Answers to Exercises 14

A Given

> 'P' expresses *Plato is a philosopher*
> 'Q' expresses *Quine is a philosopher*
> 'R' expresses *Russell is a philosopher*
> 'S' expresses *Socrates is a philosopher*

translate the following into **PLC** as best you can:

1. Quine is a philosopher only if Russell is.
   $(Q \supset R)$
2. If either Quine or Plato is a philosopher, so is Russell.
   $((Q \lor P) \supset R)$
3. Only if Plato is a philosopher is Russell one too.
   $(R \supset P)$
4. Quine and Russell are both philosophers only if Socrates is.
   $((Q \land R) \supset S)$
5. Russell's being a philosopher is a necessary condition for Quine's being one.
   $(Q \supset R)$
6. Plato is a philosopher if and only if Quine isn't.
   $(P \equiv \neg Q)$
7. If Plato is a philosopher if and only if Socrates is, then Russell is one too.
   $((P \equiv S) \supset R)$
8. Only if either Plato or Russell is a philosopher are both Quine and Socrates philosophers.
   $((Q \land S) \supset (P \lor R))$
9. That Socrates is a philosopher is a sufficient condition for it to be a necessary condition of Quine's being a philosopher that Russell is one.
   $(S \supset (Q \supset R))$
10. Quine is a philosopher unless Russell is one.
    $(Q \equiv \neg R)$
    (if you hear the claim as the saying that if Russell is a philosopher, then Quine isn't one, while in other circumstances, Quine is one: cf. Exercises 9, B7.)
11. Provided that Quine is a philosopher, Russell is one too.
    $(Q \equiv R)$
    (again, if you hear the claim as the saying that if Quine is a philosopher, then Russell is, while in other circumstances, Russell isn't.)

B Use truth-tables to determine which of the following arguments are tautologically valid:

1. $P, (P \supset Q), (Q \supset R) \therefore R$

P	Q	R	P	$(P \supset Q)$	$(Q \supset R)$	R
T	T	T				T
T	T	F	T	T	F	F
T	F	T				T
T	F	F	T	F		F
F	T	T				T
F	T	F	F			F
F	F	T				T
F	F	F	F			F

We use the now standard plan for cutting down working (see p. 111 if you are in any doubts about this). So we evaluate the conclusion first: we are left with four candidate bad lines. Next evaluate the simplest premiss on those four lines. That eliminates two of the candidate bad lines. Then we continue in the obvious way. When we complete the table, we find that there is no bad line, so the inference is tautologically valid.

2. P, (P ≡ Q), (Q ≡ R) ∴ R

P	Q	R	P	(P ≡ Q)	(Q ≡ R)	R
T	T	T				T
T	T	F	T	T	F	F
T	F	T				T
T	F	F	T	F		F
F	T	T				T
F	T	F	F			F
F	F	T				T
F	F	F	F			F

Interestingly, this is the same truth-table as for (1). How can this be when the argument is different? Because the *full* tables, with every wff evaluated on each line will indeed be different, but the differences will be on lines which don't need to be worked out on our short working.

3. (P ⊃ R) ∴ ((P ∧ Q) ⊃ R)

P	Q	R	(P ⊃ R)	((P ∧ Q) ⊃ R)
T	T	T		T t
T	T	F	F	t F f
T	F	T		T t
T	F	F		f T f
F	T	T		T t
F	T	F		f T f
F	F	T		T t
F	F	F		f T f

Tautologically valid.

4. ¬R, (¬P ⊃ R) ∴ P

P	R	¬R	(¬P ⊃ R)	P
T	T			T
T	F			T
F	T	F		F
F	F	T	F	F

Tautologically valid.

5. (P ≡ R) ∴ (¬R ≡ ¬P)

P	R	(P ≡ R)	(¬R ≡ ¬P)
T	T		T
T	F	F	F
F	T	F	F
F	F		T

Tautologically valid.

6. (P ⊃ R) ∴ (¬R ⊃ ¬P)

P	R	(P ⊃ R)	(¬R ⊃ ¬P)
T	T		T
T	F	F	F
F	T		T
F	F		T

Tautologically valid.

7. ¬R, (P ⊃ R), (¬P ⊃ Q) ∴ Q

P	Q	R	¬R	(P ⊃ R)	(¬P ⊃ Q)	Q
T	T	T				T
T	T	F				T
T	F	T	F			F
T	F	F	T	F		F
F	T	T				T
F	T	F				T
F	F	T	F			F
F	F	F	T	T	F	F

Tautologically valid.

8. (P ∨ Q), (P ⊃ R), ¬(Q ∧ ¬R) ∴ R

P	Q	R	(P ∨ Q)	(P ⊃ R)	¬(Q ∧ ¬R)	R
T	T	T				T
T	T	F	T	F	F	F
T	F	T				T
T	F	F	T	F		F
F	T	T				T
F	T	F	T	T		F
F	F	T				T
F	F	F	F			F

Tautologically valid.

9. (R ≡ (¬P ∨ Q)), ¬(P ∧ ¬R) ∴ ¬(¬R ∨ Q)

P	Q	R	(R ≡ (¬P ∨ Q))	¬(P ∧ ¬R)	¬(¬R ∨ Q)
T	T	T	T	T	F t
T	T	F		F	F t
T	F	T			T f
T	F	F		F	F t
F	T	T		T	F t
F	T	F		T	F t
F	F	T			T f
F	F	F		T	F t

Here we've worked methodically from the right, conclusion first; then the simpler premiss, at which point there are still four potentially bad lines. Starting work on the first premiss the very first line gives us a true premiss – so we can stop work at this point. The inference is revealed as tautologically invalid.

10. (¬P ∨ Q), ¬(Q ∧ ¬R) ∴ (P ⊃ R)

P	Q	R	(¬P ∨ Q)	¬(Q ∧ ¬R)	(P ⊃ R)
T	T	T			T
T	T	F	T	F	F
T	F	T			T
T	F	F	F		F
F	T	T			T
F	T	F			T
F	F	T			T
F	F	F			T

Tautologically valid.

11. $(P \wedge \neg R), (Q \supset R) \therefore \neg(P \supset Q)$

P	Q	R	$(P \wedge \neg R)$	$(Q \supset R)$	$\neg(P \supset Q)$
T	T	T	F		
T	T	F	T	F	
T	F	T	F		
T	F	F	T	T	T
F	T	T	F		
F	T	F	F		
F	F	T	F		
F	F	F	F		

Of course you can evaluate premisses and conclusion in a different order, and you'll get the same result that this inference is tautologically valid. Why have I chosen this order? Well, note that '$\neg(P \supset Q)$' is mostly false (since on most lines – six to be precise – '$(P \supset Q)$' is true). So if we start with that, we'd have lots of lines with false conclusions and a lot more work to do. '$(P \wedge \neg R)$' is also often false (again on six lines), so evaluating that first immediately rules out six lines as potential bad lines.

12 $\neg(P \equiv (Q \wedge R)), (S \vee \neg Q) \therefore \neg(S \supset P)$

P	Q	R	S	$\neg(P \equiv (Q \wedge R))$	$(S \vee \neg Q)$	$\neg(S \supset P)$
T	T	T	T	F	T	F
T	T	T	F	F	F	F
T	T	F	T	T	T	F
T	T	F	F		F	F
T	F	T	T		T	F
T	F	T	F		T	F
T	F	F	T		T	F
T	F	F	F		T	F
F	T	T	T			T
F	T	T	F		F	F
F	T	F	T			T
F	T	F	F		F	F
F	F	T	T			T
F	F	T	F		T	F
F	F	F	T			T
F	F	F	F		T	F
				3	2	1

Here we've worked methodically from the right, conclusion first; then the simpler premiss, at which point there are still eight potentially bad lines. Starting work on the first premiss the third line gives us a true premiss – so we can stop work at this point. The inference is revealed as tautologically invalid.

C Which of the arguments that you have just shown to be tautologically valid correspond to intuitively valid arguments when '$\supset$' is replaced by 'if …, then …', and '$\equiv$' is replaced by '… if and only if …'?

The inference (3), $(P \supset R) \therefore ((P \wedge Q) \supset R)$, is perhaps problematic: cf. if I flip the switch, the light will go on; hence if I flip the switch and the fuse is removed, then the light will go on. But perhpas this sort of counterexample shows again how future conditionals line up with subjunctives rather than the other indicatives that can be regimented by the material conditional. The other tautologically valid examples look plausibly valid if the material (bi)conditionals are replaced by vernacular indicative conditionals.

D Which of the following are true and why?

1. If A doesn't tautologically entail B, then $\vDash \neg(A \equiv B)$.
   False. 'P' doesn't tautologically entail 'Q': but it isn't the case that $\vDash \neg(\mathsf{P} \equiv \mathsf{Q})$, i.e. the wff '$\neg(\mathsf{P} \equiv \mathsf{Q})$' is not a tautology.

2. If A and B tautologically entail each other, then $\vDash (A \equiv B)$.
   True. If A and B tautologically entail each other, then on all valuations where A is true, B is true too. And on all valuations where A is false, B is false too. Hence on all valuations, A and B have the same value. Hence on all valuations $(A \equiv B)$ is true.

3. If $\neg A \vDash \neg B$, then $\vDash (B \supset A)$.
   True. Suppose $\neg A \vDash \neg B$. That is, suppose every valuation which makes $\neg A$ true makes $\neg B$. Then, any valuation either (i) makes A true or else (ii) makes $\neg A$ and so $\neg B$ true. So, any valuation makes A true or B false. So, any valuation makes $(B \supset A)$ true.

4. If $A, B \vDash C$ then $A \vDash (B \supset C)$
   True. Suppose $A, B \vDash C$. That is, suppose every valuation which makes A and B true makes C true. Then, any valuation which makes A true either (i) makes B false or (ii) makes B true and hence C true. So, any valuation which makes A true makes B-false-or-C-true, i.e. makes $(B \supset C)$ true.

5. If $\vDash (A \supset B)$ and $\vDash (B \supset C)$, then $\vDash (A \supset C)$.
   True. If every valuation makes $(A \supset B)$ true and also makes $(B \supset C)$, then every valuation which makes A true must make B true and hence make C true too. So every valuation either (i) makes A false or (ii) makes A true and hence makes C true. So every valuation makes $(A \supset C)$ true.

6. If $\vDash (A \supset B)$ and $\vDash (A \supset \neg B)$, then $\vDash \neg A$.
   True. If every valuation makes $(A \supset B)$ true and also makes $(A \supset \neg B)$, then any valuation which makes A true must make B true *and* $\neg B$ true – so there can be no valuation which makes A true, so every valuation makes $\neg A$ true.

7. If $\vDash (A \equiv B)$, then A, B are tautologically consistent.
   False. Suppose $A = B = (\mathsf{P} \wedge \neg\mathsf{P})$, then $\vDash (A \equiv B)$, but there is no valuation which makes A, B are true together.

8. If $\vDash (A \equiv \neg B)$, then A, B are tautologically inconsistent.
   True. Suppose $\vDash (A \equiv \neg B)$, then on every valuation $(A \equiv \neg B)$ is true, i.e. A, $\neg B$ have the same value, i.e. A, B have the different values, i.e. there is no valuation which makes A, B are true together – so A, B are tautologically inconsistent.