An Introduction to **Formal Logic**

Answers to Exercises 32

Question

In addition to the properties of relations already defined, we say that a relation *R* is *Euclidean* just if, whenever *a* has *R* to *b*, and *a* has *R* to *c*, then *b* has *R* to *c*. *R* is *asymmetric* if, whenever *a* has *R* to *b*, then *b* does *not* have *R* to *a*. *R* is *irreflexive* if no object has *R* to itself. These properties, along with the properties of being *reflexive*, *symmetric* and *transitive*, can all be captured by **QL** wffs. For example, *R*'s being symmetric is captured by

$$\forall x \forall y(Rxy \supset Ryx)$$

Give wffs that similarly capture *R*'s having each of the other five properties. Then give both informal arguments and **QL** trees to show

1. If *R* is asymmetric, it is irreflexive.
2 If *R* is transitive and irreflexive, it is asymmetric.
3. If *R* is an equivalence relation, it is Euclidean.
4. If *R* is Euclidean and reflexive, it is an equivalence relation.

What about

5. If *R* is transitive and symmetric, it is reflexive?

Answer

Translations:

R is symmetric: $\forall x \forall y(Rxy \supset Ryx)$
R is asymmetric: $\forall x \forall y(Rxy \supset \neg Ryx)$
R is reflexive: $\forall x Rxx$
R is irreflexive: $\forall x \neg Rxx$
R is transitive: $\forall x \forall y \forall z((Rxy \wedge Ryz) \supset Rxz)$
R is Euclidean: $\forall x \forall y \forall z((Rxy \wedge Rxz) \supset Ryz)$

1. To show that *if R is asymmetric, it is irreflexive*: By definition, *R* is *asymmetric* if, for any *a* and *b*, whenever *a* has *R* to *b*, then *b* does *not* have *R* to *a*. Take the particular case where *a* is *b*. Then in particular, if *a* has *R* to *a*, then *a* does not have *R* to *a* – hence (by the principle that *if P then not-P* implies *not-P*) *a* does not have *R* to *a*, and *R* is irreflexive.

Tree version:

(1)	$\forall x \forall y(Rxy \supset \neg Ryx)$		Assymmetry assumption
(2)	$\neg \forall x \neg Rxx$	√	Neg. conclusion, irreflexivity
(3)	$\exists x \neg\neg Rxx$	√	From 2, by $\neg\forall$ rule
(4)	$\neg\neg Raa$	√	Instantiate with new name
(5)	Raa	√	Remove double negation
(6)	$\forall y(Ray \supset \neg Rya)$		From 1, instantiate $\forall x$
(7)	$(Raa \supset \neg Raa)$	√	From 6, instantiate $\forall y$
(8)	$\neg Raa$ | $\neg Raa$		
	* | *		

2 To show that *if R is transitive and irreflexive, it is asymmetric*. If *R* is transitive, then if *a R b* and *b R c*, then *a R c*. So in particular, if *a R b* and *b R a*, then *a R a*. Hence, if not-(*a R a*), then not-(*a R b* and *b R a*). In other words, if *R* is irreflexive, then if *a R b* then not-(*b R a*), i.e. *R* is asymmetric.

Tree version:

(1)	∀x∀y∀z((Rxy ∧ Ryz) ⊃ Rxz)		Transitivity
(2)	∀x¬Rxx		Irreflexivity
(3)	¬∀x∀y(Rxy ⊃ ¬Ryx)	√	Neg. concl.: Assymmetry
(4)	∃x¬∀y(Rxy ⊃ ¬Ryx)	√	From 3, y ¬∀ rule
(5)	¬∀y(Ray ⊃ ¬Rya)	√	From 4, instantiating ∃x
(6)	∃y¬(Ray ⊃ ¬Rya)	√	From 5, by ¬∀ rule
(7)	¬(Rab ⊃ ¬Rba)	√	From 6, instantiating ∃y
(8)	Rab		Unpacking the neg. ⊃
(9)	¬¬Rba	√	
(10)	Rba		From 9

So far, so automatic! There is nothing else to do at steps (3) to (7) if we stick to the basic principle of not instantiating universals with new names. Now we do instantiate (1) and (2) in the obvious ways.

(11)	∀y∀z((Ray ∧ Ryz) ⊃ Raz)		From 1, instantiating ∀x
(12)	∀z((Rab ∧ Rbz) ⊃ Raz)		From 12, instantiating ∀y
(13)	((Rab ∧ Rba) ⊃ Raa)	√	From 13, instantiating ∀z
(14)	¬Raa		From 2, instantiating ∀x

(15) ¬(Rab ∧ Rba) √ | Raa
*

(16) ¬Rab | ¬Rba
* *

3. A relation is an equivalence relation if it is transitive, symmetric and reflexive. Suppose *R* is an equivalence relation; and suppose that *a R b* and *a R c*. Since R is symmetric, it follows that *b R a*. And from *b R a* and *a R c*, transitivity yields *b R c*. So we've shown that if *a R b* and *a R c* then *b R c*. So *R* is Euclidean.

∀x∀y∀z((Rxy ∧ Ryz) ⊃ Rxz)	Transitivity
∀x∀y(Rxy ⊃ Ryx)	Symmetry
¬∀x∀y∀z((Rxy ∧ Rxz) ⊃ Ryz)	Neg. Euclidean
∃x¬∀y∀z((Rxy ∧ Rxz) ⊃ Ryz)	By ¬∀ rule
¬∀y∀z((Ray ∧ Raz) ⊃ Ryz)	Instantiating ∃x
∃y¬∀z((Ray ∧ Raz) ⊃ Ryz)	Etc.
¬∀z((Rab ∧ Raz) ⊃ Rbz)	
∃z¬((Rab ∧ Raz) ⊃ Rbz)	
¬((Rab ∧ Rac) ⊃ Rbc)	
(Rab ∧ Rac)	Unpacking ¬⊃
¬Rbc	
Rab	
Rac	
∀y∀z((Rby ∧ Ryz) ⊃ Rbz)	Instantiating ∀x
∀z((Rba ∧ Raz) ⊃ Rbz)	Instantiating ∀y
((Rba ∧ Rac) ⊃ Rbc)	Instantiating ∀z

¬(Rba ∧ Rac) | Rbc
*

¬Rba | ¬Rac
∀y(Ray ⊃ Rya) | * — Instantiating ∀x in Symmetry
(Rab ⊃ Rba) — Instantiating ∀y

¬Rab | Rba
* *

(Now insert the check marks in the proper places!)

4. *We want to show: if R is Euclidean and reflexive, it is an equivalence relation.* (1) Being Euclidean, if *a R b* and *a R c*, then *b R c*; so in particular, if *a R b*, and *a R a*, then *b R a*. But we are assuming *R* is also reflexive so we always have *a R a* so that means that if *a R b* then *b R a* – so *R* is *symmetric.* (2) Now suppose *R* is Euclidean and reflexive and therefore symmetric. Then if *a R b* and *b R c*, then (by symmetry) *b R a* and *b R c,* so (by Euclidean property) *a R c* – hence *R* is *transitive.* (3) In sum, if *if R is Euclidean and reflexive* then *R* is *reflexive, symmetric* and *transitive*, so is an equivalence relation.

For the tree (again, left to you to insert check marks!):

∀x∀y∀z((Rxy ∧ Rxz) ⊃ Ryz) Euclidean
∀xRxx Reflexivity
¬((∀x∀y∀z((Rxy ∧ Ryz) ⊃ Rxz) ∧ ∀x∀y(Rxy ⊃ Ryx)) ∧∀xRxx) Neg. of Equivalence

Branch: ¬(∀x∀y∀z((Rxy ∧ Ryz) ⊃ Rxz) ∧ ∀x∀y(Rxy ⊃ Ryx)) | ¬∀xRxx

Under the left branch:

Branch: ¬∀x∀y∀z((Rxy ∧ Ryz) ⊃ Rxz) | ¬∀x∀y(Rxy ⊃ Ryx)

Left:
¬∀x∀y∀z((Rxy ∧ Ryz) ⊃ Rxz)
∃x¬∀y∀z((Rxy ∧ Ryz) ⊃ Rxz)
¬∀y∀z((Ray ∧ Ryz) ⊃ Raz)
∃y¬∀z((Ray ∧ Ryz) ⊃ Raz)
¬∀z((Rab ∧ Rbz) ⊃ Raz)
∃z¬((Rab ∧ Rbz) ⊃ Raz)
¬((Rab ∧ Rbc) ⊃ Rac)
(Rab ∧ Rbc)
¬Rac
Rab
Rbc
∀y∀z((Rby ∧ Rbz) ⊃ Ryz) Using initial Euclidean premiss
∀z((Rba ∧ Rbz) ⊃ Raz)
((Rba ∧ Rbc) ⊃ Rac)

Branch: ¬(Rba ∧ Rbc) | Rac ✻

Under ¬(Rba ∧ Rbc):

Branch: ¬Rba | ¬Rbc ✻

Under ¬Rba:
∀y∀z((Ray ∧ Raz) ⊃ Ryz) Using initial Euclidean premiss again!
∀z((Rab ∧ Raz) ⊃ Rbz)
((Rab ∧ Raa) ⊃ Rba)

Branch: ¬(Rab ∧ Raa) | Rba

Under ¬(Rab ∧ Raa):

Branch: ¬Rab ✻ | ¬Raa, Raa ✻ (Raa: From the initial reflexivity premiss!)

Right:
¬∀x∀y(Rxy ⊃ Ryx)
∃x¬∀y(Rxy ⊃ Ryx)
¬∀y(Ray ⊃ Rax)
∃y¬(Ray ⊃ Rya)
¬(Rab ⊃ Rba)
Rab
¬Rba
∀y∀z((Ray ∧ Raz) ⊃ Ryz)
∀z((Rab ∧ Raz) ⊃ Rbz)
((Rab ∧ Raa) ⊃ Rba)

The complication here is in the way we choose to instantiate the variables in the Euclidean premiss. Our choice the first time is designed to deliver quickly closing branches (so ‘Ryz’ needs to become ‘Rac’ to give a contradiction with ‘¬Ra’).

5. What about *if R is transitive and symmetric, it is reflexive?*

If *R* is transitive, then if *a R b* and *b R c*, then *a R c*. So in particular, if *a R b* and *b R a*, then *a R a*. But if *R* is symmetric, if *a R b,* then *a R b* and *b R a*. So – putting those together – if *a R b*, then if *a R a*. That tells us that if *R* relates *a* to anything in the domain, then it relates *a* to itself. But maybe *R* relates nothing in the domain of discourse!

Take an extreme example: every *a*, *b*, *c* is such that IF *a* is the same god as *b*, and *b* is the same god as *c*, then *a* is the same god as *c* (transitivity), and IF *a* is the same god as *b,* then *b* is the same god as *a* (symmetry). But it doesn't follow that every *a* is the same god as itself (for that can only be true if *a* is a god, and there may be none).

Consider now the tree:

$\forall x \forall y \forall z((Rxy \wedge Ryz) \supset Rxz)$	Transitivity
$\forall x \forall y(Rxy \supset Ryx)$	Symmetry
$\neg \forall x Rxx$	Neg. conclusion, Reflexivity
$\exists x \neg Rxx$	
$\neg Raa$	

Now, we only have the name 'a' in play. If we instantiate the universal quantifications using the only name we have in play every time, we'll get …

$$((Raa \wedge Raa) \supset Raa)$$
$$(Raa \supset Raa)$$

Press on and the tree won't close!

On the other hand the tree for

$\forall x \forall y \forall z((Rxy \wedge Ryz) \supset Rxz)$	Transitivity
$\forall x \forall y(Rxy \supset Ryx)$	Symmetry
$\forall x \exists y Rxy$	*R* relates everything to something
$\neg \forall x Rxx$	Neg. conclusion, Reflexivity

Does close:

$\exists x \neg Rxx$	By $\neg\forall$ rule
$\neg Raa$	
$\exists y Ray$	Instantiating new premiss
Rab	
$\forall y \forall z((Ray \wedge Ryz) \supset Raz)$	
$\forall z((Rab \wedge Rbz) \supset Raz)$	
$((Rab \wedge Rba) \supset Raa)$	
$\forall y(Ray \supset Rya)$	
$(Rab \supset Rba)$	

And the rest is easy!