# YONEDA WITHOUT (TOO MANY) TEARS

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*Abstract* This note aims to make the proof of the full Yoneda Lemma as unpuzzling as possible by dividing the argument into three stages, each of which *almost* writes itself ...[1]

## 1. Introduction

"The level of abstraction in the Yoneda Lemma means that many people find it quite bewildering." That's from Tom Leinster, who gives us a challenge then – to make the Lemma as unbewildering as possible.

I'll do my best here, at least to make the *proof* as clear as I can. Explaining the *significance* of the Lemma and some *applications* is for another day.

After a preamble, reminding you of some necessary background, I'll take things in three stages:

(1) Assuming you know just a bit about hom-functors and natural transformations, we bring things together and look at natural transformations between hom-functors. And we very quickly arrive at what we can think of as a decaffeinated version of the Yoneda Lemma; we will also meet the related Yoneda embedding.

(2) We then show that what I call the Restricted Yoneda Lemma can be easily generalized to get the Intermediate Lemma. We essentially just recycle the same proof ideas, so there is a bit of repetition here – but my excuse is that it is much easier to appreciate what it going on if you have already met the ideas in the simpler, stage (1), context.

(3) The Intermediate Lemma tells us about two objects being isomorphic. The third step taking us to the full-strength Lemma is to show that what we have here is a *natural* isomorphism in the official sense.

By the end, I hope you'll agree that (even if the *point* of it all isn't yet clear) the proofs are really not too bewildering at all. It is only a bit of exaggeration to say that at each stage, we just have to 'do the obvious thing'.

## 2. What do you need to bring to the party?

Fixing notation as I go along, let's assume you are familiar with the following:

(1) The idea of covariant and contravariant functors; you know e.g. that a fully faithful functor $F$ reflects isomorphisms (i.e. if $Fj$ is an isomorphism so is $j$).

(2) I use $\mathsf{C}(A, B)$ for the collection of $\mathsf{C}$-arrows from $A$ to $B$: we'll be concerned with locally small categories where such collections are sets, hom-sets as the jargon has it.

(3) There are two flavours of associated hom-functors. There's e.g. the hom-functor $\mathsf{C}(A, -)\colon \mathsf{C} \to \mathsf{Set}$, which sends a $\mathsf{C}$-object $X$ to the hom-set $\mathsf{C}(A, X)$, and sends a $\mathsf{C}$-arrow $j\colon X \to Y$ to the function we can notate $j \circ -\colon \mathsf{C}(A, X) \to \mathsf{C}(A, Y)$, i.e. the function which maps $h\colon A \to X$ in $\mathsf{C}(A, X)$ to $j \circ h\colon A \to Y$ in $\mathsf{C}(A, Y)$.

[1]This version: Oct 30, 2023. Comments and corrections to ps218@cam.ac.uk. For the notes on category theory from which this is excerpted, see logicmatters.net/categories.

Then there is a contravariant hom-functor such as $\mathsf{C}(-, A)\colon \mathsf{C} \to \mathsf{Set}$. This sends the $\mathsf{C}$-object $X$ to the hom-set $\mathsf{C}(X, A)$, and sends the $\mathsf{C}$-arrow $j\colon X \to Y$ to the function that we can notate $- \circ j\colon \mathsf{C}(Y, A) \to \mathsf{C}(X, A)$, i.e. the function which maps an arrow $h\colon Y \to A$ in $\mathsf{C}(Y, A)$ to the arrow $h \circ j\colon X \to A$ in $\mathsf{C}(X, A)$.

(4) A natural transformation $\alpha\colon F \Rightarrow G$ between two covariant functors $F, G\colon \mathsf{C} \to \mathsf{D}$, is a suite of $\mathsf{D}$-arrows $\alpha_A$, one for each $\mathsf{C}$-object $A$, such that for any $\mathsf{C}$-arrow $f\colon A \to B$ the following naturality square commutes in $\mathsf{D}$:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \Big\downarrow{\scriptstyle \alpha_A} & & \Big\downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

Similarly with the horizontal arrows reversed for natural transformations between contravariant functors.

Natural transformations can 'vertically' compose to give more transformations.

(5) A natural transformation whose components are all isomorphisms is a natural isomorphism. The objects $C$ and $D$ in the category $\mathsf{D}$ are said to be naturally isomorphic if there are naturally isomorphic functors $F, G\colon \mathsf{C} \to \mathsf{D}$ and a $\mathsf{C}$-object $A$ such that $FA = C$ and $GA = D$.

(6) We use $[\mathsf{C}, \mathsf{D}]$ to denote the category whose objects are functors $F\colon \mathsf{C} \to \mathsf{D}$ and whose arrows are the natural transformations between such functors.

(7) $Nat(F, G)$ will denote the set of natural transformations from $F$ to $G$ (that's the hom-set of arrows from $F$ to $G$ in the appropriate category $[\mathsf{C}, \mathsf{D}]$).

Where there are hom-sets, there are hom-functors. Just applying the same definitions as before, $Nat(F, -)\colon [\mathsf{C}, \mathsf{D}] \to \mathsf{Set}$ is the covariant functor which sends a $[\mathsf{C}, \mathsf{D}]$-object such as the functor $G\colon \mathsf{C} \to \mathsf{D}$ to the corresponding set $Nat(F, G)$. And it sends a $[\mathsf{C}, \mathsf{D}]$-arrow such as a natural transformation $\gamma\colon G \Rightarrow H$ to the function $\alpha \circ -$, which takes an arrow $\alpha$ from $Nat(F, G)$ and returns the arrow $\gamma \circ \alpha$ from $Nat(F, H)$.

And likewise, $Nat(-, F)\colon [\mathsf{C}, \mathsf{D}] \to \mathsf{Set}$ is the contravariant functor which sends a $[\mathsf{C}, \mathsf{D}]$-object such as the functor $E\colon \mathsf{C} \to \mathsf{D}$ to the corresponding set $Nat(E, F)$. And it sends a $[\mathsf{C}, \mathsf{D}]$-arrow such as a natural transformation $\gamma\colon D \Rightarrow E$ to the function $- \circ \gamma$, which takes an arrow $\alpha$ from $Nat(E, F)$ and returns the arrow $\alpha \circ \gamma$ from $Nat(D, F)$.

So far, so good? If so, proceed! But if some of that is too mysterious, then Yoneda is probably not (yet) for you!

## 3. Natural transformations between hom-functors

**3.1.** There is now a pair of stories to be told, one about natural transformations between covariant hom-functors, one about natural transformations between contravairant hom-functors. I will spell out the first story, leaving its companion as an exercise in dualizing.[2]

Take, then, a locally small category $\mathsf{C}$. And let's think how can we construct a natural transformation $\alpha$ from the hom-functor $\mathsf{C}(A, -)\colon \mathsf{C} \to \mathsf{Set}$ to the hom-functor $\mathsf{C}(B, -)\colon \mathsf{C} \to \mathsf{Set}$.

[2] A story of the second kind involves the added twist of needing to keep track of the direction of arrows when we deal with contravariant functors. But otherwise, the key ideas are just the same.

By definition, if $\alpha$ is to be a natural transformation, its components must be such that the following diagram commutes, given any $\mathsf{C}$-arrow $j\colon X \to Y$:

$$\begin{array}{ccc} \mathsf{C}(A,X) & \xrightarrow{\mathsf{C}(A,j)} & \mathsf{C}(A,Y) \\ \big\downarrow{\scriptstyle \alpha_X} & & \big\downarrow{\scriptstyle \alpha_Y} \\ \mathsf{C}(B,X) & \xrightarrow{\mathsf{C}(B,j)} & \mathsf{C}(B,Y) \end{array}$$

where $\mathsf{C}(A,j)$ is short for the result of applying the functor $\mathsf{C}(A,–)$ to the $\mathsf{C}$-arrow $j$. That result, we've just reminded ourselves, is a function operating on $\mathsf{C}$-arrows, namely $j \circ –\colon \mathsf{C}(A,X) \to \mathsf{C}(A,Y)$. Similarly, using the same notation, $\mathsf{C}(B,j)$ is the function $j \circ –\colon \mathsf{C}(B,X) \to \mathsf{C}(B,Y)$

Now, $\alpha_X$ has got to send any arrow $g\colon A \to X$ to some arrow $h\colon B \to X$. *And the very easiest way of doing this is for $\alpha_X$ to ensure $h = g \circ f$ for some given $f\colon B \to A$.*

So: fix on some arrow $f\colon B \to A$. And let's try not only defining the component $\alpha_X$ to be the function we can notate $– \circ f$ that sends an arrow $g\colon A \to X$ to the composite $g \circ f\colon B \to X$, but also defining $\alpha_Y$ etc. similarly. For any $Y$, then, set $\alpha_Y\colon \mathsf{C}(A,Y) \to \mathsf{C}(B,Y)$ to be the function we can again notate $– \circ f$ that sends an arrow $h\colon A \to Y$ to the composite $h \circ f\colon B \to Y$.

And lo and behold, our easy first guess at suitable $\alpha_X$, $\alpha_Y$, etc., makes our diagram commute. Take any arrow $g\colon A \to X$ in $\mathsf{C}(A,X)$, and chase it round the diagram in both directions, and we end up with the same result. Thus:

$$\begin{array}{ccc} g\colon A \to X & \xrightarrow{\quad j\circ– \quad} & j \circ g\colon A \to Y \\ \big\downarrow{\scriptstyle –\circ f} & & \big\downarrow{\scriptstyle –\circ f} \\ g \circ f\colon B \to X & \xrightarrow{\quad j\circ– \quad} & j \circ g \circ f\colon B \to Y \end{array}$$

Then generalizing: for any $X$, $Y$, and $j$, our first diagram always commutes when $\alpha$'s components are defined like $\alpha_X$. So $\alpha$ is indeed a natural transformation.

Let's sum this up, and also introduce some new notation:

**Theorem 1.** *Suppose $\mathsf{C}$ is a locally small category, and $\mathsf{C}(A,–)$, $\mathsf{C}(B,–)$ are hom-functors (for objects $A$, $B$ in $\mathsf{C}$).*

*Then, given an arrow $f\colon B \to A$, there exists a corresponding natural transformation which we will now notate $\mathsf{C}(f,–)\colon \mathsf{C}(A,–) \Rightarrow \mathsf{C}(B,–)$, where for each $X$, the component $\mathsf{C}(f,–)_X\colon \mathsf{C}(A,X) \to \mathsf{C}(B,X)$ sends an arrow $g\colon A \to X$ to $g \circ f\colon B \to X$.* □

**3.2.** An additional remark: if $f$ in our theorem is an isomorphism, then each component of our natural transformation $(– \circ f)$ has an inverse (i.e. $– \circ f^{-1}$), so is an isomorphism. Therefore the induced transformation $\mathsf{C}(f,–)$ will be a natural isomorphism.

Which is just as it should be. Arm-waving: if $f\colon B \to A$ is an isomorphism, then the members of the hom-sets $\mathsf{C}(A,X)$ and $\mathsf{C}(B,X)$ will line up nicely one-to-one, and so the corresponding hom-functors $\mathsf{C}(A,–)$ and $\mathsf{C}(B,–)$ will behave in exactly parallel ways, and so there should be a suitable isomorphism between the functors.

**3.3.** To check understanding and for future use, show the following:

**Theorem 2.** *Given a locally small category $\mathsf{C}$ including objects $A, B, C$, and arrows $f\colon B \to A$ and $g\colon C \to B$, then*

*(1)* $\mathsf{C}(f \circ g, –) = \mathsf{C}(g, –) \circ \mathsf{C}(f, –)$.
*(2)* $\mathsf{C}(f, –)_A 1_A = f$.
*(3)* $\mathsf{C}(1_A, –) = 1_{\mathsf{C}(A,–)}$.

Not that this is much of a challenge! – we just have to apply definitions. So:

*Proof of (1).* By the definition of $\mathsf{C}(f \circ g, –)$, a component $\mathsf{C}(f \circ g, –)_X$ sends any arrow $k\colon A \to X$ to $k \circ (f \circ g)$. However, $\mathsf{C}(f, –)_X$ sends $k$ to $= k \circ f$, and $\mathsf{C}(g, –)_X$ send that on to $(k \circ f) \circ g$. Hence, component by component, $\mathsf{C}(f \circ g, –)$ acts the same as $\mathsf{C}(g, –) \circ \mathsf{C}(f, –)$ – which makes them the same natural transformation. □

*Proof of (2).* By definition $\mathsf{C}(f, –)_A$ sends any $k\colon A \to A$ to $k \circ f\colon B \to A$. So in particular it sends $1_A$ to $f$. □

*Proof of (3).* By definition, $\mathsf{C}(1_A, –)_X$ sends any $k\colon A \to X$ to $k \circ 1_A\colon A \to X$ – i.e. it sends $k$ to itself.

And what is $1_{\mathsf{C}(A,–)}$? I haven't said! But the notation indicates the identity arrow on the object $\mathsf{C}(A, –)$, where the appropriate category will be a functor category including such hom-functors as objects. But then the identity object on such an object will be the identity natural transformation from $\mathsf{C}(A, –)$ to itself. So what does the $X$-component of that identity natural transformation do to an arrow such as $k\colon A \to X$? It will send the arrow to itself. Which shows that the $X$-components of $\mathsf{C}(1_A, –)$ and $1_{\mathsf{C}(A,–)}$ agree on their actions; and that holds for all $X$ and so the transformations are identical. □

**3.4.** The result (2) has an immediate corollary, which we can add to our earlier main theorem:

**Theorem 1 (cont'd).** *If $f, f'\colon B \to A$ are distinct arrows, then the corresponding natural transformations $\mathsf{C}(f, –)$ and $\mathsf{C}(f', –)$ are also distinct.*

*Proof.* We know from the result just proved that

$$\mathsf{C}(f, –)_A 1_A = f \neq f' = \mathsf{C}(f', –)_A 1_A$$

Hence the $A$-components of $\mathsf{C}(f, –)$ and $\mathsf{C}(f', –)$ can't be the same, hence the natural transformations can't overall be the same either. □

**3.5.** The obvious next question to ask is this: are *all* possible natural transformations between the hom-functors $\mathsf{C}(A, –)$ and $\mathsf{C}(B, –)$ generated from arrows $f\colon B \to A$ in the way described in Theorem 1?

Suppose a natural transformation $\alpha\colon \mathsf{C}(A, –) \Rightarrow \mathsf{C}(B, –)$ is given as being of the form $\mathsf{C}(f, –)$ for some $f\colon B \to A$, then we know that $f = \mathsf{C}(f, –)_A 1_A = \alpha_A 1_A$. Hence in *this* case, we already have a candidate for $f$. It would be nice if the same idea always works. So here's a hopeful conjecture:

**Theorem 3.** *Suppose $\mathsf{C}$ is a locally small category, and consider the hom-functors $\mathsf{C}(A, –)$ and $\mathsf{C}(B, –)$, for objects $A, B$ in $\mathsf{C}$. Then if there is a natural transformation $\alpha\colon \mathsf{C}(A, –) \Rightarrow \mathsf{C}(B, –)$, there is a unique arrow $f\colon B \to A$ such that $\alpha = \mathsf{C}(f, –)$, namely $f = \alpha_A(1_A)$.*

And this indeed is right. We just have to think about what happens when we chase $1_A$ round a naturality square involving the component $\alpha_A$ (what else?). Where does $1_A$ live? – in the hom-set $\mathsf{C}(A, A)$. So the obvious thing to do is look again at the sort of square we met before, but now putting $X = A$. Off we go:

*Proof.* Since $\alpha$ is a natural transformation, the following diagram in particular must commute, for any $X$ and any $j\colon A \to X$,

$$\begin{array}{ccc} \mathsf{C}(A,A) & \xrightarrow{\mathsf{C}(A,j)} & \mathsf{C}(A,X) \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_X} \\ \mathsf{C}(B,A) & \xrightarrow{\mathsf{C}(B,j)} & \mathsf{C}(B,X) \end{array}$$

And here, recalling the definitions, $\mathsf{C}(A, j)$ is the map that (among other things) sends an arrow $h\colon A \to A$ to the arrow $j \circ h\colon A \to X$, and $\mathsf{C}(B, j)$ sends an arrow $k\colon B \to A$ to the arrow $j \circ k\colon B \to X$.

Chase that identity arrow $1_A$ round the diagram from the top left to bottom right nodes. The top route sends it to $\alpha_X(j)$. The bottom route sends it to $j \circ (\alpha_A(1_A))$, which equals $\mathsf{C}(\alpha_A(1_A), –)_X(j)$ (check how we set up the notation in Theorem 1).

Since our square always commutes we have $\alpha_X(j) = \mathsf{C}(\alpha_A(1_A), –)_X(j)$, for all objects $X$ and for all arrows $j\colon A \to X$. So the $X$-components of $\alpha$ and $\mathsf{C}(\alpha_A(1_A), –)$ agree on their application to all arrows $j\colon A \to X$, hence must be the same. Since $X$ was arbitrary, that means all the components of $\alpha$ and $\mathsf{C}(\alpha_A(1_A), –)$ are the same.

So those natural transformations are identical, which is just what we need for the existence part of our theorem: we've found our $f$, i.e. $\alpha_A(1_A)$, such that $\alpha = \mathsf{C}(f, –)$. Look at it this way: fixing what (the relevant component of) $\alpha$ does to $1_A$ fixes the whole natural transformation.

Now suppose both $f$ and $f'$ are such that $\alpha = \mathsf{C}(f, –) = \mathsf{C}(f', –)$. Then by Theorem 2 (2)

$$f = \mathsf{C}(f, –)_A(1_A) = \mathsf{C}(f', –)_A(1_A) = f'$$

which shows $f$'s uniqueness. □

**3.6.** The theorems so far in this section have been about covariant hom-functors. Entirely predictably, there are dual results for contravariant hom-functors $\mathsf{C} \to \mathsf{Set}$ (or equivalently, covariant hom-functors $\mathsf{C}^{op} \to \mathsf{Set}$).

Here's a summary theorem (proofs can be left as routine exercises in dualization, paying attention to the direction of arrows):

**Theorem 4.** *Suppose* $\mathsf{C}$ *is a locally small category, and* $\mathsf{C}(–, A)$, $\mathsf{C}(–, B)$ *are contravariant hom-functors (for objects* $A, B$ *in* $\mathsf{C}$*). Then:*

(1) *If there exists an arrow* $f\colon A \to B$*, there is a natural transformation* $\mathsf{C}(–, f)\colon \mathsf{C}(–, A) \Rightarrow \mathsf{C}(–, B)$*, where for each* $X$*, the component* $\mathsf{C}(–, f)_X\colon \mathsf{C}(X, A) \to \mathsf{C}(X, B)$ *sends an arrow* $j\colon X \to A$ *to* $f \circ j\colon X \to B$.

(2) $\mathsf{C}(–, g \circ f) = \mathsf{C}(–, g) \circ \mathsf{C}(–, f)$.

(3) *Different arrows* $f, f'\colon A \to B$ *give rise to different corresponding natural transformations* $\mathsf{C}(–, f)$, $\mathsf{C}(–, f)$.

(4) *If there is a natural transformation* $\alpha\colon \mathsf{C}(–, A) \Rightarrow \mathsf{C}(–, B)$*, there is a unique arrow* $f\colon A \to B$ *such that* $\alpha = \mathsf{C}(–, f)$*, namely* $f = \alpha_A(1_A)$. □

## 4. The Restricted Yoneda Lemma

**4.1.** Recall, $Nat(F, G)$ denotes the collection of natural transformations from $F$ to $G$.

Now, in proving Theorems 1 and 3, we have shown that for a locally small category $\mathsf{C}$ its arrows $B \to A$ line up one-to-one with natural transformations $\mathsf{C}(A, –) \Rightarrow \mathsf{C}(B, –)$ In other words, there is a bijection between the hom-set $\mathsf{C}(B, A)$ and the collection $Nat(\mathsf{C}(A, –), \mathsf{C}(B, –))$. And ah-ha! – since we are dealing with a locally small $\mathsf{C}$, $\mathsf{C}(B, A)$ is set-sized, so since it is the same size we can happily treat $Nat(\mathsf{C}(A, –), \mathsf{C}(B, –))$ as a set too.

Likewise Theorem 4 tells us that its arrows $A \to B$ line up one-to-one with natural transformations $\mathsf{C}(–, A) \Rightarrow \mathsf{C}(–, B)$. In other words, there is a bijection between the hom-set $\mathsf{C}(A, B)$ and the set $Nat(\mathsf{C}(–, A), \mathsf{C}(–, B))$.

But bijections between sets, of course, count as isomorphisms in $\mathsf{Set}$. So we have established the following key theorem:

**Theorem 5** (The Restricted Yoneda Lemma)**.** *Suppose* $\mathsf{C}$ *is a locally small category, and* $A, B$ *are objects of* $\mathsf{C}$*. Then* $Nat(\mathsf{C}(A, –), \mathsf{C}(B, –)) \cong \mathsf{C}(B, A)$ *and* $Nat(\mathsf{C}(–, A), \mathsf{C}(–, B)) \cong \mathsf{C}(A, B))$*.*

The rationale for this theorem's – non-standard! – label will become clear shortly.

**4.2.** It is worth spelling out the justification for our theorem in a slightly different style. Or rather, I'll do the first part, leaving the other part as another exercise in dualizing.

Fix on the objects $A$ and $B$. Then we've shown that there is a function $\mathcal{X}_{AB}$ with source $\mathsf{C}(B, A)$ and target $Nat(\mathsf{C}(A, –), \mathsf{C}(B, –))$, which sends an arrow $f\colon B \to A$ to $\mathsf{C}(f, –)$. And there is a function $\mathcal{E}_{AB}$ in the reverse direction, from $Nat(\mathsf{C}(A, –), \mathsf{C}(B, –))$ to $\mathsf{C}(B, A)$, which sends a natural transformation $\alpha\colon \mathsf{C}(A, –) \Rightarrow \mathsf{C}(B, –)$ to $\alpha_A(1_A)$.

Then we immediately have:

(1) Given any $f\colon B \to A$,

$$(\mathcal{E}_{AB} \circ \mathcal{X}_{AB})f = \mathcal{E}_{AB}(\mathsf{C}(–, f)) = \mathsf{C}(–, f)_A(1_A) = f.$$

But $f$ was arbitrary. Whence $\mathcal{E}_{AB} \circ \mathcal{X}_{AB} = 1$ (that's the identity on $\mathsf{C}(B, A)$).

(2) Given any $\alpha\colon \mathsf{C}(–, A) \Rightarrow \mathsf{C}(–, B)$,

$$(\mathcal{X}_{AB} \circ \mathcal{E}_{AB})\alpha = \mathcal{X}_{AB}(\alpha_A(1_A)) = \mathsf{C}(\alpha_A(1_A), –) = \alpha$$

where the last identity is from Theorem 3. But $\alpha$ was arbitrary. Whence $\mathcal{X}_{AB} \circ \mathcal{E}_{AB} = 1$ (that's the identity on $Nat(\mathsf{C}(–, A), \mathsf{C}(–, B))$).

Having a two-sided inverse, $\mathcal{X}_{AB}$ is therefore an isomorphism, and we have half our last theorem again.

The proof of the other half is dual. There is a function $\mathcal{Y}_{AB}$, with source $\mathsf{C}(A, B)$ and target $Nat(\mathsf{C}(–, A), \mathsf{C}(–, B))$, which sends an arrow $f\colon A \to B$ to $\mathsf{C}(–, f)$. And there is a function $\mathcal{E}_{AB}$ from $Nat(\mathsf{C}(–, A), \mathsf{C}(–, B))$ to $\mathsf{C}(A, B)$, which sends a natural transformation $\alpha$ to $\alpha_A(1_A)$. As before, we can show these two functions are inverses. And so $\mathcal{Y}_{AB}$ is also an isomorphism, and we have the other half our theorem.

## 5. The Yoneda embedding

**5.1.** A moment ago we fixed objects $A$ and $B$ and defined $\mathcal{X}_{AB}$ as a map from arrows $B \to A$ to natural transformations $\mathsf{C}(A, –) \Rightarrow \mathsf{C}(B, –)$.

But there was nothing special about the objects $A$ and $B$ here. So we can in fact think of a more general operation on arrows $\mathcal{X}$ which, now for *any* arrow $f$ at all living in $\mathsf{C}$, sends it to a corresponding natural transformation $\mathsf{C}(f, –)$.

Similarly, we can define an operation on objects which we will also label $\mathcal{X}$ that takes any $\mathsf{C}$-object $A$ and sends it to the corresponding hom-functor $\mathsf{C}(A, –)$.

That double use of '$\mathcal{X}$' for an operation on objects and operation on arrows promises that the two components assemble into a functor! Which they do.

For consider: hom-functors like $\mathsf{C}(X, –)$ are objects of the functor category $[\mathsf{C}, \mathsf{Set}]$. And natural transformations like $\mathsf{C}(f, –)$ are arrows in that same category. So, the $\mathcal{X}$ operation on objects and the $\mathcal{X}$ operation on arrows are at least of just the right types to be components of a contravariant functor $\mathcal{X}\colon \mathsf{C} \to [\mathsf{C}, \mathsf{Set}]$ (contravariant, of course, because an arrow $f\colon B \to A$ is sent to a natural transformation $\mathsf{C}(f, –)\colon \mathsf{C}(A, –) \Rightarrow \mathsf{C}(B, –)$).

And we can easily confirm that the two key conditions for functoriality are indeed satisfied. First, identities are preserved:

$$\mathcal{X}(1_A) = \mathsf{C}(1, A, –) = 1_{\mathsf{C}(A,–)} = 1_{\mathcal{X}(A)}.$$

Second, composition is respected. In other words, for any composable $f, g$ in $\mathsf{C}$,

$$\mathcal{X}(g \circ f) = \mathsf{C}(g \circ f, –) = \mathsf{C}(–, f) \circ \mathsf{C}(–, g) = \mathcal{X}(f) \circ \mathcal{X}(g),$$

reversing the order of composition as is required for a contravariant function.

So to summarize this important result, and state its dual:

**Theorem 6.** *For any locally small category $\mathsf{C}$, there is a contravariant functor $\mathcal{X}\colon \mathsf{C} \to [\mathsf{C}, \mathsf{Set}]$ (equivalently, covariant functor $\mathcal{X}\colon \mathsf{C}^{op} \to [\mathsf{C}, \mathsf{Set}]$ ) such that*

*(1) for any $\mathsf{C}$-object $A$, $\mathcal{X}A = \mathsf{C}(A, –)$,*
*(2) for any arrow $\mathsf{C}$-arrow $f\colon B \to A$, $\mathcal{X}f = \mathsf{C}(f, –)\colon \mathsf{C}(A, –) \Rightarrow \mathsf{C}(B, –)$.*

*Dually, there is a covariant functor $\mathcal{Y}\colon \mathsf{C} \to [\mathsf{C}^{op}, \mathsf{Set}]$ such that*

*(1) for any $\mathsf{C}$-object $A$, $\mathcal{Y}A = \mathsf{C}(–, A)$,*
*(2) for any $\mathsf{C}$-arrow $f\colon A \to B$, $\mathcal{Y}f = \mathsf{C}(–, f)\colon \mathsf{C}(–, A) \Rightarrow \mathsf{C}(–, B)$.*

Challenge: prove the second half of this theorem.

**5.2.** It is immediate that the functors $\mathcal{X}$ and $\mathcal{Y}$ behave nicely:

**Theorem 7.** *$\mathcal{X}$ and $\mathcal{Y}$ are fully faithful and injective on objects.*

*Proof.* Let's work through the second case. By definition, $\mathcal{Y}$ is full just in case, for any $\mathsf{C}$-objects $A, B$, and any natural transformation $\alpha\colon \mathsf{C}(–, A) \Rightarrow \mathsf{C}(–, B)$, there is an arrow $f\colon A \to B$ in $\mathsf{C}$ such that $\alpha = \mathcal{Y}f = \mathsf{C}(–, f)$. Which is given in Theorem 4.

By definition, $\mathcal{Y}$ is faithful just in case, for any $\mathsf{C}$-objects $A, B$, and any pair of arrows $f, g\colon A \to B$ in $\mathsf{C}$, then if $\mathsf{C}(–, f) = \mathsf{C}(–, g)$ then $f = g$. But that also follows immediately from Theorem 4.

So the only new claim is that $\mathcal{Y}$ is injective on objects, meaning that if $\mathcal{Y}(A) = \mathcal{Y}(B)$ then $A = B$. Suppose then that we are given $\mathcal{Y}(A) = \mathcal{Y}(B)$, i.e. $\mathsf{C}(–, A) = \mathsf{C}(–, B)$. Then for any object $C$ we'll have $\mathsf{C}(C, A) = \mathsf{C}(C, B)$. But that can't be so if $A \neq B$, since hom-sets on different pairs of objects must be disjoint (on the standard view), for no arrow $g\colon C \to A$ can equal some $h\colon C \to B$, having distinct targets. □

**5.3.** The situation, then, is this. The functor $\mathcal{Y}$ injects a one-to-one copy of the $\mathsf{C}$-objects into the objects of the functor category $[\mathsf{C}^{op}, \mathsf{Set}]$; and then it fully and faithfully matches up the arrows between $\mathsf{C}$-objects with arrows between the corresponding objects in $[\mathsf{C}^{op}, \mathsf{Set}]$. In other words:

**Theorem 8** (Yoneda Embedding)**.** *The image of $\mathsf{C}$ under the functor $\mathcal{Y}$ is an isomorphic copy of $\mathsf{C}$, embedded inside the functor category $[\mathsf{C}^{op}, \mathsf{Set}]$ as a full sub-category.*

And by the way, the '$\mathcal{Y}$' notation – in upper or lower case, in one font or another – is pretty standard for the Yoneda embedding.[3]

[3]It is customary to emphasize this result rather than its dual. To be sure, $\mathcal{X}$ embeds a copy of $\mathsf{C}^{op}$ in the functor category $[\mathsf{C}, \mathsf{Set}]$ – but obviously we are more likely to be interested in finding copies of the category $\mathsf{C}$ we start off with rather than a copy of its opposite.

**5.4.** As an important corollary, we also have

**Theorem 9.** *For any objects $A, B$ in the locally small category $\mathsf{C}$, $A \cong B$ iff $\mathcal{Y}A \cong \mathcal{Y}B$.*

*Proof.* Suppose $A \cong B$. Then there is an isomorphism $f\colon A \xrightarrow{\sim} B$. So there is a natural transformation $\mathsf{C}(-,f)\colon \mathsf{C}(-,A) \Rightarrow \mathsf{C}(-,B)$, which by the remark after Theorem 1 is a natural isomorphism. So in our alternative notation, $\mathcal{Y}f\colon \mathcal{Y}A \overset{\sim}{\Longrightarrow} \mathcal{Y}B$. Hence $\mathcal{Y}A \cong \mathcal{Y}B$.

Now suppose $\mathcal{Y}A \cong \mathcal{Y}B$. So there exists a natural isomorphism $\alpha\colon \mathsf{C}(-,A) \overset{\sim}{\Longrightarrow} \mathsf{C}(-,B)$. By Theorem 4, $\alpha$ is $\mathsf{C}(-,f)$ for some $f\colon A \to B$, i.e. is $\mathcal{Y}f$. But $\mathcal{Y}$ is fully faithful and reflects isomorphisms; so since $\mathcal{Y}f$ is an isomorphism, is $f$ too. Hence $A \cong B$.

That shows $A \cong B$ iff $\mathcal{Y}A \cong \mathcal{Y}B$. □

Exercise: what's the corresponding result involving the dual functor $\mathcal{X}$?

## 6. Onwards to the full Yoneda Lemma!

In the terms of the Introduction, that's stage (1) done! – we have established what I called the Restricted Yoneda Lemma and shown that the Yoneda Embedding functor is indeed an embedding. And the *proofs* of those initial results are, I hope, not at all bewildering. Basically, we just asked what the natural transformations between two hom-functors might look like, and then followed our noses, doing the obvious things.

True, as we have been going along you might have perhaps have been puzzled about the *point* of it all: why do the theorems matter? Good question. But here in this note I want to press on with the technicalities, and prove the full-power, unrestricted, Yoneda Lemma. What does this involve?

Assume again that $\mathsf{C}$ is locally small. Then here again is one half of our restricted Theorem 5:

> Let $F$ be the hom-functor $\mathsf{C}(B,-)\colon \mathsf{C} \to \mathsf{Set}$. Then there is an isomorphism between $Nat(\mathsf{C}(A,-),F)$ and $FA$.

Now, to get from that to the Yoneda Lemma proper involves – as announced – two more stages. And we can now give a little more detail:

(2) *Generalizing on F.* We look again at the ingredients of the proof of the restricted version and ask 'Did we essentially depend on the fact that the second functor in the story, now notated simply '$F$', was actually a hom-functor $\mathsf{C}(B,-)$ for some $B$?'

Inspection reveals that we didn't. So we in fact have the more general result that for *any* functor $F\colon \mathsf{C} \to \mathsf{Set}$, and any $\mathsf{C}$-object $A$, there is an isomorphism between $Nat(\mathsf{C}(A,-),F)$ and $FA$.

(3) *Confirming it's all natural.* Our proof of this general result – like the proof of the original Restricted Lemma – provides a recipe for constructing the required isomorphism that doesn't involve any arbitrary choices, and doesn't depend on any special features of $A$ or $F$.

In an *intuitive* sense, then, we've constructed a natural isomorphism between the objects $Nat(\mathsf{C}(A,-),F)$ and $FA$. And so hopefully we should be able to show that these objects are naturally isomorphic in the official *categorial* sense.

In short, we will get from the Restricted Yoneda Lemma to the full-dress Yoneda Lemma by generalizing a construction, and then recasting in category-theoretic terms our intuitive judgement of the naturality of our construction. Neither stage involves anything conceptually very difficult. It is forgivable to skip the details, though you'll probably want to grasp the general proof ideas.

And needless to say, it's really a two-for-one deal: once we have done all the work of strengthening one half of the Restricted Lemma, we can leave strengthening the dual half as an exercise.

## 7. The generalizing move

**7.1.** We continue working in a locally small category $\mathsf{C}$. And let's restate some of what we already know, again using '$F$' to abbreviate '$\mathsf{C}(B,–)$':

(i) $F$ sends a $\mathsf{C}$-object $A$ to the set $FA = \mathsf{C}(B, A)$, and there is a bijection between elements of $FA$ and natural transformations $\mathsf{C}(A,–) \Rightarrow F$ – this bijection sends $f\colon B \to A$ in $FA$ to the transformation whose $X$-component maps an arrow $g\colon A \to X$ to $g \circ f\colon B \to X$.

(ii) $F$ sends a $\mathsf{C}$-arrow $g\colon A \to X$ to a function $Fg$, where this takes a $\mathsf{C}$-arrow $f\colon B \to A$ to the $\mathsf{C}$-arrow $g \circ f\colon B \to X$. In short, $Fg(f) = g \circ f$. (That's just from Theorem **??**, re-lettered.)

(iii) Hence, putting (i) and (ii) together, we have: there's a bijection which sends an element $f$ in $FA$ to the natural transformation whose $X$-component maps $g\colon A \to X$ to $Fg(f)$.

And note that (iii) potentially makes sense for *any* functor $F\colon \mathsf{C} \to \mathsf{Set}$. For $FA$, where $A$ is a $\mathsf{C}$-object, is a set. And $Fg$, where $g$ is a $\mathsf{C}$-arrow will be a $\mathsf{Set}$-arrow, i.e. a (total!) set-function. And that function $Fg$ can be applied to any element $f$ of the set $FA$.

**7.2.** We can now use the idea in (iii) to prove the following generalization of half of the Restricted Lemma (again my label is non-standard):

**Theorem 10** (The Intermediate Yoneda Lemma)**.** *For any object $A$ of the locally small category $\mathsf{C}$, and any functor $F\colon \mathsf{C} \to \mathsf{Set}$, $Nat(\mathsf{C}(A,–), F) \cong FA$.*

*Proof, step 1: Defining $\mathcal{X}_{AF}$, a candidate bijection.* Pick an element $f$ (in general, *not* a function!) from the set $FA$.

And taking up the idea in (iii), define $\alpha^f_X\colon \mathsf{C}(A, X) \to FX$ as the function that maps any $\mathsf{C}$-arrow $g\colon A \to X$ to $Fg(f)$. Similarly, define $\alpha^f_Y\colon \mathsf{C}(A, Y) \to FY$ as the function that maps any $\mathsf{C}$-arrow $h\colon A \to Y$ to $Fh(f)$; and so on.

Then it is immediate that a diagram like this commutes for any $j\colon X \to Y$:

$$\begin{array}{ccc}
\mathsf{C}(A,X) & \xrightarrow{\mathsf{C}(A,j)} & \mathsf{C}(A,Y) \\
\Big\downarrow{\scriptstyle \alpha^f_X} & & \Big\downarrow{\scriptstyle \alpha^f_Y} \\
FX & \xrightarrow{\quad Fj \quad} & FY
\end{array}$$

The upper route takes a $\mathsf{C}$-arrow $g\colon A \to X$ to $j \circ g\colon A \to Y$. And $\alpha^f_Y$ sends that on to $F(j \circ g)(f)$ which equals $Fj((Fg(f))$ by functoriality. While the lower route takes $g$ to $Fg(f)$ to $Fj(Fg(f))$. So we get the same result either way.

Hence, as defined, the components $\alpha^f_X, \alpha^f_Y, \ldots$ assemble into a natural transformation $\alpha^f\colon \mathsf{C}(A,–) \Rightarrow F$. Great!

So we have brought into play a nice function $\mathcal{X}_{AF}\colon FA \to Nat(\mathsf{C}(A,–), F)$ which sends an element $f$ to $\alpha^f$. It just remains to show that this function is bijective (as was its analogue in §4). □

*Proof step 2: Showing $\mathcal{X}_{AF}$ is surjective.* We want to prove that every natural transformation $\alpha\colon \mathsf{C}(A,–) \Rightarrow F$ is some $\alpha^f$ generated by an element $f$ in $FA$. Given the proof of Theorem 3, we know exactly how to do that!

We start by noting that, given $\alpha$ is a natural transformation, the following diagram in particular must commute, for any $Y$ and any $j\colon A \to Y$:

$$\begin{array}{ccc} \mathsf{C}(A,A) & \xrightarrow{\mathsf{C}(A,j)} & \mathsf{C}(A,Y) \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_Y} \\ FA & \xrightarrow{Fj} & FY \end{array}$$

Now chase the identity arrow $1_A$ round the diagram from the top left to bottom right nodes. The top route sends it first to $j$ and then on to $\alpha_Y(j)$. The bottom route sends it to $Fj(\alpha_A(1_A))$ – which, by definition, equals $\alpha^f_Y j$ for $f = \alpha_A(1_A)$. Since this holds for any $j$, we have $\alpha_Y = \alpha^f_Y$.

But $Y$ was arbitrary, so the equality holds for all components, therefore $\alpha = \alpha^f$ when $f = \alpha_A(1_A)$. □

*Proof step 3: Showing $\mathcal{X}_{AF}$ is injective.* We use the same pattern of argument as for Theorem 1 (cont'd), except that where we previously used the fact that $\mathsf{C}(f,–)_A 1_A = f$, we now use the fact that $\alpha^f_A(1_A) = f$. And why is that a fact? By definition, $\alpha^f_A$ sends an arrow $g\colon A \to A$ to $Fg(f)$. So $\alpha^f_A(1_A)$ yields $F1_A(f)$. But the functoriality of $F$ ensures that $F1_A$ is an identity function.

So if $f \neq f'$ we have $\alpha^f_A(1_A) = f \neq f' = \alpha^{f'}_A(1_A)$, and hence $\alpha^f \neq \alpha^{f'}$ □

Which establishes our announced theorem, and we've got a bijection $\mathcal{X}_{AF}$ between sets, showing they are isomorphic.

Of course, there's a companion dual result which I can leave as an exercise to check:[4]

**Theorem 10 (cont'd).** *For any object $A$ of the locally small category $\mathsf{C}$, and any functor $F\colon \mathsf{C}^{op} \to \mathsf{Set}$, $Nat(\mathsf{C}(–,A),F) \cong FA$.* □

**7.3.** Let's take up another idea from Theorem 3. We'll show that $\mathcal{X}_{AF}\colon FA \to Nat(\mathsf{C}(A,–),F)$ has a two-sided inverse $\mathcal{E}_{AF}\colon Nat(\mathsf{C}(A,–),F) \to FA$ where that is the function which sends a natural transformation $\alpha\colon \mathsf{C}(A,–) \Rightarrow FA$ to the element $\alpha_A(1_A)$. Proceeding more or less as before,

(1) Given any element $f$ of $FA$,

$$(\mathcal{E}_{AF} \circ \mathcal{X}_{AF})f = \mathcal{E}_{AF}\alpha^f = \alpha^f_A(1_A) = f.$$

But $f$ was arbitrary. Whence $\mathcal{E}_{AF} \circ \mathcal{X}_{AF} = 1$ (that's the identity on $FA$).

(2) Given any $\alpha\colon \mathsf{C}(–,A) \Rightarrow F)$,

$$(\mathcal{X}_{AF} \circ \mathcal{E}_{AF})\alpha = \mathcal{X}_{AF}(\alpha_A(1_A)) = \alpha^{\alpha_A(1_A)} = \alpha$$

(for the last identity, see the end of the Proof step 2 above). But $\alpha$ was arbitrary. Whence $\mathcal{X}_{AF} \circ \mathcal{E}_{AF} = 1$ (that's the identity on $Nat(\mathsf{C}(A,–),F)$).

[4]Hint: I'll state this dual in the conventional way, in terms of a covariant functor $F\colon \mathsf{C}^{op} \to \mathsf{Set}$. But you may well find it easier to keep track of which arrows go in which direction, and avoid dancing between $\mathsf{C}$ and $\mathsf{C}^{op}$, if you set out the proof thinking in terms of the equivalent contravariant $F\colon \mathsf{C} \to \mathsf{Set}$.

Having a two-sided inverse, $\mathcal{X}_{AF}$ is therefore (as we know!) an isomorphism – as is its inverse $\mathcal{E}_{AF}$, a point we'll need in a moment.

There is of course a dual story to be told about an isomorphism $\mathcal{Y}_{AF}\colon FA \to Nat(\mathsf{C}(-,A),F)$ which sends an element $f$ of $FA$ to a suitable $\alpha^f\colon \mathsf{C}(-,A) \Rightarrow F$. But I'll leave it as another challenge to fill in the details.

## 8. Making it all natural

**8.1.** So where have we got to?

To concentrate again on half the story, Theorem 10 tells us that – when $\mathsf{C}$ is a locally small category, $A$ is any object in that category, and $F\colon \mathsf{C} \to \mathsf{Set}$ is some functor – then $FA$ is isomorphic to $Nat(\mathsf{C}(A,-),F)$. I called that the Intermediate Yoneda Lemma. And the earlier Restricted Lemma, Theorem 5, is what get when we restrict to the cases where $F$ has the form $\mathsf{C}(B,-)$ for some $B$.

Now, our proof of Theorem 10 didn't depend on any special facts about $A$ or $F$, and didn't depend on any arbitrary choices. So, at least in an *intuitive* sense, we have found a natural isomorphism. But when we find an intuitively natural isomorphism between objects, what I like to call the Eilenberg/Mac Lane Thesis (compare the Church/Turing Thesis) enjoins us to see this as generated by a natural isomorphism in the *categorial* sense, an isomorphism between functors.

And so this is going to be our next move, stage (3) of the argument taking us to the full Yoneda Lemma. We want to show how the intuitively natural isomorphism we've found between between the objects $FA$ and $Nat(\mathsf{C}(A,-)),F)$ can be seen as arising, in fact in two ways, from natural isomorphisms between functors.

**8.2.** A quick observation before continuing, however.

It is often remarked that the Yoneda lemma is perhaps the most used result in category theory. And taking a look at monographs on category theory which go rather beyond entry level we do indeed find multiple invocations of Yoneda.

However, I think it is correct to say that the appeal is frequently either to our easy Yoneda Embedding result, Theorem 8, or to the simple existence of some isomorphism of the kind reported in the Intermediate Theorem 10. The further fact that such an isomorphism between objects can officially be seen as arising from a natural isomorphism between functors is quite often not needed.

**8.3.** Down to work again. Two objects in $\mathsf{Set}$ are said to be *naturally isomorphic in A* if they are the images $FA$ and $GA$ of the same object $A$ under a couple of naturally isomorphic functors $F,G\colon \mathsf{C} \to \mathsf{Set}$.

So to show that $FA$ and $Nat(\mathsf{C}(A,-),F)$ are naturally isomorphic in $A$, we need to find a functor $G\colon \mathsf{C} \to \mathsf{Set}$ such that $GA = Nat(\mathsf{C}(A,-),F)$, and $G$ is naturally isomorphic to $F$.

But now recall two functors we already know about:

(1) Theorem 6 defined the contravariant functor $\mathcal{X}\colon \mathsf{C} \to [\mathsf{C},\mathsf{Set}]$ which sends an object $A$ to the hom-functor $\mathsf{C}(A,-)$ (and sends a $\mathsf{C}$ arrow $f\colon A \to B$ to the natural transformation $\mathsf{C}(f,-)\colon \mathsf{C}(B,-) \Rightarrow \mathsf{C}(A,-)$).

(2) §2 gives us a contravariant functor, $Nat(-,F)\colon [\mathsf{C},\mathsf{Set}] \to \mathsf{Set}$. This sends a functor $\mathsf{C}(A,-)\colon \mathsf{C} \to \mathsf{Set}$ to the set $Nat(\mathsf{C}(A,-),F)$ (and sends a natural transformation $\mathsf{C}(f,-)\colon \mathsf{C}(B,-) \Rightarrow \mathsf{C}(A,-)$ to the corresponding function which takes a natural transformation $\alpha\colon \mathsf{C}(A,-) \Rightarrow F$ and outputs $\alpha \circ \mathsf{C}(f,-)\colon \mathsf{C}(B,-) \Rightarrow F$).

Hence, if we put $G = Nat(\text{–}, F) \circ \mathcal{X}$ we'll get, as we wanted, a covariant functor $G\colon \mathsf{C} \to \mathsf{Set}$ (because two contravariant functors compose to a covariant functor), and as required $GA = Nat(\mathsf{C}(A, \text{–}), F)$.

Then, to prove $GA$ and $FA$ are naturally isomorphic in $A$ we need to show the following:

**Theorem 11.** *Let $\mathsf{C}$ be a locally small category, and $F$ a functor $F\colon \mathsf{C} \to \mathsf{Set}$. Then the functors $G = Nat(\text{–}, F) \circ \mathcal{X}$ and $F$ are naturally isomorphic.*

And it wouldn't be absurd to set this as a challenge to prove for yourself. You just need to write down a potential naturality square of the kind we need. But what will be the components of the candidate natural isomorphism taking us from the likes of $GA$ (i.e. $Nat(\mathsf{C}(A, \text{–}), F)$) to $FA$? $\mathcal{E}_{AF}$ as defined in the previous section of course. Then you just need to prove that the square commutes. Try it before reading on!

*Proof.* Given some arrow $j\colon A \to B$, consider the following square:

$$\begin{array}{ccc} GA = Nat(\mathsf{C}(A, \text{–}), F) & \xrightarrow{\;Gj\;} & GB = Nat(\mathsf{C}(B, \text{–}), F) \\ \downarrow{\scriptstyle \mathcal{E}_{AF}} & & \downarrow{\scriptstyle \mathcal{E}_{BF}} \\ FA & \xrightarrow{\;Fj\;} & FB \end{array}$$

Take any $\alpha\colon \mathsf{C}(A, \text{–}) \Rightarrow F$ in $GA$. Then we have:

(1) $\mathcal{E}_{BF} \circ Gj\alpha = \mathcal{E}_{BF}(\alpha \circ \mathsf{C}(j, \text{–}) = (\alpha \circ \mathsf{C}(j, \text{–}))_B(1_B) = \alpha_B(\mathsf{C}(j, \text{–}))_B(1_B)) = \alpha_B(j)$.

(2) But also $Fj \circ \mathcal{E}_{AF}(\alpha) = Fj \circ \alpha_A(1_A) = \alpha_B \circ \mathsf{C}(A, j)(1_A) = \alpha_B(j)$ (for the middle equation we note that $Fj \circ \alpha_A = \alpha_B \circ \mathsf{C}(A, j)$ by a naturality square for $\alpha$).

So our diagram will always commute, and hence there is a natural isomorphism $\mathcal{E}_F\colon G \Rightarrow F$ with components $(\mathcal{E}_F)_A = \mathcal{E}_{AF}$ for each $A$ in $\mathsf{C}$, and our theorem is proved. □

**8.4.** That captures in categorial terms the intuition that the isomorphism between $FA$ and $Nat(\mathsf{C}(A, \text{–}), F)$ depends in a natural way on $A$. Now for the companion intuition that it depends in a natural way on $F$ too. Keeping $A$ fixed, we want to prove $Nat(\mathsf{C}(A, \text{–}), F) \cong FA$ naturally in $F$.

Now, I haven't actually said what it is for such an isomorphism to be natural in a functor like $F$ (as opposed to an object like $A$). But the idea is the predictable one. We want to show that our isomorphism arises again from a natural isomorphism between two functors.

So first, we want a functor which sends $F$ to $Nat(\mathsf{C}(A, \text{–}), F)$. But we've met such a functor in §2: it's the covariant hom functor $Nat(\mathsf{C}(A, \text{–}), \text{–})$, which is a functor from $[\mathsf{C}, \mathsf{Set}]$ to $\mathsf{Set}$.[5]

And second, we want a functor which sends $F$ to $FA$. And the obvious candidate is the simple evaluate-at-$A$ functor $ev_A\colon [\mathsf{C}, \mathsf{Set}] \to \mathsf{Set}$ which sends any functor $F\colon \mathsf{C} \to \mathsf{Set}$ to $FA$ and sends any natural transformation $\alpha\colon F \Rightarrow G$ to $\alpha_A\colon FA \to GA$.

So, to show that $Nat(\mathsf{C}(A, \text{–}), F)$ is isomorphic to $FA$ naturally in $F$, we want to prove the following:

**Theorem 12.** *Let $\mathsf{C}$ be a locally small category. Then $Nat(\mathsf{C}(A, \text{–}), \text{–})$ and $ev_A$ are naturally isomorphic.*

Again, here's a challenge: prove that before reading on ...

[5] The original definition of $Nat(F, \text{–})$ took a functor $F\colon \mathsf{C} \to \mathsf{D}$ to give us a hom-functor from $[\mathsf{C}, \mathsf{D}]$ to $\mathsf{Set}$. So yes, $Nat(\mathsf{C}(A, \text{–}), \text{–})$ takes the functor $\mathsf{C}(A, \text{–})\colon \mathsf{C} \to \mathsf{Set}$, and gives us a hom-functor from $[\mathsf{C}, \mathsf{Set}]$ to $\mathsf{Set}$!

*Proof.* Given any $\gamma\colon F \Rightarrow G$, consider the following diagram,

$$\begin{array}{ccc} Nat(\mathsf{C}(A,–),F) & \xrightarrow{Nat(\mathsf{C}(A,–),\gamma)} & Nat(\mathsf{C}(A,–),G) \\ \downarrow{\scriptstyle \mathcal{E}_{AF}} & & \downarrow{\scriptstyle \mathcal{E}_{AG}} \\ ev_A(F) = FA & \xrightarrow{ev_A(\gamma)} & ev_A(G) = GA \end{array}$$

Take any $\alpha\colon \mathsf{C}(A,–) \Rightarrow F$, and recall that $Nat(\mathsf{C}(A,–),\gamma)$ sends $\alpha$ to $\gamma \circ \alpha$. Then we have:

(1) $\mathcal{E}_{AG} \circ Nat(\mathsf{C}(A,–),\gamma)(\alpha) = \mathcal{E}_{AG}(\gamma \circ \alpha) = (\gamma \circ \alpha)_A(1_A) = \gamma_A(\alpha_A(1_A))$.
(2) But also $ev_A(\gamma) \circ \mathcal{E}_{AF}(\alpha) = \gamma_A(\alpha_A(1_A))$.

Hence the diagram always commutes. Therefore there is a natural isomorphism $\mathcal{E}_A\colon K \Rightarrow ev_A$ with components $(\mathcal{E}_A)_F = \mathcal{E}_{AF}$ for each $F$ from $[\mathsf{C}, \mathsf{Set}]$. □

## 9. Putting everything together

Our last two theorems have duals – which I'll leave you to state and prove. But taking those as read, we can now combine all the ingredients from the last three theorems . . .

Cue drum-roll!

. . . and at last we get the fully caffeinated Lemma:

**Theorem 13** (Yoneda Lemma)**.** *For any locally small category* $\mathsf{C}$*, object* $A$ *in* $\mathsf{C}$*, and covariant functor* $F\colon \mathsf{C} \to \mathsf{Set}$*,* $Nat(\mathsf{C}(A,–),F) \cong FA$*, both naturally in* $A$ *and naturally in* $F$*.*

*And for any contravariant functor* $F\colon \mathsf{C} \to \mathsf{Set}$ *(equivalently, covariant functor* $F\colon \mathsf{C}^{op} \to \mathsf{Set}$*),* $Nat(\mathsf{C}(–,A),F) \cong FA$*, both naturally in* $A$ *and naturally in* $F$*.* □

. . . So we are done!

. . . And that wasn't *too* bewildering, was it?[6]

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[6]To aid comparison with other presentations, two quick notational/terminological points. First, where I have written $\mathsf{C}(A,–)$ and $\mathsf{C}(–,A)$ for, respectively, the covariant and contravariant hom-functors, you will often find others writing simply $h_A$ and $h^A$.

Second, for reasons I won't go into here, a contravariant functor from $\mathsf{C}$ to $\mathsf{Set}$, i.e. a functor $F\colon \mathsf{C}^{op} \to \mathsf{Set}$, is standardly called a *presheaf* on $\mathsf{C}$. And then the presheaves on $\mathsf{C}$ (as objects) together with the natural transformations between them (as arrows) form the presheaf category on $\mathsf{C}$, often denoted simply $\widehat{\mathsf{C}}$ (where we wrote $[\mathsf{C}^{op}, \mathsf{Set}]$). So in this notation, the Yoneda embedding is a functor $\mathcal{Y}\colon \mathsf{C} \to \widehat{\mathsf{C}}$.