A Use the truth-table test to determine which of the following arguments are tautologically valid:

1. $\neg(P \lor Q) \therefore \neg P$

P	Q	$\neg(P \lor Q)$	$\neg P$
T	T	F	F
T	F	F	F
F	T		T
F	F		T

We evaluate the conclusion first, and then ignore lines 3 and 4 which are already shown not to be bad. There are no bad lines, so the inference is tautologically valid.

2. $(P \land Q) \therefore (P \lor \neg Q)$

P	Q	$(P \land Q)$	$(P \lor \neg Q)$
T	T		T
T	F		T
F	T	F	F
F	F		T

Again, we evaluate the conclusion first, and immediately we are left with only one candidate bad line with a false conclusion. So we only need to evaluate the premiss on just one line. When we complete the table, we find that there are no bad lines, so the inference is tautologically valid.

3. $(P \lor \neg Q), \neg P \therefore \neg\neg Q$

P	Q	$(P \lor \neg Q)$	$\neg P$	$\neg\neg Q$
T	T			T
T	F		F	F
F	T			T
F	F	T	T	F

Once more, we evaluate the conclusion first: we are left with only two candidate bad lines. Next evaluate the simpler premiss on those two lines. That eliminates one of the candidate bad lines. So we only need to evaluate the remaining premiss on just one line. When we complete the table, we find that there is a bad line, so the inference is not tautologically valid.

4. $(P \land S), \neg(S \land \neg R) \therefore (R \lor \neg P)$

P	R	S	$(P \land S)$	$\neg(S \land \neg R)$	$(R \lor \neg P)$
T	T	T			T
T	T	F			T
T	F	T	T	F	F
T	F	F	F		F
F	T	T			T
F	T	F			T
F	F	T			T
F	F	F			T

Once more, we evaluate the conclusion first: we are left with only two candidate bad lines. Next evaluate the simpler premiss on those two lines. That eliminates one of the candidate bad lines. So we only need to evaluate the remaining premiss on just one line. When we complete the table, we find that there is no bad line, so the inference is tautologically valid.

5. P, ¬(P ∧ ¬Q), (¬Q ∨ R) ∴ R

P	Q	R	P	¬(P ∧ ¬Q)	(¬Q ∨ R)	R
T	T	T				T
T	T	F	T	T	F	F
T	F	T				T
T	F	F	T	F		F
F	T	T				T
F	T	F	F			F
F	F	T				T
F	F	F	F			F

The same plan of action, we evaluate the conclusion first: we are left with four candidate bad lines. Next evaluate the simpler premiss on those four lines. That eliminates two of the candidate bad lines. Then we continue in the obvious way. When we complete the table, we find that there is no bad line, so the inference is tautologically valid.

6. (P ∨ Q), ¬(P ∧ ¬R), (R ∨ ¬Q) ∴ R

P	Q	R	(P ∨ Q)	¬(P ∧ ¬R)	(R ∨ ¬Q)	R
T	T	T				T
T	T	F	T	F		F
T	F	T				T
T	F	F	T	F		F
F	T	T				T
F	T	F	T	T	F	F
F	F	T				T
F	F	F	F			F

Similar order of evaluations to (5): again a tautologically valid inference.

7. (¬P ∨ ¬(Q ∨ R)), (Q ∨ (P ∧ R)) ∴ (¬P ∨ Q)

P	Q	R	(¬P ∨ ¬(Q ∨ R))	(Q ∨ (P ∧ R))	(¬P ∨ Q)
T	T	T			T
T	T	F			T
T	F	T	f F f		F
T	F	F	f T t	f F f	F
F	T	T			T
F	T	F			T
F	F	T			T
F	F	F			T

Evaluate the conclusion first. Then the first premiss – the overall value is signified by the capital 'T' or 'F', the lower case letters indicate working towards the overall valuation. Tautologically valid.

8. (P ∨ Q), ¬(Q ∧ ¬¬R) ∴ ¬(R ∨ P)

P	Q	R	(P ∨ Q)	¬(Q ∧ ¬¬R)	¬(R ∨ P)
T	T	T	T	F t	F
T	T	F	T	T f	F
T	F	T	T		F
T	F	F	T		F
F	T	T	T		F
F	T	F			T
F	F	T	F		F
F	F	F			T

Evaluate the conclusion first. Then the first premiss. At that point there are still five possible bad lines. We start evaluating the second premiss, and the second line gives us a bad line – so we can stop here, having shown the argument not to be valid.

9. (P ∧ (Q ∨ R)) ∴ ((P ∧ Q) ∨ (P ∧ R))

P	Q	R	(P ∧ (Q ∨ R))			((P ∧ Q) ∨ (P ∧ R))		
T	T	T				t	T	
T	T	F				t	T	
T	F	T				f	T	t
T	F	F	t	F	f	f	F	f
F	T	T	f	F		f	F	f
F	T	F	f	F		f	F	f
F	F	T	f	F		f	F	f
F	F	F	f	F		f	F	f

Tautologically valid.

10. (P ∨ Q), (¬P ∨ R), ¬(Q ∧ S) ∴ ¬(¬R ∧ S)

P	Q	R	S	(P ∨ Q)	(¬P ∨ R)	¬(Q ∧ S)	¬(¬R ∧ S)
T	T	T	T				T
T	T	T	F				T
T	T	F	T	T	F		F
T	T	F	F				T
T	F	T	T				T
T	F	T	F				T
T	F	F	T	T	F		F
T	F	F	F				T
F	T	T	T				T
F	T	T	F				T
F	T	F	T	T	T	F	F
F	T	F	F				T
F	F	T	T				T
F	F	T	F				T
F	F	F	T	F			F
F	F	F	F				T
				3	2	4	1

Tautologically valid.

B Evaluate the following arguments:

1. Either Jack went up the hill or Jill did. Either Jack didn't go up the hill or the water got spilt. Hence, either Jill went up the hill or the water got spilt.

   Put

   P Jack went up the hill.
   Q Jill went up the hill.
   R The water got spilt.

   then the translation is

   (P ∨ Q), (¬P ∨ R) ∴ (Q ∨ R)

P	Q	R	(P ∨ Q)	(¬P ∨ R)	(Q ∨ R)
T	T	T			T
T	T	F			T
T	F	T			T
T	F	F	T	F	F
F	T	T			T
F	T	F			T
F	F	T			T
F	F	F	F		F

Tautologically valid – and hence plain valid.

2. It isn't the case the Jack went up the hill and Jill didn't. It isn't the case that Jill went up the hill and the water got spilt. So either Jack didn't go up the hill or the water got spilt.

P Jack went up the hill.
Q Jill went up the hill.
R The water got spilt.

$\neg(P \wedge \neg Q), \neg(Q \wedge R) \therefore (\neg P \vee R)$

P	Q	R	¬(P ∧ ¬Q)	¬(Q ∧ R)	(¬P ∨ R)
T	T	T			T
T	T	F	T	T	F
T	F	T			T
T	F	F	F		F
F	T	T			T
F	T	F			T
F	F	T			T
F	F	F			T

Tautologically invalid – and the situation which makes the premisses true and conclusion false is evidently possible, so the argument is plain invalid (see §13.5).

3. Either Jack is lazy or stupid. It isn't true that he is both lazy and a good student. Either Jack isn't a good student or he isn't stupid. Hence he isn't a good student.

P Jack is lazy.
Q Jack is stupid.
R Jack is a good student.

$(P \vee Q), \neg(P \wedge R), (\neg R \vee \neg Q) \therefore \neg R$

P	Q	R	(P ∨ Q)	¬(P ∧ R)	(¬R ∨ ¬Q)	¬R
T	T	T	T	F		F
T	T	F				T
T	F	T	T	F		F
T	F	F				T
F	T	T	T	T	F	F
F	T	F				T
F	F	T	F			F
F	F	F				T

Tautologically valid.

4. Either Jill hasn't trained hard or she will win the race. It isn't true that she'll win the race and not be praised. Either Jill has trained hard or she deserves to lose. Hence either Jill will be praised or she deserves to lose.

P Jill has trained hard.
Q Jill will win the race.
R Jill will be praised.
S Jill deserves to lose.

$(\neg P \vee Q), \neg(Q \wedge \neg R), (P \vee S) \therefore (R \vee S)$

P	Q	R	S	(¬P ∨ Q)	¬(Q ∧ ¬R)	(P ∨ S)	(R ∨ S)
T	T	T	T				T
T	T	T	F				T
T	T	F	T				T
T	T	F	F	T	F		F
T	F	T	T				T
T	F	T	F				T
T	F	F	T				T
T	F	F	F	F			F
F	T	T	T				T
F	T	T	F				T
F	T	F	T				T
F	T	F	F	T	F		F
F	F	T	T				T
F	F	T	F				T
F	F	F	T				T
F	F	F	F	T	T	F	F
				2	3	4	1

Tautologically valid so plain valid.

C Which of the following are true; which false; and why?

1. Any two tautologies are truth-functionally equivalent.
   True — wffs are truth-functionally equivalent just if they take the same value on all valuations of all the atoms occurring in them (see p. 94). But two tautologies will always be true on any valuation of all the atoms in them, so always take the same value as each other.

2. If *A* tautologically entails *C*, then *A* entails any wff truth-functionally equivalent to *C*.
   True — if C^* is equivalent to *C*, then whenever *C* is true, C^* is true. So if *A* tautologically entails *C*, i.e. if whenever *A* is true, *C* is true, then whenever *A* is true C^* is true, i.e. *A* tautologically entails C^*

3. If *A* and *B* tautologically entail *C* and *B* is a tautology, then *A* by itself tautologically entails *C*.
   True — *A* and *B* tautologically entail *C* just in case there is no valuation which makes *A* and *B* true and *C* . That is to say, any way of making *C* false must make either *A* or *B* false. But nothing makes *B* false since it is a tautology. So any way of making *C* false must make *A* false. So every way of making *A* true, must make *C* true.

4. If $A \vDash B$, then $\neg A \vDash \neg B$.
   False. P and Q entails P, but not-(P and Q) doesn't entail not-P.

5. If $A \vDash B$, then $\neg B \vDash \neg A$.
   True — if *A* tautologically entails *B*, there is no valuation which makes *A* true and *B* false, so there is no valuation which makes $\neg A$ false and $\neg B$ true, so $\neg B$ tautologically entails $\neg A$.

6. If *A* is a contradiction, then $A \vDash B$, for any *B*.
   True — if *A* is a contradiction, then there is no valuation which makes *A* true; so there is no valuation which makes *A* true and *B* false, so $A \vDash B$ irrespective of what *B* is.

7. If the set of wffs $A_1, A_2, \ldots, A_n$ is unsatisfiable, the set $\neg A_1, \neg A_2, \ldots, \neg A_n$ is satisfiable.

False — the set of wffs P, ¬P is unsatisfiable, and so is ¬P, ¬¬P is unsatisfiable.

8. If the set of wffs $A_1, A_2, \ldots, A_n$ is unsatisfiable, then $A_1, A_2, \ldots, A_n \vDash B$, for any B.
   True — if the set of wffs $A_1, A_2, \ldots, A_n$ is unsatisfiable, then there is no valuation which makes all the A_i true together; so there is no valuation which makes all the A_i true together and B false, so $A_1, A_2, \ldots, A_n \vDash B$ irrespective of what B is.

9. If $A \vDash C$, then $\vDash (\neg A \lor C)$.
   True — if A tautologically entails C, there is no valuation which makes A true and C false, i.e. every valuation either makes $\neg A$ false and C true, so every valuation makes $(\neg A \lor C)$ true.

10. If A, B, C are tautologically inconsistent, then $A \vDash (\neg B \lor \neg C)$.
    True — if A, B, C are tautologically inconsistent, then there is no valuation which makes A true and also makes B and C true together, i.e. every valuation which makes A true makes one of B and C false, so every valuation makes A true makes $(\neg B \lor \neg C)$ true.

11. If A, B are tautologically inconsistent, so are $\neg A, \neg B$.
    False — Suppose A is a contradiction. Then it is inconsistent with any B. But the negation of A is a tautology and can be consistent with not-B for suitable B.

12. If the wffs A, B together make a satisfiable set, and B, C do too, then the wffs A, C together make a satisfiable set.
    False — the set of wffs P, Q is unsatisfiable, and so is Q, ¬P, is satisfiable. But P, ¬P isn't.