Tennenbaum's Theorem*

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We are going to prove a key theorem that tells us just a bit more about the structure of the non-standard countable models of first-order Peano Arithmetic; and then we will *very* briefly consider whether any broadly philosophical morals can be drawn from the technical result.

We'll state the theorem after ...

1 Fixing notation and terminology

1. Take the primitive non-logical vocabulary of *PA* to be: '0', '1', '+', '×', '<'.

2. We are only interested in distinguishing models of *PA* 'up to isomorphism'; and since any countable model is isomorphic with a model whose domain is the natural numbers, we'll concentrate on countable models which have $\mathbb{N}$ as their domain. Further, we can take the interpretations of '0', '1' to be standard too (for any given countable model $\mathcal{M}$ with domain $\mathbb{N}$ that gives a deviant interpretation n_0 for '0' and n_1 for '1', there will be an isomorphic model $\mathcal{M}'$ which permutes n_0 with 0 and n_1 with 1, making compensating adjustments to the interpretations of '+', '×', '≤'). So, without loss of generality, we can regard a countable model $\mathcal{M}$ henceforth to be a structure $\langle \mathbb{N}, 0, 1, \oplus, \otimes, \circledless \rangle$.

3. A model $\mathcal{M}$ is, of course, non-standard if it is not isomorphic to the standard model $\mathcal{N} = \{\mathbb{N}, 0, 1, +, \times, <\}$. It is familiar that there are continuum-many pairwise non-isomorphic countable models of *PA*.

4. Any model $\mathcal{M}$ of *PA* must supply elements to be the denotations for the *standard numerals* i.e. for '0', '1', '$\overline{2}$' (i.e. '$\overline{1} + \overline{1}$'), '$\overline{3}$' (i.e. '$\overline{1} + \overline{1} + \overline{1}$'), etc. – schematically $\overline{n}$. These denotations will be the elements $0, 1, 2^{\mathcal{M}} = 1 \oplus 1, 3^{\mathcal{M}} = 1 \oplus 1 \oplus 1$, etc. – schematically $n^{\mathcal{M}}$. Call those $\mathcal{M}$'s *standard elements*. Only in the standard model, however, is $n^{\mathcal{M}}$ always the number n.

5. Suppose element e in model $\mathcal{M}$ satisfies the open wff $\varphi(x)$. Then, for brevity, we'll write $\mathcal{M} \vDash \varphi([e])$.

*These notes were written for a reading group on Kaye's *Models of Peano Arithmetic*, in particular for our discussion of his Ch. 11. The basic line of proof is essentially as in Kaye's treatment: but the stripped-down presentation here aims to be reasonably stand-alone and hence, perhaps, more accessible. The usual warning applies: just because it is prettily LaTeXed, that doesn't mean that it's right! All corrections and suggestions for improvement welcome, to ps218@cam.ac.uk.

6. For future use, the n-th prime is π_n (we count from zero, so $\pi_0 = 2, \pi_1 = 3, \ldots$). The function $\pi : n \to \pi_n$ is primitive recursive, so can be represented in *PA* by a Σ_1 wff $\Pi(x, y)$, so $\mathcal{N} \vDash \Pi(\overline{n}, \overline{\pi_n})$.

   Suppose we want to say in *PA* that the k-th prime multiplied by l equals m. Then we'd officially need to write (i) $\exists y(\Pi(\overline{k}, y) \wedge y \times \overline{l} = \overline{m})$. For ease of notation, however, we'll unofficially suppose that we've definitionally added to *PA* a function sign π defined in the terms of Π so we can write (ii) $\pi(\overline{k}) \times \overline{l} = \overline{m}$. Take wffs like (ii) just to be slang shorthand for wffs like (i).

2 Recursive models and Tennenbaum's Theorem

Since we are taking the domain of a countable model $\mathcal{M}$ always to be $\mathbb{N}$, $\mathcal{M}$'s interpretations for '+' and '×' are two-place *numerical* functions, and its interpretation for '<' is a two-place numerical relation. So it makes sense to define a *recursive model* $\mathcal{M}$ to be a countable model $\langle \mathbb{N}, 0, 1, \oplus, \otimes, \otimes\!\!\!<\rangle$ where the functions $\oplus$ and $\otimes$ and the relation $\lessdot$ are all recursive. Obviously, *only* countable models can be candidates for having recursive functions/relations as constituents.

And now we can state

Tennenbaum's Theorem *The standard model is the only recursive model of PA.*

We will prove this by showing that, in any non-standard countable model, $\oplus$ is not recursive. We could also show that $\otimes$ can't be recursive either by a variant argument, but the non-recursiveness of $\oplus$ will suffice for us. (For the record, however, there *are* non-standard models where $\lessdot$ is recursive.)

Before getting to the meat of the proof, we need a few very quick background reminders.

3 A reminder about recursive inseparability

Start by recalling a standard definition from computability theory.

Two sets of numbers $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ are recursively inseparable if and only if (i) they are disjoint but (ii) there is no recursive set $X \subseteq \mathbb{N}$ such that $A \subseteq X$ and $B \subseteq \overline{X}$. In other words, A and B are disjoint, but you can't separate them by throwing a recursively-defined lasso around all of A (and maybe more) while capturing none of B. An elementary theorem invoking this notion is

Theorem 1. *There are recursively enumerable sets of numbers A and B which are not recursively separable.*

Proof. Let $\varphi_e(n)$ be the one-place partial function computed by the e-th Turing machine when given n as input. Put $A = \{e \mid \varphi_e(e) = 0\}$ and $B = \{e \mid \varphi_e(e) = 1\}$. Plainly, A and B are disjoint. They are also both recursively enumerable (just start doing a zigzag though steps of the computations of $\varphi_0(0)$, $\varphi_1(1)$, $\varphi_2(2)$ ...).

Suppose X is a recursive set separating A from B, so $A \subseteq X$ and $B \subseteq \overline{X}$. Then some total computable function φ_x is X's characteristic function, which is to say $X = \{n \mid \varphi_x(n) = 1\}$, $\overline{X} = \{n \mid \varphi_x(n) = 0\}$. But then both $x \in X$ and $x \in \overline{X}$ immediately lead to contradiction. □

4 A reminder about overspill and Δ_0-absoluteness

Turning to talk about non-standard models of *PA*, let's recall a simple form of 'overspill' theorem:

Theorem 2. *Suppose that* $\mathcal{M}$ *is a non-standard model of PA such that, for all* n, $\mathcal{M} \vDash \varphi(\overline{n})$. *Then there is a non-standard element* e *which also satisfies* $\varphi(x)$, *i.e. such that* $\mathcal{M} \vDash \varphi([e])$.

Proof. Suppose that all but only standard elements satisfy $\varphi(x)$. Then, trivially, we have both $\mathcal{M} \vDash \varphi(0)$ and $\mathcal{M} \vDash \forall x(\varphi(x) \to \varphi(x+1))$. But $\mathcal{M}$ must also satisfy *PA*'s induction axiom for φ. So we can conclude $\mathcal{M} \vDash \forall x \varphi(x)$, so φ is satisfied by the non-standard elements after all. Contradiction. □

Next, a reminder of the very basic fact that non-standard models of *PA* are well-behaved with respect to Δ_0 truths: that is to say.

Theorem 3. *If* φ *is a* Δ_0 *sentence and* $\mathcal{N} \vDash \varphi$, *then for any model* $\mathcal{M}$ *of PA*, $\mathcal{M} \vDash \varphi$.

Proof. If $\mathcal{N} \vDash \varphi$, then $PA \vdash \varphi$ because *PA* is Δ_0 complete. Therefore any model of *PA* must make φ true. □

5 Coding sets of numbers

Think informally for a moment. Fix on a monadic property C. Then the following is true for any natural number bound m: there is a number c such that, for all $k < m$, k has property C if and only if π_k divides c. Why? Well, just set c to be product of the primes π_k such that both $k < m$ and k has C. In an obvious sense, we can treat c as a code for numbers with property C which are less than m: we decode by prime factorization.

Going formal, we have the following. Given a $\varphi(x)$ and a number m, let S be the set of numbers less than m that satisfy φ. Then there is a code number c for the set such that $S = \{n \mid \mathcal{N} \vDash \exists y(\pi(n) \times y = \overline{c})\}$. Or equivalently, of course, we could write $S = \{n \mid \mathcal{N} \vDash \exists y(\pi(n) \times y = [c])\}$.[1]

So far so elementary. But of course in $\mathcal{N}$, elements c – being finite, and having only a finite number of prime divisors – can only be used to code up finite sets. However, now let's generalize and think about what can happen in non-standard models. Let's say that $S \subseteq \mathbb{N}$ is *canonically coded* in the model $\mathcal{M}$ just if there is an element a in the model such that $S = \{n \mid \mathcal{M} \vDash \exists y(\pi(\overline{n}) \times y = [a])\}$. A non-standard element a might have an infinite number of 'prime divisors' (where we here interpret that notion inside $\mathcal{M}$ terms of $\otimes$), and so the coded set S can now be infinite.

OK. Having got the basic idea of coding a (possibly infinite) set of numbers by an element in a model, we might now wonder about liberalizing it beyond coding-by-primes. Let's say, more generally, that $S \subseteq \mathbb{N}$ can be *coded* in $\mathcal{M}$ just if there is a *PA*-formula $\varphi(x, y)$ and element b in the model such that $S = \{n \mid \mathcal{M} \vDash \varphi(\overline{n}, [b])\}$. Does this allow us to code more sets of numbers in $\mathcal{M}$?

No: if a set S can be coded in $\mathcal{M}$ at all, it can be canonically coded in the model. However, we don't need that general claim. For our purposes we are in fact only going

[1] Reality check: the condition on the right is just short hand for saying that in the model $\mathcal{N}$, c satisfies the open wff $\exists y(\pi(n) \times y = x)$.

to interested in one other kind of coding, where we define a set S using a formula of the kind $(\exists k < x)A(k, y)$ with Δ_0 kernel A; in other words, we are interested in cases where we have $S = \{n \mid \mathcal{M} \vDash (\exists k < [b])A(k, \overline{n})\}$.

So all we are going to need is the following result, which shows that any set of numbers that can be coding in this new way can be canonically coded using primes:

Theorem 4. *If A is Δ_0, and $\mathcal{M}$ is a non-standard model of PA, then for any element b there is an a such that for any n, $\mathcal{M} \vDash (\exists k < [b])A(k, \overline{n}) \leftrightarrow \exists y(\pi(\overline{n}) \times y = [a])$.*

Proof. Evidently, given our reasoning above, the following is a tolerably elementary informal arithmetical truth, for given Δ_0 wff A and any number n

$$\forall b \exists a (\forall u < n)[(\exists k < b)A(k, u) \leftrightarrow \exists y(\pi(u) \times y = a)].$$

For put finite limits on k and u, and there will be a finite number of instances where $A(k, u)$ holds, so a finite number of relevant values u and we just set a to be the product of the relevant $\pi(u)$. Being a tolerably elementary arithmetical truth, its formal counterpart will certainly be provable in *PA*.[2] So we'll have, for any n,

$$PA \vdash \forall b \exists a (\forall u < \overline{n})[(\exists k < b)A(k, u) \leftrightarrow \exists y(\pi(u) \times y = a)].$$

And hence, since $\mathcal{M}$ is a model of *PA*, we of course have, for any n,

$$\mathcal{M} \vDash \forall b \exists a (\forall u < \overline{n})[(\exists k < b)A(k, u) \leftrightarrow \exists y(\pi(u) \times y = a)].$$

But now we can use overspill. There will be a non-standard element e such that

$$\mathcal{M} \vDash \forall b \exists a (\forall u < [e])[(\exists k < b)A(k, u) \leftrightarrow \exists y(\pi(u) \times y = a)].$$

But that requires that for any element b there is an a such that

$$\mathcal{M} \vDash (\forall u < [e])[(\exists k < [b])A(k, u) \leftrightarrow \exists y(\pi(u) \times y = [a])].$$

However, every standard element $n^{\mathcal{M}}$ is 'less than' a non-standard element in $\mathcal{M}$ (i.e. $n^{\mathcal{M}} \olessthan e$). So for any element b there is an a such that we certainly have for all n

$$\mathcal{M} \vDash (\exists k < [b])A(k, \overline{n}) \leftrightarrow \exists y(\pi(\overline{n}) \times y = [a]).$$

Which is just what we set out to prove. $\square$

6 Non-standard models code non-recursive sets

Recall the standard theorem that the only sets straightforwardly definable in *PA* are recursive. We are now going to show that, in contrast, coding-by-non-standard-elements can pick out non-recursive sets, by giving an example which proves:

Theorem 5. *For any non-standard model $\mathcal{M}$ of PA, there is a non-recursive set S which is canonically coded in $\mathcal{M}$.*

[2] OK, there's strictly speaking a gap in the proof here; but for present purposes, we can take this on trust – we know that unprovable truths in *PA* are more complicated than the elementary truth in play here! But what's the nicest way of avoiding this slightly arm-waving appeal?

Proof. Suppose A and B are recursively enumerable yet recursively inseparable. Then there will be Σ_1 wffs $\exists u A(u, x)$ and $\exists u B(u, x)$ that define these sets, where the kernels A and B are Δ_0.

Since A and B are disjoint, for any n,

$$\mathcal{N} \vDash (\forall v < \overline{n})(\forall w < \overline{n})(\forall x < \overline{n})\neg(A(v, x) \wedge B(w, x)).$$

But now remark that the wff we've just constructed out of the Δ_0 kernels is still Δ_0, so by the Δ_0-absoluteness Theorem 3,[3] for any non-standard $\mathcal{M}$ and any n,

$$\mathcal{M} \vDash (\forall v < \overline{n})(\forall w < \overline{n})(\forall x < \overline{n})\neg(A(v, x) \wedge B(w, x)).$$

So by the overspill Theorem 2, for some non-standard element e belonging to M, we get the exclusion principle

$$\mathcal{M} \vDash (\forall v < [e])(\forall w < [e])(\forall x < [e])\neg(A(v, x) \wedge B(w, x)).$$

Let's now define $X \subseteq \mathbb{N}$ to be the set $\{n \mid \mathcal{M} \vDash (\exists v < [e])A(v, \overline{n})\}$. And note the following to easy facts about X:

1. $A \subseteq X$. For if $n \in A$, then for some m, $\mathcal{N} \vDash A(\overline{m}, \overline{n})$. So by Theorem 3, $\mathcal{M} \vDash A(\overline{m}, \overline{n})$, whence $\mathcal{M} \vDash (\exists v < [e])A(v, \overline{n})\}$, since any standard element is less than any non-standard element.

2. $B \cap X = \emptyset$. For if $n \in B$, then for some m, $\mathcal{N} \vDash B(\overline{m}, \overline{n})$. So, arguing similarly, $\mathcal{M} \vDash (\exists w < [e])B(w, \overline{n})\}$. Whence $\mathcal{M} \vDash \neg(\exists v < [e])A(v, \overline{n})\}$, by the exclusion principle, so $n \notin X$.

Hence X can't be recursive, else A and B would be recursively separable, contrary to hypothesis.

Now apply Theorem 4 to our definition of the set X, and we know there will be an a in the model such that $X = \{n \mid \mathcal{M} \vDash \exists y(\pi(\overline{n}) \times y = [a])\}$.

Hence X will serve as the desired example of a set canonically coded in $\mathcal{M}$ which is not recursive. □

7 And so, at last, to Tennenbaum's Theorem

We know there will be an a in the non-standard model $\mathcal{M}$ such that $X = \{n \mid \mathcal{M} \vDash \exists y(\pi(\overline{n}) \times y = [a])\}$ and X is not recursive.

Now note that $\mathcal{M} \vDash \pi(\overline{n}) \times [c] = [c] + [c] + [c] + \ldots + [c]$, with π_n summands on the right, i.e. $\pi_n^{\mathcal{M}} \otimes c = c \oplus c \oplus c \oplus c \oplus \ldots \oplus c$. (That's because *PA* proves the corresponding $\forall x(\pi(\overline{n}) \times x = x + x + x + \ldots + x$.)

So $n \in X$ iff there is a c in the model $\mathcal{M}$ such that $a = c \oplus c \oplus c \oplus c \oplus \ldots \oplus c$ (for π_n summands).

Now suppose, for reductio, that $\oplus$ is recursive. And now we set off on a search through the natural numbers taken in their *natural* order (remember, the naturals constitute the domain of $\mathcal{M}$, and a is just a particular number among them). We search until we hit a number c such the following disjunction holds,

[3] So *here* we don't have to appeal to that arm-waving thought, 'if it is elementary, *PA* can prove it'.

either $a = c \oplus c \oplus c \oplus c \oplus \ldots \oplus c$ (with π_n cs)

or $a = c \oplus c \oplus c \oplus c \oplus \ldots \oplus c \oplus 1$ (with π_n cs)

or $a = c \oplus c \oplus c \oplus c \oplus \ldots \oplus c \oplus 1 \oplus 1$ (with π_n cs)

or $a = c \oplus c \oplus c \oplus c \oplus \ldots \oplus c \oplus 1 \oplus 1 \oplus 1$ (with π_n cs)

$\vdots$

or $a = c \oplus c \oplus c \oplus c \oplus \ldots \oplus c \oplus 1 \oplus 1 \oplus 1 \oplus \ldots \oplus 1$ (with π_n cs and $\pi_n - 1$ 1s).

By our assumption that $\oplus$ is recursive, we can mechanically check whether the disjunction holds.

Now, our search for a c which makes the disjunction true must terminate. Why? Because *PA* of course proves the standard result about division, i.e. that for any a, b, where $b \neq 0$, there are unique numbers, a divisor c and remainder $r < b$, such that $a = b \times c + r$. Hence $\mathcal{M}$ must make this true, i.e. must ensure that for any a and $b \neq 0$, there is a c and $r \otimes c$ such that $a = (b \otimes c) \oplus r$. Now take the particular case where b is the standard element $\pi_n^{\mathcal{M}}$. But $\pi_n^{\mathcal{M}} \otimes c = c \oplus c \oplus c \oplus c \oplus \ldots \oplus c$ with π_n summands. And if $r \otimes \pi_n^{\mathcal{M}}$, then $r = 0$ or $r = 1 \oplus 1 \oplus 1 \oplus \ldots \oplus 1$ for some number of summands less than π_n (since *PA* proves $\forall x(x < \overline{\pi_n} \leftrightarrow x = 0 \vee x = 1 \vee x = 1+1 \vee \ldots x = 1+1+1+\ldots+1)$).

Suppose, then, that the search terminates verifying a disjunct of the first kind (with no remainder): then $n \in X$. Otherwise, the search terminates with one of the other disjuncts verified, and then $n \notin X$.

So, in sum, if $\oplus$ recursive, we have a decision procedure for membership of X, and so X will be recursive. But we know that X isn't recursive. Hence $\oplus$ is not recursive. Which, as promised, proves a version of Tennenbaum's theorem:

Theorem 6. *If $\mathcal{M} = \langle \mathbb{N}, 0, 1, \oplus, \otimes, \otimes \rangle$ is a non-standard model of PA with domain the natural numbers, then $\oplus$ is not recursive.*

8 Concluding philosophical remarks

Our result is pretty and has technical interest: but does it have any especial *conceptual* significance?

Well, it might be suggested that we have a grip on the idea of a function's being computable-by-algorithm and an understanding of addition and multiplication in particular as indeed being computable-by-algorithm. So, the thought continues, assuming Church's Thesis that computability-by-algorithm implies recursiveness, Tennenbaum's Theorem shows that our arithmetical talk must indeed be latching on to the only possible recursive model, i.e. the so-called standard model of arithmetic.[4]

In response to this suggested line of thought, I'll make just two quick comments, one very general, one much more specific.

(1) The general comment is that it is in fact quite unclear that there is, hereabouts, a cogent question to which an appeal to Tennenbaum's Theorem could be a cogent

[4]For discussions around and about this line of thought, see Walter Dean (2002), 'Models and recursivity', http://www.walterdean.com/DEANmodelsAndRecursivity.pdf, Volker Halbach and Leon Horsten (2005) 'Computational structuralism', *Philosophia Mathematica* 13, pp. 174–186, also at http://users.ox.ac.uk/~sfop0114/pdf/compstruct.pdf, and Paula Quinon and Konrad Zdanowski (2006), 'The Intended Model of Arithmetic. An Argument from Tennenbaum's Theorem', http://www.impan.pl/~kz/files/PQKZ_Tenn.pdf.

answer. Suppose that we are looking from the outside at aliens, so to speak, an area of whose idealized practice seems to amount to endorsing some as-yet-uninterpreted-by-us first-order theory T. Then we can ask the interpretative question: what do *their* terms refer to, and what extensions do *their* predicates have? And our mathematical knowledge of the multiplicity of models for a consistent first-order T, i.e. of the multiplicity of available interpretations that charitably make them come out speaking the truth, does raise a seemingly sensible question: is there, for all that, a determinate answer to the question of what our aliens are talking about, and if so, what determines it? If, for example, their theory looks like PA, what if anything fixes that they really are talking about the natural numbers rather than something non-standard?

So far, so good, perhaps. But it doesn't follow that it makes equal sense to start treating *ourselves* as aliens and to wonder whether *we* are really talking about the natural numbers or some non-standard model. We can work inside our own mathematical language: here we have a working understanding of talk of numbers and their natural order, and (piggy-backing on that basic understanding) we can go on to reach an understanding of – inter alia – talk about other structures built from the numbers with a deviant ordering. Alternatively, we can bracket our understanding, and try to can stand outside our practice. But we can't simultaneously do *both*: we can't understand the idea of standard vs other models because we are working *inside* maths, and at the same time treat our own mathamatical discourse as alien, awaiting radical interpretation from *outside*.

The general thought, then, is that what we need in order to sooth the itch that tempts us to Skolemite scepticism is not more technicalities to weigh against the technicalities of a non-categoricity proof, but something quite different – namely some reminders of where we are supposed to be in the story when we wonder about our own arithmetical practice. And if that is the right kind of therapy, then an appeal to Tennenbaum's Theorem is necessarily going to be distractingly beside the point.

(2) But let's try to set aside those sorts of general concern and pretend we understand the project of explaining our own grasp of the standard model of arithmetic. There's still a special problem about appealing to Tennenbaum's Theorem in any explanation of the supposedly desired kind.

The idea, recall, was that we can appeal to our given understanding of addition and multiplication as computable-by-algorithm, and argue via Church's Thesis and Tennenbaum. But there's trouble at the very start here. For what is uncontentiously given is only that we have a practice of working out sums and products for tractably small numbers. What entitles us to project from this finite base and credit ourselves with an understanding of how 'in principle' to compute on any number-inputs at all? If there are possible divergent extensions of the finite basis – which there are! – then there are possible divergent interpretations of our original finite practice:[5] are we algorithmically computing or deviantly quomputing? I don't see how it could be legitimate, in the current context where there is supposed to be a genuine question how we grasp $+$ rather some deviant $\otimes$, to suppose that we can blithely assume that there is no similar problem about how we grasp the notion of a computation. So an appeal to the sketched argument that went via Tennenbaum's Theorem would seem to limp from the outset.

[5] I'm here consciously echoing Putnam's 'If there are possible divergent extensions of our practice, then there are possible divergent interpretations of even the natural number sequence – our practice, or our mental representations, etc., do not single out a unique 'standard model' of the natural number sequence.' (*Reason Truth and History*, p. 67)