The Philosophical Significance of Tennenbaum's Theorem

Some First Thoughts, a Moral and a Conjecture

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October 31, 2011

- The Tennenbaum Argument
- The Initial Segment Argument
- The Arguments simply rebutted
- Do the rebuttals miss the point?
- Philosophical problems, philosophical solutions

Fixing ideas, fixing notation

- Assume first-order Peano Arithmetic *PA* is presented with primitive non-logical vocabulary: '0', '1', '$+$', '$\times$', '$<$'. [Choice of primitives is non-critical.]
- As usual, put '$\overline{n}$' for '$1 + 1 + \ldots + 1$' ($n > 1$ summands).
- The standard model of *PA* is a structure $\mathcal{N} = \langle \mathbb{N}, 0, 1, +, \times, < \rangle$.
- More generally, a model of *PA* is a structure $\mathcal{M} = \langle M, 0^{\mathcal{M}}, 1^{\mathcal{M}}, \oplus, \otimes, \lessdot \rangle$
- Put $n^{\mathcal{M}} = 1^{\mathcal{M}} \oplus 1^{\mathcal{M}} \oplus \ldots \oplus 1^{\mathcal{M}}$ ($n > 1$ summands).
- Without loss of generality, we can consider any countable model *PA* to be a structure $\langle \mathbb{N}, 0, 1, \oplus, \otimes, \lessdot \rangle$. [We can take the domain to be $\mathbb{N}$ as we are only interested in identifying countable models up to isomorphism. And if we initially choose a non-standard zero and one, we can just permute them with 'real' zero and one.]

Tennenbaum's Theorem

- Observation: If $\mathcal{M}$ is a countable model $\langle \mathbb{N}, 0, 1, \oplus, \otimes, \lessdot \rangle$, it makes sense to ask whether the functions $\oplus$ and $\otimes$ are recursive.
- Theorem [Tennenbaum]: If $\mathcal{M}$ is a countable model of *PA* and either $\oplus$ or $\otimes$ is recursive, then $\mathcal{M}$ is isomorphic to the standard model.
- Proof: see e.g. Kaye's *Models of Peano Arithmetic*.
- No non-countable model can have recursive addition or multiplication. So the standard model is the only model with recursive addition and/or multiplication.

A tempting philosophical response

- The Tennenbaum Argument: "In learning arithmetic, we learn in particular how to compute addition. Reflecting on our practice, we see that genuine addition over the numbers must be a recursive function. But of course *PA* is intended to be, inter alia, true of genuine addition. So by Tennenbaum's Theorem, the intended model of *PA* just has to be the standard model."
- Versions of this line of thought have been suggested in the literature with various degrees of commitment and various bells and whistles (going back to Horsten in 2001). But we don't want to get too tangled in exegesis.
- We agree that the argument is sound. Our beef is that we don't think it that it should convince someone with sceptical worries about our grasp of the standard model.

Who could the Tennenbaum Argument be aimed at – 1?

- Someone with nominalist worries about the very idea of such abstracta as the standard model?
- No – you can't use mathematics to refute nominalism!
- Someone with general Benacerraf-style worries about whether can get any epistemic handle on abstracta?
- Again no – if you doubt whether we can ever really know about the entities postulated in a mathematical discipline like model theory, then more mathematics isn't what you need.

Who could the Tennenbaum Argument be aimed at – 2?

- The model-theoretic sceptic – henceforth 'Thoralf'?
- Thoralf isn't swayed by a global nominalism, and he is perhaps happy to allow us knowledge of some abstracta. But he is discomfited by the discovery that *PA* has multiple non-isomorphic models.
- " If our practice of endorsing *PA*-theorems doesn't determine an interpretation, than what price now the idea of the 'right', the 'standard', interpretation?"
- Perhaps at least Thoralf's doubts, arising as they do from a specifically model-theoretic argument, can be quieted by appeal to more model theory?

The strategy

- We state (and prove) another, much simpler, model-theoretic result, the Initial Segment Theorem.
- We then introduce the what we'll call the Initial Segment Argument, which parallels the Tennenbaum Argument.
- We show that the Initial Segment Argument evidently begs the question against Thoralf. We then argue that the same applies mutatis mutandis to the Tennenbaum Argument.
- We briefly consider a possible line of response.

Initial segments

- Since PA proves that $<$ is a total order, the elements of any model $\mathcal{M} = \langle M, 0^{\mathcal{M}}, 1^{\mathcal{M}}, \oplus, \otimes, \otimes \rangle$ are totally ordered by $\otimes$.
- $PA \vdash \forall x(x < \overline{n} \rightarrow x = 0 \lor x = 1 \lor x = \overline{2} \lor \ldots \lor x = \overline{n-1})$. So if $m \in M$ is such that $m \otimes n^{\mathcal{M}}$, then $m = 0^{\mathcal{M}}$ or $m = 1^{\mathcal{M}}$ or $m = 2^{\mathcal{M}}$ or $\ldots$ or $m = (n-1)^{\mathcal{M}}$.
- Hence $N^{\mathcal{M}} = 0^{\mathcal{M}}, 1^{\mathcal{M}}, 2^{\mathcal{M}}, 3^{\mathcal{M}}, 4^{\mathcal{M}}, \ldots$ in their $\otimes$-ordering form an initial segment of the $\otimes$-ordering of M. [Any M element $\otimes$-earlier than some element of $N^{\mathcal{M}}$ is already in $N^{\mathcal{M}}$.]
- $PA \vdash \overline{m} + \overline{n} = \overline{m+n}$. Hence in $\mathcal{M}$, $\oplus$ always takes members of the initial segment $N^{\mathcal{M}}$ back into $N^{\mathcal{M}}$. Similarly for $\otimes$.
- It follows that $\mathcal{N}^{\mathcal{M}} = \langle N^{\mathcal{M}}, 0^{\mathcal{M}}, 1^{\mathcal{M}}, \oplus', \otimes', \otimes' \rangle$ (primes indicating restrictions to $N^{\mathcal{M}}$) is a submodel of $\mathcal{M}$.
- But evidently $\mathcal{N}^{\mathcal{M}}$ is isomorphic to $\mathcal{N}$.

The Initial Segment Theorem

- So, informally: every model $\mathcal{M} = \langle M, 0^{\mathcal{M}}, 1^{\mathcal{M}}, \oplus, \otimes, \otimes\rangle$ of *PA*, however fancy and cluttered, embeds a nice copy of the standard model, namely $\mathcal{N}^{\mathcal{M}} = \langle N^{\mathcal{M}}, 0^{\mathcal{M}}, 1^{\mathcal{M}}, \oplus', \otimes', \otimes'\rangle$, as an initial part.
- Take an element $m \in M$ where $m \notin N^{\mathcal{M}}$. Then for all n, $n^{\mathcal{M}} \otimes m$. [Otherwise, $n^{\mathcal{M}} = m$ or $m \otimes n^{\mathcal{M}}$, and $m \in N^{\mathcal{M}}$ after all.] So m has an infinite number of $\otimes$-predecessors.
- So if no $m \in M$ has an infinite number of $\otimes$-predecessors, $\mathcal{M} = \mathcal{N}^{\mathcal{M}}$.
- Initial Segment Theorem: If $\mathcal{M}$ is a model of *PA* and for every $m \in M$ there are only finitely many n such that $n \otimes m$, then $\mathcal{M}$ is isomorphic to the standard model.

The Initial Segment Argument

- The Initial Segment Argument: "In learning arithmetic, we learn in particular how to count backwards. Reflecting on our practice, we see that, under the genuine order relation over the numbers, any number has a finite number of predecessors. But of course *PA* is intended to be, inter alia, true of genuine order over the numbers. So by the Initial Segment Theorem, the intended model of *PA* just has to be the standard model."
- There are obvious slight variants. E.g. in learning arithmetic we learn that we can count (eventually) to any number. So we again see that under the genuine order relation over the numbers, any number has a finite number of predecessors. Etc.

What's wrong with the Initial Segment Argument

- Thoralf is evidently not going to be convinced by an argument appealing to the Initial Segment Theorem. For the statement of the Theorem involves the phrase 'there are only finitely many n such that ...'. But to understand this phrase, we need to have a determinate grasp on the idea of what counts as a finite cardinal number.
- However, that is precisely what we were trying to *supply* to the model-theoretic sceptic who thinks that the idea is open to multiple interpretations.
- So if Thoralf has a genuine worry about how we grasp the numbers in their standard ordering, this argument plainly can't help him.

What's wrong with the Tennenbaum Argument.

- Similarly Thoralf is going to ask us to explain what 'recursive' means when we appeal to the Tennenbaum Argument.
- We say [roughly] that a total function is recursive so long as, for any inputs, a computer could determine in an arbitrary but finite number of steps what the output of the function is.
- So to understand even the statement of the Argument, Thoralf needs to have a determinate grasp of the idea of an arbitrary but finite numbers of steps.
- But to grasp this, he again already needs to have a determinate grasp on the idea of a finite number. However, that is precisely what we were trying to supply to Thoralf. So if the sceptic genuinely doesn't understand how we grasp the standard model, the argument from Tennenbaum's Theorem plainly can't help him.

Another argument, another response

- Walter Dean has shown the nice result that in every nonstandard countable model, every (standardly) recursive set is decidable in (what the model claims is) polynomial time.
- A natural thought now is that non-standard models get ‘polynomial time’ wrong too, and we can use this fact to home in on the standard model.
- An algorithm runs in polynomial time just in case there are fixed c, k such that, for any input of size n, the time required for a solution is bounded by cn^k.
- But if someone is to grasp this definition in its intended sense, where c, k and n are all standard numbers in $\mathbb{N}$, it is because they have already homed in on the standard model. And Thoralf is asking, precisely, how we can do that.

'Recursive' as a practical notion

- Are we being unfair in treating 'recursive' as a theoretical notion, up for reinterpretation?
- It might be suggested that we can understand recursiveness as a practical notion which concerns our ability to manipulate signs, a notion we can determinately grasp by reflecting on our real world limitations.
- This doesn't help. Our *actual* practice reveals only that we are pretty good at working out sums, products, and the like, for tractably small numbers. But if we want to talk more sweepingly, about what can be done in a computation (but without setting any upper time limit), we need to talk about what we could do in principle – i.e. done in an *arbitrary finite number* of stages.
- But do we grasp that idea determinately? That's what Thoralf is challenging!

The Initial Segment Argument again

- The same 'going practical' move could have been offered to shore up the Initial Segment Argument. For it might be said: the notion of having only finitely many predecessors can surely be given in terms of 'real world limitations'.
- If you imagine writing down the numbers using Hilbert's stroke notation, and understand 'predecessor' in terms of 'substring', we could suggest that 'finitely many' simply means 'anything which you can have written down'.
- But our *actual* limitations mean that we only ever succeed in writing down a few such numbers. If we want to talk about what can be written down, without upper limit, we need to talk about what we could do in principle.
- And by 'in principle', we mean 'given world enough, and time', i.e. given an *arbitrary finite number* of stages to concatenate strokes. Again, we beg the question against Thoralf.

The perfect parallel

- There is an exact parallel, then, between the ripostes to the Tennenbaum's Argument and to the Initial Segment Theorem.
- Some who have written about the Tennenbaum Argument have missed this. They've agreed that we can't use the Initial Segment Argument as that would presuppose an ability to understand finitude and an ability to distinguish standard from non-standard elements. But they've gone onto suppose that we can distinguish between standard and nonstandard arbitrary computation procedures, which is at least as hard.
- The point is that, just as there are divergent interpretations of the terms involved in stating Tennenbaum's Theorem (and the Initial Segment Theorem), so there are divergent interpretations of our fragmentary practices of computation (and of writing down predecessors).

An aside on Kripkenstein

- If Thoralf is so awkward as to worry about the interpretation of 'recursive', have his concerns collapsed into a generic Kripkensteinian rule-following scepticism about interpretation generally?
- No. Kripke's worry: for suitably large n and m, I might – consistently with all my practices to date – end up saying '$m + n = 5$', i.e. I might come out with a (quantifier-free) sentence which is *false* (false in $\mathcal{N}$).
- By contrast, we are considering re-interpreting practices which involve endorsing PA; accordingly, there will be no deviation concerning quantifier-free sentences. Indeed, we can restrict our attention to practices which deviate nowhere from True Arithmetic.
- Which is is why we are dealing with an essentially *model-theoretic* worry, and not a generic rule-following problem.

To summarize, and generalize

- If Thoralf has a genuine problem about our ability to grasp the standard model, then it is not to be tackled by appeal to Tennenbaum's Theorem. After all, that Theorem and its proof make heavy use of precisely the problematic notions.
- Moreover, the 'practical turn' doesn't help: our standard mathematical notions radically idealise on actual practice.
- Generalizing: If there is a genuine problem about our grasp of the standard model, then it is not to be tackled by offering any further model theoretic results either.
- For our model theory will extend *PA* precisely in order to give itself such notions as 'an arbitrary finite number', or whatever; but then Thoralf's problem about how we fix the intended interpretation of *PA* will just recur as a problem about how we fix the intended interpretation of the richer model theory.

Does this mean we give the game to Thoralf?

- In its most simple-minded form, Thoralf's position seems downright confused. He starts by asking us to treat all mathematical discourse as a wholly alien language, *awaiting* interpretation.
- But, to convince us that there are many possible interpretations of this language, Thoralf plays free and fast with the idea of standard and nonstandard models.
- In so doing, he must employ some *already* determinately interpreted mathematical theory, namely model theory; and that is *precisely* what he prohibited us all from doing at the outset!
- Thoralf is now mired in a dialectical tangle, and only philosophical therapy (not more model theory) will help him out of it.

Thoralf de-bugged?

- Thoralf might try to avoid this tangle by claiming that, despite appearances, he is not *really* employing some already-interpreted model theory. He is simply offering a *reductio* against his realist opponent:
- "Since you believe that model theory is true, you are committed (by your own lights) to the existence of *many* models of *PA*. But *you* can't tell a plausible story which explains *how* you are able to refer to one model, rather than any other. That's totally unacceptable."
- Why is this Thoralf so confident that no plausible story is to be had? Very likely because he has general worries about how we ever manage to talk about abstracta ["Numbers are so pure, so unsustained by the cement of the universe, that reference to them and their ilk seems quite *sui generis*." Hodes.]

A moral ... and a conjecture

- But Thoralf de-bugged now looks as if he has quite general philosophical worries about talking of abstracta, and the broad philosophical issues these raise can't be met by offering him more maths. He needs more philosophical therapy.
- Our discussion of at least simple-minded appeals to Tennenbaum's Theorem therefore illustrates a familiar moral: philosophical problems which are supposedly generated by mathematical results can rarely be tackled by offering more mathematics.
- Our conjecture is that the moral won't be dented by any fancier-minded appeals to Tennenbaum's Theorem either.