further reading).

5. You could also look at the opening chapters of the pretty encyclopaedic Sanjeev Arora and Boaz Barak *Computational Complexity: A Modern Approach*** (CUP, 2009). The authors say 'Requiring essentially no background apart from mathematical maturity, the book can be used as a reference for self-study for anyone interested in complexity, including physicists, mathematicians, and other scientists, as well as a textbook for a variety of courses and seminars.' And it at least starts very readably. A late draft of the book can be freely downloaded.

14.4 Incompleteness and related matters

(a) If you have looked at my book and/or Boolos and Jeffrey you should now be in a position to appreciate the terse elegance of

1. Raymond Smullyan, *Gödel's Incompleteness Theorems*, Oxford Logic Guides 19 (Clarendon Press, 1992). This is delightully short – under 140 pages – proving some beautiful, slightly abstract, versions of the in-

completeness theorems. This is a modern classic which anyone with a taste for mathematical elegance will find rewarding.

2. Equally short and equally elegant is Melvin Fitting's, *Incompleteness in the Land of Sets** (College Publications, 2007). This approaches things from a slightly different angle, relying on the fact that there is a simple correspondence between natural numbers and 'hereditarily finite sets' (i.e. sets which have a finite number of members which in turn have a finite number of members which in turn ... where all downward membership chains bottom out with the empty set).

In terms of difficulty, these two lovely brief books could easily have appeared among our introductory readings in Chapter **??**. I have put them here because (as I see it) the simpler, more abstract, stories they tell can probably only be fully appreciated if you've first met the basics of computability theory and the incompleteness theorems in a more conventional treatment.

You ought also at some stage read an even briefer, and still officially introductory, treatment of the incompleteness theorems,

3. Craig Smoryński, 'The incompleteness theorems' in J. Barwise, editor, *Handbook of Mathematical Logic*, pp. 821–865 (North-Holland, 1977), which covers a lot very compactly.

After these, where should you go if you want to know more about matters more or less directly to do with the incompleteness theorems?

4. Raymond Smullyan's *Diagonalization and Self-Reference*, Oxford Logic Guides 27 (Clarendon Press 1994) is an investigation-in-depth around and about the idea of diagonalization that figures so prominently in proofs of limitative results like the unsolvability of the halting problem, the arithmetical undefinability of arithmetical truth, and the incompleteness of arithmetic. Read at least Part I.

5. Torkel Franzén, *Inexaustibility: A Non-exhaustive Treatment* (Association for Symbolic Logic/A. K. Peters, 2004). The first two-thirds of the book gives another take on logic, arithmetic, computability and incompleteness. The last third notes that Gödel's incompleteness results have a positive consequence: 'any system of axioms for mathematics that we recognize as correct can be properly extended by adding as a new axiom a formal statement expressing that the original system is consistent. This suggests that

our mathematical knowledge is inexhaustible, an essentially philosophical topic to which this book is devoted.' Not always easy (you will need to know something about ordinals before you read this), but very illuminating.

6. Per Lindström, *Aspects of Incompleteness* (Association for Symbolic Logic/ A. K. Peters, 2nd edn., 2003). This is probably for enthusiasts. A terse book, not always reader-friendly in its choices of notation and the brevity of its argument, but the more mathematical reader will find that it again repays the effort.

7. Craig Smoryński, *Logical Number Theory I, An Introduction* (Springer, 1991). There are three long chapters. Ch. I discusses pairing functions and numerical codings, primitive recursion, the Ackermann function, computability, and more. Ch. II concentrates on 'Hilbert's tenth problem' – showing that we can't mechanically decide the solubility of certain equations. Ch. III considers Hilbert's Programme and contains proofs of more decidability and undecidability results, leading up to a version of Gödel's First Incompleteness Theorem. (The promised Vol. II which would have discussed the Second Incompleteness Theorem has never appeared.)

   The level of difficulty is rather varied, and there are a lot of historical disgressions and illuminating asides. So this is an idiosyncratic book, a bumpy ride, but is still an enjoyable and very instructive read.

(b) Going in a rather different direction, you will recall from my *IGT2* or other reading on the second incompleteness theorem that we introduced the so-called derivability conditions on $\Box\varphi$ where this is an abbreviation for (or at any rate, is closely tied to) $\mathsf{Prov}(\ulcorner\varphi\urcorner)$, which expresses the claim that the wff φ, whose Gödel number is $\ulcorner\varphi\urcorner$, is provable in some given theory. The '$\Box$' here functions rather like a modal operator: so what is its modal logic? This is investigated in:

8. George Boolos, *The Logic of Provability* (CUP, 1993). From the blurb: "What [the author] does is to show how the concepts, techniques, and methods of modal logic shed brilliant light on the most important logical discovery of the twentieth century: the incompleteness theorems of Kurt Gödel and the 'self-referential' sentences constructed in their proof. The book explores the effects of reinterpreting the notions of necessity and possibility to mean provability and consistency." This is a wonderful modern classic.

14.5 Theories of arithmetic

The readings in §?? will have introduced you to the canonical first-order theory of arithmetic, first-order Peano Arithmetic, as well as to some subsystems of PA (in particular, Robinson Arithmetic) and second-order extensions. And you will already know that first-order PA has non-standard models (in fact, it even has uncountably many non-isomorphic models which can be built out of natural numbers!).

So what to read next? You should get to more about models of PA. For a taster, you could look at these nice lecture notes:

1. Jaap van Oosten, 'Introduction to Peano Arithmetic: Gödel Incompleteness and Nonstandard Models'. (1999)

But for a fuller story, you need

2. Richard Kaye's *Models of Peano Arithmetic* (Oxford Logic Guides, OUP, 1991) which tells us a great deal about non-standard models of PA. This will reveal more about what PA can and can't prove, and will introduce you to some non-Gödelian examples of incompleteness. This does get pretty challenging in places, and it is probably best if you've already done a little model theory. Still, this is a terrific book, and deservedly a modern classic.

(There's also another volume in the Oxford Logic Guides series which can be thought of as a sequel to Kaye's for real enthusiasts with more background in model theory, namely Roman Kossak and James Schmerl, *The Structure of Models of Peano Arithmetic*, OUP, 2006. But this is much tougher.)

Next, going in a rather different direction, and explaining a lot about arithmetics weaker than full PA, here's another modern classic:

3. Petr Hájek and Pavel Pudlák, *Metamathematics of First-Order Arithmetic*** (Springer 1993). Now freely available from projecteuclid.org. This is pretty encyclopaedic, but the long first three chapters, say, actually do remain surprisingly accessible for such a work. This is, eventually, a must-read if you have a serious interest in theories of arithmetic and incompleteness.

And what about going beyond first-order PA? We know that full second-order PA (where the second-order quantifiers are constrained to run over all possible sets of numbers) is unaxiomatizable, because the underlying second-order logic is unaxiomatiable. But there are axiomatizable subsystems of second order arithmetic. These are wonderfully investigated in another encyclopaedic modern classic:

4. Stephen Simpson, *Subsystems of Second-Order Logic* (Springer 1999; 2nd edn CUP 2009). The focus of this book is the project of 'reverse mathematics' (as it has become known): that is to say, the project of identifying the weakest theories of numbers-and-sets-of-numbers that are required for proving various characteristic theorems of classical mathematics.

   We know that we can reconstruct classical analysis in pure set theory, and rather more neatly in set theory with natural numbers as unanalysed 'urelemente'. But just *how much* set theory is needed to do the job, once we have the natural numbers? The answer is: stunningly little. The project of exploring what's needed is introduced very clearly and accessibly in the first chapter, which is a must-read for anyone interested in the foundations of mathematics. This introduction is freely available at the book's website.

Chapter 15

Serious set theory

In §**??**, we gave suggestions for readings on the elements of set theory. These will have introduced you to the standard set theory ZFC, and the iterative hierarchy it seeks to describe. They also explained e.g. how we can construct the real number system in set theoretic terms (so giving you a sense of what might be involved in saying that set theory can be used as a 'foundation' for another mathematical theory). You will have in addition learnt something about the role of the axiom of choice, and about the arithmetic of infinite cardinal and ordinal numbers.

If you looked at the books by Fraenkel/Bar-Hillel/Levy or by Potter, however, you will also have noted that while standard ZFC is the market leader, it is certainly not the only set theory on the market.

So where do we go next? We'll divide the discussion into three.

- We start by focusing again on our canonical theory, ZFC. The exploration eventually becomes seriously hard mathematics – and, to be honest, it becomes of pretty specialist interest (very well beyond 'what every logician ought to know'). But it isn't clear where to stop in a Guide like this, even if I have no doubt overdone it!

- Next we backtrack from those excursions towards the frontiers to consider old questions about the Axiom of Choice (as this is of particular conceptual and mathematical interest).

- Then we will say something about non-standard set theories, rivals to ZFC (again, the long-recognised possibility of different accounts, with different degrees of departure from the canonical theory, is of considerable conceptual interest and you don't need a huge mathematical background to understand some of the options).

15.1 ZFC, with all the bells and whistles

15.1.1 A first-rate overview

One option is immediately to go for broke and dive in to the modern bible, which is highly impressive not just for its size:

1. Thomas Jech, *Set Theory*, The Third Millennium Edition, Revised and Expanded (Springer, 2003). The book is in three parts: the first, Jech says, every student should know; the second part every budding set-theorist should master; and the third consists of various results reflecting 'the state of the art of set theory at the turn of the new millennium'. Start at page 1 and keep going to page 705 (or until you feel glutted with set theory, whichever comes first).

   This is indeed a masterly achievement by a great expositor. And if you've happily read e.g. the introductory books by Enderton and then Moschovakis mentioned earlier in the Guide, then you should be able to cope pretty well with Part I of the book while it pushes on the story a little with some material on small large cardinals and other topics. Part II of the book starts by telling you about independence proofs. The Axiom of Choice is consistent with ZF and the Continuum Hypothesis is consistent with ZFC, as proved by Gödel using the idea of 'constructible' sets. And the Axiom of Choice is independent of ZF, and the Continuum Hypothesis is independent with ZFC, as proved by Cohen using the much more tricky idea of 'forcing'. The rest of Part II tells you more about large cardinals, and about descriptive set theory. Part III is indeed for enthusiasts.

Now, Jech's book is wonderful, but let's face it, the sheer size makes it a trifle daunting. It goes quite a bit further than many will need, and to get there it does in places speed along a bit faster than some will feel comfortable with. So what other options are there for if you want to take things more slowly?

15.1.2 Rather more slowly, towards forcing

(a) Why not start with some preliminary historical orientation. If you looked at the old book by Fraenkel/Bar-Hillel/Levy which was recommended earlier in the Guide, then you will already know something of the early days. Alternatively, there is a nice short introductory overview

2. José Ferreirós, 'The early development of set theory', *The Stanford Encycl. of Philosophy.*

I should mention that Ferreirós has also written a book *Labyrinth of Thought: A History of Set Theory and its Role in Modern Mathematics* (Birkhäuser 1999). But I found this rather heavy going, though your mileage may vary.

You could also browse through the substantial article

3. Akhiro Kanamori, 'The Mathematical Development of Set Theory from Cantor to Cohen', *The Bulletin of Symbolic Logic* (1996) pp. 1-71, a revised version of which is downloadable here. (You will very probably need to skip chunks of this at a first pass: but even a partial grasp will help give you a good sense of the lie of the land for when you work on the technicalities.)

(b) The divide between the 'entry level' books on set theory discussed in §**??** and the more advanced books we are considering in this chapter is rather artificial, of course. Where, for example, should we place this classic?

4. Azriel Levy, *Basic Set Theory* (Springer 1979, republished by Dover 2002). This is 'basic' in the sense of not dealing with topics like forcing. However it *is* a quite advanced-level treatment of the set-theoretic fundamentals at least in its mathematical style, and even the earlier parts are I think best tackled once you know some set theory (they could be very useful, though, as a rigorous treatment consolidating the basics – a reader comments that Levy's is his "go to" book when he needs to check set theoretical facts that don't involve forcing or large cardinals.). The last part of the book starts on some more advanced topics, including various real spaces, and finally treats some infinite combinatorics and 'large cardinals'.

However, a much admired older book remains the recommended first treatment of its topic:

5. Frank R. Drake, *Set Theory: An Introduction to Large Cardinals* (North-Holland, 1974). This overlaps with Part I of Jech's bible, though at perhaps a gentler pace. But it also will tell you about Gödel's Constructible Universe and then some more about large cardinals. Very lucid.

(c) But now the crucial next step – that perhaps marks the point where set theory gets really challenging – is to get your head around Cohen's idea of forcing used in independence proofs. However, there is not getting away from it, this is tough. In the admirable

6. Timothy Y. Chow, 'A beginner's guide to forcing',

(and don't worry if initially even this beginner's guide looks puzzling), Chow writes

> All mathematicians are familiar with the concept of *an open research problem*. I propose the less familiar concept of *an open exposition problem*. Solving an open exposition problem means explaining a mathematical subject in a way that renders it totally perspicuous. Every step should be motivated and clear; ideally, students should feel that they could have arrived at the results themselves. The proofs should be 'natural' ... [i.e., lack] any ad hoc constructions or brilliancies. I believe that it is an open exposition problem to explain forcing.

In short: if you find that expositions of forcing tend to be hard going, then join the club.

Here though is a very widely used and much reprinted textbook, which nicely complements Drake's book and which has (inter alia) a pretty good first presentation of forcing:

7. Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs* (North-Holland, 1980). If you have read (some of) the introductory set theory books mentioned in the Guide, you should actually find much of this text now pretty accessible, and can probably speed through some of the earlier chapters, slowing down later, until you get to the penultimate chapter on forcing which you'll need to take slowly and carefully. This is a rightly admired classic text.

Kunen has since published another, totally rewritten, version of this book as *Set Theory** (College Publications, 2011). This later book is quite significantly longer, covering an amount of more difficult material that has come to prominence since 1980. Not just because of the additional material, my current sense is that the earlier book may remain the slightly more approachable read.

15.1.3 Pausing for problems

At this point mathematicians could very usefully dip into the problem sets in the excellent

8. Péver Komjáte and Vilmos Totik, *Problems and Theorems in Classical Set Theory* (Springer, 2006). From the blurb: "Most of classical set theory is

covered, classical in the sense that independence methods are not used, but classical also in the sense that most results come from the period between 1920–1970. Many problems are also related to other fields of mathematics .... Rather than using drill exercises, most problems are challenging and require work, wit, and inspiration." Look at the problems that pique your interest: the authors give answers, often very detailed.

15.1.4 Pausing for more descriptive set-theory

Early on, it was discovered that the Axiom of Choice implied the existence of 'pathological' subsets of the reals, sets lacking desirable properties like being measurable. In reaction, there developed the study of nice, well-behaved, 'definable' sets – the topic of descriptive set theory. This has already been touched on in e.g. Kunen's book. For more see e.g.

9. Yiannis Moschovakis, *Descriptive Set Theory* (North Holland, 1980: second edition AMS, 2009, freely available from the author's website).

10. Alexander Kechris, *Classical Descriptive Set Theory* (Springer-Verlag, 1994).

15.1.5 Forcing further explored

To return, though, to the central theme of independence proofs and other results that can be obtained be forcing: Kunen's classic text takes a 'straight down the middle' approach, starting with what is basically Cohen's original treatment of forcing, though he does relate this to some variant approaches. Here are two of them:

11. Raymond Smullyan and Melvin Fitting, *Set Theory and the Continuum Problem* (OUP 1996, Dover Publications 2010). This medium-sized book is divided into three parts. Part I is a nice introduction to axiomatic set theory (in fact, officially in its NBG version – see §15.3). The shorter Part II concerns matters round and about Gödel's consistency proofs via the idea of constructible sets. Part III gives a different take on forcing (a variant of the approach taken in Fitting's earlier *Intuitionistic Logic, Model Theory, and Forcing*, North Holland, 1969). This is beautifully done, as you might expect from two writers with a quite enviable knack for wonderfully clear explanations and an eye for elegance.

12. Keith Devlin, *The Joy of Sets* (Springer 1979, 2nd edn. 1993) Ch. 6 introduces the idea of Boolean-Valued Models and their use in independence proofs. The basic idea is fairly easily grasped, but details perhaps get hairy. For more on this theme, see John L. Bell's classic *Set Theory: Boolean-Valued Models and Independence Proofs* (Oxford Logic Guides, OUP, 3rd edn. 2005). The relation between this approach and other approaches to forcing is discussed e.g. in Chow's paper and the last chapter of Smullyan and Fitting.

Here are three further, more recent, books which highlight forcing ideas, one very short, the others much more wide-ranging:

10. Nik Weaver, *Forcing for Mathematicians* (World Scientific, 2014) is less than 150 pages (and the first applications of the forcing idea appear after just 40 pages: you don't have to read the whole book to get the basics). From the blurb: "Ever since Paul Cohen's spectacular use of the forcing concept to prove the independence of the continuum hypothesis from the standard axioms of set theory, forcing has been seen by the general mathematical community as a subject of great intrinsic interest but one that is technically so forbidding that it is only accessible to specialists ... This is the first book aimed at explaining forcing to general mathematicians. It simultaneously makes the subject broadly accessible by explaining it in a clear, simple manner, and surveys advanced applications of set theory to mainstream topics." And this does strike me as a clear and very helpful attempt to solve Chow's basic exposition problem.

11. Lorenz J. Halbeisen, *Combinatorial Set Theory, With a Gentle Introduction to Forcing* (Springer 2011, with a late draft freely downloadable from the author's website). From the blurb "This book provides a self-contained introduction to modern set theory and also opens up some more advanced areas of current research in this field. The first part offers an overview of classical set theory wherein the focus lies on the axiom of choice and Ramsey theory. In the second part, the sophisticated technique of forcing, originally developed by Paul Cohen, is explained in great detail. With this technique, one can show that certain statements, like the continuum hypothesis, are neither provable nor disprovable from the axioms of set theory. In the last part, some topics of classical set theory are revisited and further developed in the light of forcing." True, this book gets quite hairy towards the end: but the earlier parts of the book should be more accessible. This book has been strongly recommended for its expositional

merits by more reliable judges than me; but I confess I wasn't entirely convinced when I settled down to work through it.

12. Ralf Schindler, *Set Theory: Exploring Independence and Truth* (Springer, 2014). The book's theme is "the interplay of large cardinals, inner models, forcing, and descriptive set theory". It doesn't presume you already know any set theory, though it does proceed at a cracking pace in a brisk style. But, if you already have some knowledge of set theory, this seems a clear and interesting exploration of some themes highly relevant to current research.

15.1.6 The higher infinite

And then what next? You want more?? Back to finish Jech's doorstep of a book, perhaps. And then – oh heavens! – there is another older blockbuster still awaiting you:

13. Akihiro Kanamori, *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings* (Springer, 1997, 2nd edn. 2003).

15.2 The Axiom of Choice

But now let's leave the Higher Infinite and get back down to earth, or at least to less exotic mathematics! In fact, to return to the beginning, we might wonder: is ZFC the 'right' set theory?

Start by thinking about the Axiom of Choice in particular. It is comforting to know from Gödel that AC is consistent with ZF (so adding it doesn't lead to contradiction). But we also know from Cohen's forcing argument that AC is independent with ZF (so accepting ZF doesn't commit you to accepting AC too). So why buy AC? Is it an optional extra?

Quite a few of the readings already mentioned will have touched on the question of AC's status and role. But for an overview/revision of some basics, see

1. John L. Bell, 'The Axiom of Choice', *The Stanford Encyclopedia of Philosophy.*

And for a short book also explaining some of the consequences of AC (and some of the results that you need AC to prove), see

2. Horst Herrlich, *Axiom of Choice* (Springer 2006), which has chapters rather tantalisingly entitled 'Disasters without Choice', 'Disasters with Choice' and 'Disasters either way'.

Herrlich perhaps already tells you more than enough about the impact of AC: but there's also a famous book by H. Rubin and J.E. Rubin, *Equivalents of the Axiom of Choice* (North-Holland 1963; 2nd edn. 1985) worth browsing through: it gives over two hundred equivalents of AC! Then next there is the nice short classic

3. Thomas Jech, *The Axiom of Choice** (North-Holland 1973, Dover Publications 2008). This proves the Gödel and Cohen consistency and independence results about AC (without bringing into play everything needed to prove the parallel results about the Continuum Hypothesis). In particular, there is a nice presentation of the so-called Fraenkel-Mostowski method of using 'permutation models'. Then later parts of the book tell us something about mathematics without choice, and about alternative axioms that are inconsistent with choice.

And for a more recent short book, taking you into new territories (e.g. making links with category theory), enthusiasts might enjoy

4. John L. Bell, *The Axiom of Choice** (College Publications, 2009).

15.3 Other set theories?

From earlier reading you should have picked up the idea that, although ZFC is the canonical modern set theory, there are other theories on the market. I mention just a selection here (I'm not suggesting you follow up all these – the point is to stress that set theory is not quite the monolithic edifice that some presentations might suggest).

For a brisk overview, putting many of the various set theories we'll consider below into some sort of order (and mentioning yet further alternatives) see

1. M. Randall Holmes, 'Alternative axiomatic set theories', *The Stanford Encyclopedia of Philosophy.*

At this stage, you might well find this *too* brisk and allusive, but it is useful to give you a preliminary sense of the range of possibilities here.

NBG You will have come across mention of this already (e.g. even in the early pages of Enderton's set theory book). And in fact – in many of the respects that matter – it isn't really an 'alternative' set theory. So let's get it out of the way first. We know that the universe of sets in ZFC is not itself a set. But we might think that this universe is a *sort* of big collection. Should we explicitly recognize, then, two sorts of collection, sets and (as they are called in the trade) proper classes which are too big to be sets? NBG (named for von Neumann, Bernays, Gödel: some say VBG) is one such theory of collections. So NBG in *some* sense recognizes proper classes, objects having 'members' but that cannot be members of other entities: but in some sense, these classes are merely virtual objects. NBG's principle of class comprehension is predicative; i.e. quantified variables in the defining formula can't range over proper classes but range only over sets, and we get a conservative extension of ZFC (nothing in the language of sets can be proved in NBG which can't already be proved in ZFC).

1. Abraham Fraenkel, Yehoshua Bar-Hillel and Azriel Levy, *Foundations of Set-Theory* (North-Holland, 2nd edition 1973), Ch. II §7 remains a classic general discussion of the role of classes in set theory.

2. Michael Potter, *Set Theory and Its Philosophy* (OUP 2004) Appendix C is a brisker account of NBG and on other theories with classes as well as sets.

For detailed presentations of set-theory via NBG, you can see either or both of

3. Elliott Mendelson, *Introduction to Mathematical Logic* (CRC, 4th edition 1997), Ch.4. is a classic and influential textbook presentation.

4. Raymond Smullyan and Melvin Fitting, *Set Theory and the Continuum Problem* (OUP 1996, Dover Publications 2010), Part I is another development of set theory in its NBG version.

SP This again is by way of reminder. Recall, earlier in the Guide, we very warmly recommended Michael Potter's book which we just mentioned again. This presents a version of an axiomatization of set theory due to Dana Scott (hence 'Scott-Potter set theory'). This axiomatization is consciously guided by the conception of the set theoretic universe as built up in levels (the conception that, supposedly, also warrants the axioms of ZF). What Potter's book aims to reveal is that we can get a rich hierarchy of sets, more than enough for mathematical purposes, without committing ourselves to *all* of ZFC (whose extreme richness comes from the full Axiom of Replacement). If you haven't read Potter's book before, now is the time to look at it.

ZFA (i.e. ZF − AF + AFA) Here again is the now-familiar hierarchical conception of the set universe: We start with some non-sets (maybe zero of them in the case of pure set theory). We collect them into sets (as many different ways as we can). Now we collect what we've already formed into sets (as many as we can). Keep on going, as far as we can. On this 'bottom-up' picture, the Axiom of Foundation is compelling (any downward chain linked by set-membership will bottom out, and won't go round in a circle). But now here's another alternative conception of the set universe. Think of a set as a gadget that points you at some some things, its members. And those members, if sets, point to *their* members. And so on and so forth. On this 'top-down' picture, the Axiom of Foundation is not so compelling. As we follow the pointers, can't we for example come back to where we started? It is well known that in much of the usual development of ZFC the Axiom of Foundation AF does little work. So what about considering a theory of sets which drops AF and instead has an Anti-Foundation Axiom (AFA), which allows self-membered sets? To explore this idea,

1. Start with Lawrence S. Moss, 'Non-wellfounded set theory', *The Stanford Encycl. of Philosophy.*

2. Keith Devlin, *The Joy of Sets* (Springer, 2nd edn. 1993), Ch. 7. The last chapter of Devlin's book, added in the second edition of his book, starts with a very lucid introduction, and develops some of the theory.

3. Peter Aczel's, *Non-well-founded sets*, (CSLI Lecture Notes 1988) is a very readable short classic book.

4. Luca Incurvati, 'The graph conception of set' *Journal of Philosophical Logic* (2014) pp. 181-208, very illuminatingly explores the motivation for such set theories.

NF Now for a much more radical departure from ZF. Standard set theory lacks a universal set because, together with other standard assumptions, the idea that there is a set of all sets leads to contradiction. But by tinkering with those other assumptions, there are coherent theories with universal sets. For a readable presentation concentrating on Quine's NFU ('New Foundations' with urelements), and explaining motivations as well as technical details, see

1. M. Randall Holmes, *Elementary Set Theory with a Universal Set*** (Cahiers du Centre de Logique No. 10, Louvain, 1998). Now freely available here.

The following is rather tougher going, though with some interesting ideas:

2. T. F. Forster, *Set Theory with a Universal Set* Oxford Logic Guides 31 (Clarendon Press, 2nd edn. 1995).

ETCS Famously, Zermelo constructed his theory of sets by gathering together some principles of set-theoretic reasoning that seemed actually to be used by working mathematicians (engaged in e.g. the rigorization of analysis or the development of point set topology), hoping to get a theory strong enough for mathematical use while weak enough to avoid paradox. The later Axiom of Replacement was added in much the same spirit. But does the result overshoot? We've already noted that SP is a weaker theory which may suffice. For a more radical approach, see

1. Tom Leinster, 'Rethinking set theory', gives an advertising pitch for the merits of Lawvere's Elementary Theory of the Category of Sets, and ...

2. F. William Lawvere and Robert Rosebrugh, *Sets for Mathematicians* (CUP 2003) gives a presentation which in principle doesn't require that you have already done any category theory. But I suspect that it won't be an easy ride if you know no category theory (and philosophers will find it conceptually puzzling too – what *are* these 'abstract sets' that we are supposedly theorizing about?). In my judgement, to really appreciate what's going on, you will have to start engaging with more category theory.

IZF, CZF ZF/ZFC has a classical logic: what if we change the logic to intuitionistic logic? what if we have more general constructivist scruples? The place to start exploring is

1. Laura Crosilla, 'Set Theory: Constructive and Intuitionistic ZF', *The Stanford Encyclopedia of Philosophy.*

Then for one interesting possibility, look at the version of constructive ZF in

2. Peter Aczel and Michael Rathjen, *Constructive Set Theory* (Draft, 2010).

IST Leibniz and Newton invented infinitesimal calculus in the 1660s: a century and a half later we learnt how to rigorize the calculus without invoking infinitely small quantities. Still, the idea of infinitesimals retains a certain intuitive appeal, and in the 1960s, Abraham Robinson created a theory of hyperreal numbers: this yields a rigorous formal treatment of infinitesimal calculus (you will have seen this mentioned in e.g. Enderton's *Mathematical Introduction to Logic*, §2.8, or van Dalen's *Logic and Structure*, p. 123). Later, a simpler and arguably more natural approach, based on so-called Internal Set Theory, was invented by Edward Nelson. As put it, 'IST is an extension of Zermelo-Fraenkel set theory in that alongside the basic binary membership relation, it introduces a new unary predicate 'standard' which can be applied to elements of the mathematical universe together with some axioms for reasoning with this new predicate.' Starting in this way we can recover features of Robinson's theory in a simpler framework.

1. Edward Nelson, 'Internal set theory: a new approach to nonstandard analysis' *Bulletin of The American Mathematical Society* 83 (1977), pp. 1165–1198.

2. Nader Vakin, *Real Analysis through Modern Infinitesimals* (CUP, 2011). A monograph developing Nelson's ideas whose early chapters are quite approachable and may well appeal to some.

Yet more? Well yes, we *can* keep on going. Take a look, for example, at SEAR. But we must call a halt! Though you could round things out by taking a look at a piece that could be thought of as an expanded version of Randall Holmes's *Stanford Encyclopedia* piece that we mentioned at the the beginning of this section:

2. M. Randall Holmes, Thomas Forster and Thierry Libert. 'Alternative Set Theories'. In Dov Gabbay, Akihiro Kanamori, and John Woods, eds. *Handbook of the History of Logic, vol. 6, Sets and Extensions in the Twentieth Century*, pp. 559-632. (Elsevier/North-Holland 2012).

Chapter 16

What else?

16.1 Missing topics!

Mathematical logicians and philosophers interested in the philosophy of maths will want to know about yet more areas that fall outside the traditional math logic curriculum. For example:

- It is worth knowing about at least core aspects of *type theory*;
- Relatedly, we should explore something of the *the lambda calculus*.
- Even in elementary model theory we relax the notion of a language to allow e.g. for uncountably many names: what if we further relax and allow for e.g. sentences which are infinite conjunctions? Pursuing such questions leads us to consider *infinitary logics*.

These areas – in particular, the first of them – may or may not get a section in a later version of this Guide.

16.2 Category theory

But there is one more topic I should mention here. For if set theory traditionally counts as part of mathematical logic, because of its generality, breadth and foundational interest, then there is surely an argument for including some too.

So that topic *does* get its own supplementary webpage, including a reading list for philosophers, and links to a lot of freely available material, and a link to my own rough-and-ready work-in-progress towards some useful notes.

Index of authors