An Introduction to
Formal Logic

Answers to Exercises 17

A Use the tree test to determine which of the following arguments (from Exercises 13A) are tautologically valid:

1. $\neg(P \lor Q) \therefore \neg P$

$\neg(P \lor Q)$	√
$\neg\neg P$	
$\neg P$	
$\neg Q$	
✼	

The tree closes, so the inference is tautologically valid. Note you should have ticked off the one complex wff you have unpacked. NB line numbers aren't necessary!

2. $(P \land Q) \therefore (P \lor \neg Q)$

$(P \land Q)$	√
$\neg(P \lor \neg Q)$	√
P	
Q	
$\neg P$	
$\neg\neg Q$	
✼	

The tree closes, so the inference is tautologically valid. Of course, to expose the contradiction, we really only need to extract 'P' from the first line and '¬P' from the second. However, note that the unpacking rules on p.159 insist that you add both conjuncts when unpacking a conjunction etc. (That's to ensure that when you 'tick' off a wff you really have extracted all its implications.)

3. $(P \lor \neg Q), \neg P \therefore \neg\neg Q$

The tree starts as follows – NB the triple helping of negation signs in the negated conclusion!

$(P \lor \neg Q)$
$\neg P$
$\neg\neg\neg Q$

Applying the non-branching rule to the last wff first, the completed tree looks like this:

$(P \lor \neg Q)$		√
$\neg P$		
$\neg\neg\neg Q$		√
$\neg Q$		
P	$\neg Q$	
✼		

The tree is finished (we have ticked off every complex wff), and doesn't close. So the inference is invalid.

4. $(P \wedge S), \neg(S \wedge \neg R) \therefore (R \vee \neg P)$

(1)	$(P \wedge S)$		√
(2)	$\neg(S \wedge \neg R)$		√
(3)	$\neg(R \vee \neg P)$		√
(4)	P		
(5)	S		
(6)	$\neg R$		
(7)	$\neg\neg P$		
(8)	$\neg S$ ✻	$\neg\neg R$ ✻	

The inference is tautologically valid. Here we have first applied the non-branching rules to (1) and (3), before applying the branching rule to (2). We could have then unpacked (7) too – but we don't need to do so in order to get the tree to close.

5. $P, \neg(P \wedge \neg Q), (\neg Q \vee R) \therefore R$

P			
$\neg(P \wedge \neg Q)$			√
$(\neg Q \vee R)$			√
$\neg R$			
$\neg P$ ✻	$\neg\neg Q$		
	$\neg Q$ ✻	R ✻	

The inference is tautologically valid.

6. $(P \vee Q), \neg(P \wedge \neg R), (R \vee \neg Q) \therefore R$

(1)	$(P \vee Q)$			√
(2)	$\neg(P \wedge \neg R)$			√
(3)	$(R \vee \neg Q)$			√
(4)	$\neg R$			
(5)	R ✻	$\neg Q$		
(6)		P	Q ✻	
(7)	$\neg P$ ✻	$\neg\neg R$ ✻		

The inference is tautologically valid. We've chosen to unpack the three premisses in the order (3)–(1)–(2): why? Well, if we'd unpacked (1) first, no branch would immediately close, and the resulting tree would sprawl a bit.

7. (¬P ∨ ¬(Q ∨ R)), (Q ∨ (P ∧ R)) ∴ (¬P ∨ Q)

```
(1)              (¬P ∨ ¬(Q ∨ R))          √
(2)               (Q ∨ (P ∧ R))           √
(3)                ¬(¬P ∨ Q)              √
(4)                   ¬¬P
(5)                   ¬Q
                   /      \
(6)              ¬P      ¬(Q ∨ R)         √
(7)              *          ¬Q
(8)                         ¬R
                          /     \
(9)                      Q     (P ∧ R)    √
(10)                     *        P
(11)                              R
                                  *
```

Apply the 'straight' rule to (3) first; similarly at (5) where we have a choice of unpacking (2) or (5), again apply the 'straight' rule first. Tree closes, so argument is valid.

8. (P ∨ Q), ¬(Q ∧ ¬¬R) ∴ ¬(R ∨ P)

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(1)                        (P ∨ Q)                          √
(2)                       ¬(Q ∧ ¬¬R)                        √
(3)                       ¬¬(R ∨ P)                         √
(4)                        (R ∨ P)                          √
                     /                  \
(5)               P                        Q
               /      \                 /      \
(6)         ¬Q        ¬¬¬R           ¬Q        ¬¬¬R         √
(7)        /  \        ¬R             *         ¬R
                      /  \                     /  \
(8)      R     P     R    P                   R    P
                     *                        *
```

That leaves four open branches. These correspond (from the left) to the (non-exclusive!) valuations

P ⇒ T, Q ⇒ F, R ⇒ T
P ⇒ T, Q ⇒ F, and don't care about R
P ⇒ T, R ⇒ F, and don't care about Q
P ⇒ T, Q ⇒ T, R ⇒ F

which, as is easily checked, all make the premisses true and conclusion false, so the inference is indeed invalid.

9. $(P \wedge (Q \vee R)) \therefore ((P \wedge Q) \vee (P \wedge R))$

(1) $(P \wedge (Q \vee R))$ √
(2) $\neg ((P \wedge Q) \vee (P \wedge R))$ √
(3) P
(4) $(Q \vee R)$ √
(5) $\neg(P \wedge Q)$ √
(6) $\neg(P \wedge R)$ √
(7) $\neg P$ ✻ | $\neg Q$
(8) $\neg P$ ✻ | $\neg R$
(9) Q ✻ | R ✻

Valid again!

10. $(P \vee Q)$, $(\neg P \vee R)$, $\neg(Q \wedge S) \therefore \neg(\neg R \wedge S)$

$(P \vee Q)$ √
$(\neg P \vee R)$ √
$\neg(Q \wedge S)$ √
$\neg\neg(\neg R \wedge S)$ √
$(\neg R \wedge S)$ √
$\neg R$
S √

$\neg P$ | R ✻
$\neg Q$ | $\neg S$ ✻
P ✻ | Q ✻

Valid – and just compare the brevity of that with the following cut-down, minimal truth table!!

P	Q	R	S	$(P \vee Q)$	$(\neg P \vee R)$	$\neg(Q \wedge S)$	$\neg(\neg R \wedge S)$
T	T	T	T				T
T	T	T	F				T
T	T	F	T	T	F		F
T	T	F	F				T
T	F	T	T				T
T	F	T	F				T
T	F	F	T	T	F		F
T	F	F	F				T
F	T	T	T				T
F	T	T	F				T
F	T	F	T	T	T	F	F
F	T	F	F				T
F	F	T	T				T
F	F	T	F				T
F	F	F	T	F			F
F	F	F	F				T
				3	2	4	1

B Which of the following are tautologies?

2 ((Q ∧ (¬¬P ∨ R)) ∨ ¬(P ∧ Q))

¬((Q ∧ (¬¬P ∨ R)) ∨ ¬(P ∧ Q))	√
¬(Q ∧ (¬¬P ∨ R))	√
¬¬(P ∧ Q)	√
(P ∧ Q)	√
P	
Q	
¬Q ¬(¬¬P ∨ R)	√
* ¬¬¬P	√
¬R	
¬P	
*	

We negated the wff to be tested for being a tautology, and use that wff to start the tree. We then apply the 'straight' rules first, and keep going …

3 ({P ∨ ¬(Q ∧ R)} ∨ {(¬P ∧ Q) ∧ R})

(1)	¬({P ∨ ¬(Q ∧ R)} ∨ {(¬P ∧ Q) ∧ R})	
(2)	¬{P ∨ ¬(Q ∧ R)}	(from 1)
(3)	¬{(¬P ∧ Q) ∧ R}	(from 1)
(4)	¬P	(from 2)
(5)	¬¬(Q ∧ R)	(from 2)
(6)	(Q ∧ R)	(from 5)
(7)	Q	
(8)	R	
(9)	¬(¬P ∧ Q) ¬R *	(from 3)
(10)	¬¬P * ¬Q *	(from 9)

A tautology again.

4 ({P ∨ (Q ∧ ¬R)} ∨ ¬{(¬P ∨ R) ∨ Q})

(1)	¬({P ∨ (Q ∧ ¬R)} ∨ ¬{(¬P ∨ R) ∨ Q})	
(2)	¬{P ∨ (Q ∧ ¬R)}	(from 1)
(3)	¬¬{(¬P ∨ R) ∨ Q}	(from 1)
(4)	¬P	(from 2)
(5)	¬(Q ∧ ¬R)	(from 2)
(6)	{(¬P ∨ R) ∨ Q}	(from 3)
(7)	¬Q ¬¬R	(from 5)
(8)	R	(from 7))
(9)	(¬P ∨ R) Q * (¬P ∨ R) Q	(from 6)
(10)	¬P R ¬P R	

Lots of open branches – not a tautology!

5 ({P ∧ (¬Q ∨ ¬R)} ∨ ¬{(P ∨ ¬R) ∨ ¬Q})

(1)	¬({P ∧ (¬Q ∨ ¬R)} ∨ ¬{(P ∨ ¬R) ∨ ¬Q})				
(2)	¬{P ∧ (¬Q ∨ ¬R)}				(from 1)
(3)	¬¬{(P ∨ ¬R) ∨ ¬Q}				(from 1)
(4)	{(P ∨ ¬R) ∨ ¬Q}				(from 3)
(5)	¬P		¬(¬Q ∨ ¬R)		(from 2)
(6)			¬¬Q		(from 5)
(7)			¬¬R		(from 5)
(8)			Q		
(9)			R		
(6)	(P ∨ ¬R)	¬Q	(P ∨ ¬R)	¬Q *	(from 4)
(7)	P ¬R		P ¬R		(from 6)
(8)	*		*		

Lots of open branches – not a tautology!

6 ¬({¬(P ∧ ¬R) ∧ ¬(Q ∧ ¬S)} ∧ ¬{¬(P ∨ Q) ∨ (R ∨ S)})

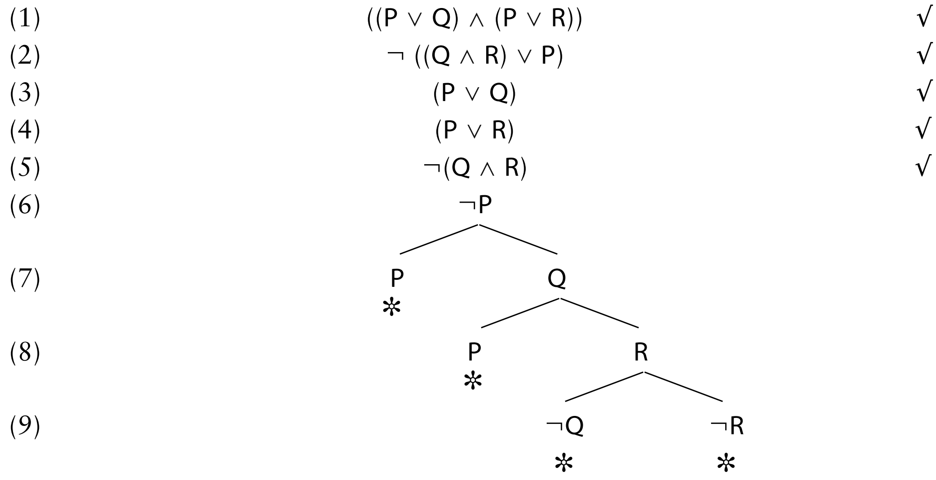
(1)	¬¬({¬(P ∧ ¬R) ∧ ¬(Q ∧ ¬S)} ∧ ¬{¬(P ∨ Q) ∨ (R ∨ S)})		
(2)	({¬(P ∧ ¬R) ∧ ¬(Q ∧ ¬S)} ∧ ¬{¬(P ∨ Q) ∨ (R ∨ S)})		(from 1)
(3)	{¬(P ∧ ¬R) ∧ ¬(Q ∧ ¬S)}		(from 2)
(4)	¬{¬(P ∨ Q) ∨ (R ∨ S)}		(from 2)
(5)	¬(P ∧ ¬R)		(from 3)
(6)	¬(Q ∧ ¬S)		(from 3)
(7)	¬¬(P ∨ Q)		(from 4)
(8)	¬(R ∨ S)		(from 4)
(9)	(P ∨ Q)		(from 7)
(10)	¬R		(from 8)
(11)	¬S		(from 8)
(12)	¬P	¬¬R *	(from 5)
(13)	¬Q ¬¬S *		(from 6)
(14)	P * Q *		(from 9)

Tautology!

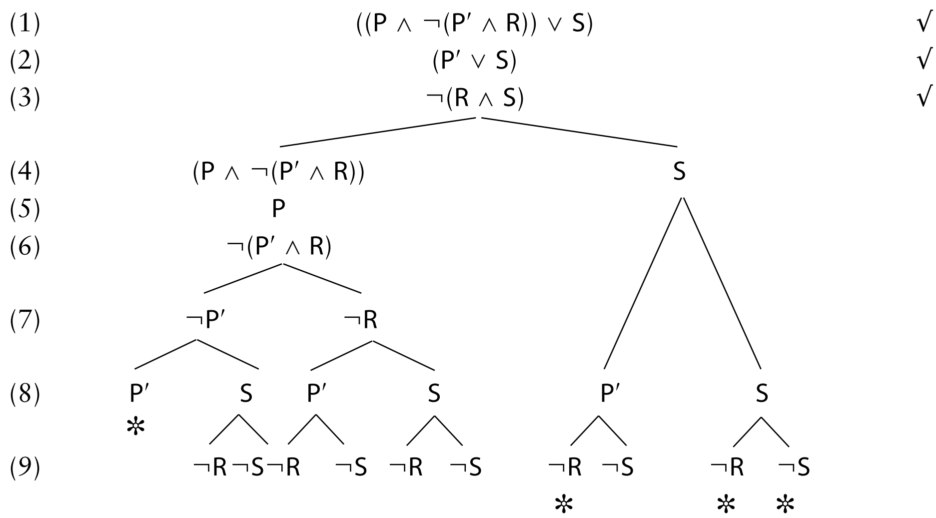
C And since practice makes perfect, use the tree test to determine which of the following are true.

1. $((P \lor Q) \land (P \lor R)) \vDash ((Q \land R) \lor P)$

(For the symbol '$\vDash$' see §13.7.) The following tree shows that this is true:

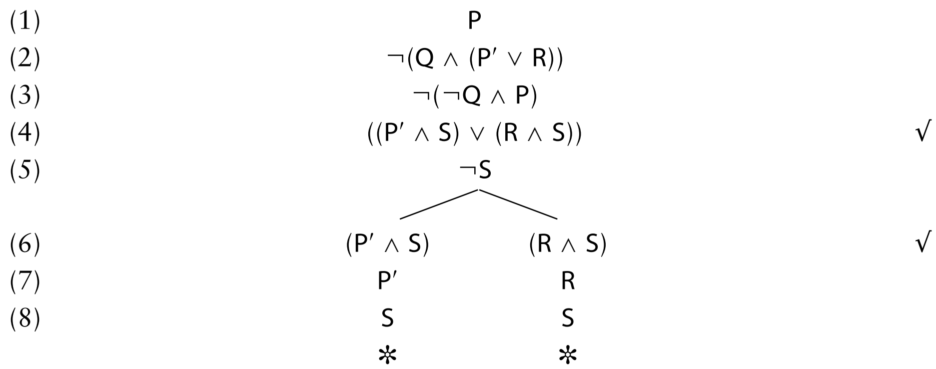


2. $((P \land \neg(P' \land R)) \lor S), (P' \lor S) \vDash (R \land S)$



So not a tautological entailment!

3. $P, \neg(Q \land (P' \lor R)), \neg(\neg Q \land P), ((P' \land S) \lor (R \land S)) \vDash S$



Tautological entailment claim is true (conclusion follows from last premiss alone!).

4. ⊨ (¬(¬(P ∧ Q) ∨ ¬(P ∧ R)) ∨ ¬(P ∨ (Q ∧ R)))

Is this wff a tautology? It is helpful to begin by bracket the wff more transparently. Rewrite it as

(¬{¬(P ∧ Q) ∨ ¬(P ∧ R)} ∨ ¬{P ∨ (Q ∧ R)})

(1) ¬(¬{¬(P ∧ Q) ∨ ¬(P ∧ R)} ∨ ¬{P ∨ (Q ∧ R)})
(2) ¬¬{¬(P ∧ Q) ∨ ¬(P ∧ R)} (from 1)
(3) ¬¬{P ∨ (Q ∧ R)} (from 1)
(4) {¬(P ∧ Q) ∨ ¬(P ∧ R)} (from 2)
(5) {P ∨ (Q ∧ R)} (from 2)
(6) P | (Q ∧ R) (from 5)
(7) Q
(8) R
(9) ¬(P ∧ Q) | ¬(P ∧ R) | ¬(P ∧ Q) | ¬(P ∧ R)
(10) ¬P | ¬Q | ¬P | ¬R | ¬P | ¬Q | ¬P | ¬R
✻ ✻ ✻ ✻

So tautological entailment claim is false.

5. ((P ∧ Q) ∨ (R ∧ P)), ¬(¬S ∧ P), ¬(Q ∧ R) ⊨ (Q ∧ S)

Another boring sprawling tree! I fear that the third premiss should have been ¬(¬Q ∧ R), then the tree would have closed.

D How would the rules and principles of tree-building need to be re-written if we had stuck to using *signed* trees, where wffs are explicitly assigned T or F?

We need ‘unpacking’ rules for conjunctions, disjunctions, and negations assigned ‘T’, and for conjunctions, disjunctions and negations assigned ‘F’ (and these will give us rules to cope with every non-atomic wff):

$(A \wedge B) \Rightarrow$ T
|
$A \Rightarrow$ T
$B \Rightarrow$ T

$(A \wedge B) \Rightarrow$ F
$A \Rightarrow$ F | $B \Rightarrow$ F

$(A \vee B) \Rightarrow$ T
$A \Rightarrow$ T | $B \Rightarrow$ T

$(A \vee B) \Rightarrow$ F
|
$A \Rightarrow$ F
$B \Rightarrow$ F

$\neg A \Rightarrow$ T
|
$A \Rightarrow$ F

$\neg A \Rightarrow$ F
|
$A \Rightarrow$ T

And the tree building rules are as before, except that we start the tree with the premisses assigned true and conclusion false; and close a branch if contains a pair $A \Rightarrow$ T, $A \Rightarrow$ F.