An Introduction to
Formal Logic

Answers to Exercises 26

Which of the following are wffs of ***QL****? Which are sentences, i.e. closed wffs? In the open wffs, which occurrences of which variables are free? What is the main logical operator of each wff?*

1 $\exists z \forall y(Myz \lor \neg\forall y(Mxy \land Lxy))$

Consider the tree

$$Mxy \qquad Lxy$$
$$(Mxy \land Lxy)$$
$$\forall y(Mxy \land Lxy)$$
$$Myz \qquad \neg\forall y(Mxy \land Lxy)$$
$$(Myz \lor \neg\forall y(Mxy \land Lxy))$$

which shows that '$(Myz \lor \neg\forall y(Mxy \land Lxy))$' is a wff; but now we can't – on our wff-building rules – prefix this with '$\forall y$' as 'y' already appears bound inside. Recall, the rule is *If A is a wff, and v is an individual variable which occurs in A, (while neither ∀v nor ∃v occurs in A), then ∀vA is also a wff*. So (1) is not a wff.

2 $\exists z(\forall yMyz \lor \neg(Mxy \land Lxy))$

$$Mxy \qquad Lxy$$
$$Myz \qquad (Mxy \land Lxy)$$
$$\forall \underline{y}M\underline{y}z \qquad \neg(Mxy \land Lxy)$$
$$(\forall \underline{y}M\underline{y}z \lor \neg(Mxy \land Lxy))$$
$$\exists \underline{z}(\forall \underline{y}M\underline{y}\underline{z} \lor \neg(Mxy \land Lxy))$$

is a well-formed construction tree: and here we've underlined variables at the point where they get bound by associated quantifiers. The non-underlined variables are still free, so this isn't a closed wff, and the main logical operator is of course the initial existential quantifier '$\exists z$'.

(*The numbering of the exercises is in error, with no example 3.*)

4 $\exists z(\forall yMyz \lor \neg\forall y(Mxy \land Lxy))$

$$Mxy \qquad Lxy$$
$$(Mxy \land Lxy)$$
$$Myz \qquad \forall \underline{y}(Mx\underline{y} \land Lx\underline{y})$$
$$\forall \underline{y}M\underline{y}z \qquad \neg\forall \underline{y}(Mx\underline{y} \land Lx\underline{y})$$
$$(\forall \underline{y}M\underline{y}z \lor \neg\forall \underline{y}(Mx\underline{y} \land Lx\underline{y}))$$
$$\exists \underline{z}(\forall \underline{y}M\underline{y}\underline{z} \lor \neg\forall \underline{y}(Mx\underline{y} \land Lx\underline{y}))$$

A well-formed construction tree: the non-underlined variables are still free, so this isn't a closed wff, and the main logical operator is again the initial existential quantifier '$\exists z$'.

5 $\neg\exists z(\forall yMyz \lor \neg\forall x\forall y(Mxy \land Lxy))$

$$Mxy \qquad Lxy$$
$$(Mxy \land Lxy)$$
$$\forall \underline{y}(Mx\underline{y} \land Lx\underline{y})$$
$$Myz \qquad \forall \underline{x}\forall \underline{y}(M\underline{xy} \land L\underline{xy})$$
$$\forall \underline{y}M\underline{y}z \qquad \neg\forall \underline{x}\forall \underline{y}(M\underline{xy} \land L\underline{xy})$$
$$(\forall \underline{y}M\underline{y}z \lor \neg\forall \underline{x}\forall \underline{y}(M\underline{xy} \land L\underline{xy}))$$
$$\exists \underline{z}(\forall \underline{y}M\underline{y}\underline{z} \lor \neg\forall \underline{x}\forall \underline{y}(M\underline{xy} \land L\underline{xy}))$$
$$\neg\exists \underline{z}(\forall \underline{y}M\underline{y}\underline{z} \lor \neg\forall \underline{x}\forall \underline{y}(M\underline{xy} \land L\underline{xy}))$$

A well-formed construction tree: every variable is underlined as it gets bound by an associated quantifier, so this is a closed wff, and the main logical operator is the initial negation sign.

6 ∃z∀x(∀yMyz ∨ ¬∀y(Mxy ∧ Lxy))

$$\mathsf{Mxy} \qquad \mathsf{Lxy}$$
$$\mathsf{(Mxy \land Lxy)}$$
$$\mathsf{Myz} \qquad \forall \underline{\mathsf{y}}\mathsf{(Mx}\underline{\mathsf{y}} \land \mathsf{Lx}\underline{\mathsf{y}}\mathsf{)}$$
$$\forall \underline{\mathsf{y}}\mathsf{M}\underline{\mathsf{y}}\mathsf{z} \qquad \neg\forall \underline{\mathsf{y}}\mathsf{(Mx}\underline{\mathsf{y}} \land \mathsf{Lx}\underline{\mathsf{y}}\mathsf{)}$$
$$\mathsf{(}\forall \underline{\mathsf{y}}\mathsf{M}\underline{\mathsf{y}}\mathsf{z} \lor \neg\forall \underline{\mathsf{y}}\mathsf{(Mx}\underline{\mathsf{y}} \land \mathsf{Lx}\underline{\mathsf{y}}\mathsf{))}$$
$$\forall \underline{\mathsf{x}}\mathsf{(}\forall \underline{\mathsf{y}}\mathsf{M}\underline{\mathsf{y}}\mathsf{z} \lor \neg\forall \underline{\mathsf{y}}\mathsf{(M}\underline{\mathsf{xy}} \land \mathsf{L}\underline{\mathsf{xy}}\mathsf{))}$$
$$\exists \underline{\mathsf{z}}\forall \underline{\mathsf{x}}\mathsf{(}\forall \underline{\mathsf{y}}\mathsf{M}\underline{\mathsf{yz}} \lor \neg\forall \underline{\mathsf{y}}\mathsf{(M}\underline{\mathsf{xy}} \land \mathsf{L}\underline{\mathsf{xy}}\mathsf{))}$$

A well-formed construction tree: every variable is underlined as it gets bound by an associated quantifier, so this is a closed wff, and the main logical operator is the initial existential quantifier.

7 ∀x(Gx ⊃ ∃y∃z(Hy ∧ Hz) ∧ Rxyz))

Not a wff – as a simple bracket count reveals (there are more right-hand than left-hand brackets).

8 (Gx ⊃ ∃x∃z((Hy ∧ Hz) ∧ Rxyz))

$$\mathsf{Hy} \qquad \mathsf{Hz}$$
$$\mathsf{(Hy \land Hz)} \qquad \mathsf{Rxyz}$$
$$\mathsf{((Hy \land Hz) \land Rxyz)}$$
$$\exists \underline{\mathsf{z}}\mathsf{((Hy \land Hz) \land Rxy}\underline{\mathsf{z}}\mathsf{)}$$
$$\mathsf{Gx} \qquad \exists \underline{\mathsf{x}}\exists \underline{\mathsf{z}}\mathsf{((Hy \land H}\underline{\mathsf{z}}\mathsf{) \land R}\underline{\mathsf{x}}\mathsf{y}\underline{\mathsf{z}}\mathsf{)}$$
$$\mathsf{(Gx \supset} \exists \underline{\mathsf{x}}\exists \underline{\mathsf{z}}\mathsf{((Hy \land H}\underline{\mathsf{z}}\mathsf{) \land R}\underline{\mathsf{x}}\mathsf{y}\underline{\mathsf{z}}\mathsf{))}$$

A well-formed construction tree: the non-underlined variables are still free, so this isn't a closed wff, and the main logical operator is again the initial existential quantifier '⊃'.

9 ∀x∃y(Gx ⊃ ∃y¬∃z((Hy ∧ Hz) ∧ Rxyz))

Not a wff: '(Gx ⊃ ∃y¬∃z((Hy ∧ Hz) ∧ Rxyz))' is a wff but already contains 'y' bound, so we can't prefix it with another quantifier '∃y'.

10 ∀x(Gx ⊃ ∃y∀x((Hy ∧ Hz) ∧ Rxyz))

Not a wff: '(Gx ⊃ ∃y∀x((Hy ∧ Hz) ∧ Rxyz))' is a wff but already contains 'x' bound, so we can't prefix it with another quantifier '∃x'.

11 (∀x(Gx ⊃ ∃z((Hy ∧ Hz) ∧ Rxyz)) ∨ ∃zHz)

$$\mathsf{Hy} \qquad \mathsf{Hz}$$
$$\mathsf{(Hy \land Hz)} \qquad \mathsf{Rxyz}$$
$$\mathsf{((Hy \land Hz) \land Rxyz)}$$
$$\mathsf{Gx} \qquad \exists \underline{\mathsf{z}}\mathsf{((Hy \land H}\underline{\mathsf{z}}\mathsf{) \land Rxy}\underline{\mathsf{z}}\mathsf{)}$$
$$\mathsf{(Gx \supset} \exists \underline{\mathsf{z}}\mathsf{((Hy \land H}\underline{\mathsf{z}}\mathsf{) \land Rxy}\underline{\mathsf{z}}\mathsf{))} \qquad \mathsf{Hz}$$
$$\forall \underline{\mathsf{x}}\mathsf{(G}\underline{\mathsf{x}} \supset \exists \underline{\mathsf{z}}\mathsf{((Hy \land H}\underline{\mathsf{z}}\mathsf{) \land R}\underline{\mathsf{x}}\mathsf{y}\underline{\mathsf{z}}\mathsf{))} \qquad \exists \underline{\mathsf{z}}\mathsf{H}\underline{\mathsf{z}}$$
$$\mathsf{(}\forall \underline{\mathsf{x}}\mathsf{(G}\underline{\mathsf{x}} \supset \exists \underline{\mathsf{z}}\mathsf{((Hy \land H}\underline{\mathsf{z}}\mathsf{) \land R}\underline{\mathsf{x}}\mathsf{y}\underline{\mathsf{z}}\mathsf{))} \lor \exists \underline{\mathsf{z}}\mathsf{H}\underline{\mathsf{z}}\mathsf{)}$$

A well-formed construction tree: the non-underlined variables are still free, so this isn't a closed wff, and the main logical operator is '∨'.

12 $\forall x \forall y \forall z((Fx \wedge Fy) \wedge Rzxy) \supset Gz)$

Not a wff as a bracket count reveals.

13 $\neg\forall x \forall y(((Fx \wedge Fy) \wedge \forall zRzxy) \supset Gz)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{Fx \qquad Fy}{(Fx \wedge Fy)} \qquad \forall \underline{z}R\underline{z}xy}{((Fx \wedge Fy) \wedge \forall \underline{z}R\underline{z}xy)} \qquad Gz}{(((Fx \wedge Fy) \wedge \forall \underline{z}R\underline{z}xy) \supset Gz)}}{\dfrac{\forall \underline{y}(((Fx \wedge F\underline{y}) \wedge \forall \underline{z}R\underline{z}x\underline{y}) \supset Gz)}{\dfrac{\forall \underline{x}\forall \underline{y}(((F\underline{x} \wedge F\underline{y}) \wedge \forall \underline{z}R\underline{z}\underline{x}\underline{y}) \supset Gz)}{\neg\forall \underline{x}\forall \underline{y}(((F\underline{x} \wedge F\underline{y}) \wedge \forall \underline{z}R\underline{z}\underline{x}\underline{y}) \supset Gz)}}}
$$

A well-formed construction tree: the final non-underlined ‘z’ is still free, so this isn’t a closed wff, and the main logical operator is the initial negation.

14 $\forall z \forall x \forall y(((Fx \wedge Fy) \wedge \forall wRwxy) \supset Gz)$
A closed wff with every variable bound, and the initial ‘$\forall z$’ is the main operator.

15 $\forall x \forall y \forall z((Fx \wedge Fy) \supset (Rzxy \supset Gz))$
Also a closed wff with every variable bound, and the initial ‘$\forall x$’ as the main operator.