Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 13: Expressive adequacy and DNF

The following questions are longwinded: but the answers are comfortingly quick and easy!

(a) *We could introduce a new* four-*place connective '$\bigsqcup$', where $\bigsqcup(\alpha, \beta, \gamma, \delta)$ is true when exactly two of $\alpha, \beta, \gamma, \delta$ are true, and is false otherwise. Show that doing this would be redundant because we can already define the new connective using the standard three connectives.*

We are given that (for any wffs $\alpha, \beta, \gamma, \delta$) the wff $\bigsqcup(\alpha, \beta, \gamma, \delta)$ is true just when one of the corresponding following expressions is true:

$(\alpha \land \beta \land \neg\gamma \land \neg\delta)$
$(\alpha \land \neg\beta \land \gamma \land \neg\delta)$
$(\alpha \land \neg\beta \land \neg\gamma \land \delta)$
$(\neg\alpha \land \neg\beta \land \gamma \land \delta)$
$(\neg\alpha \land \beta \land \neg\gamma \land \delta)$
$(\neg\alpha \land \beta \land \gamma \land \neg\delta)$

is true, and not otherwise – assume that these expressions are internally bracketed to make them kosher wffs.

Now disjoin these six expressions, then (again bracketing carefully), and you will get a wff which is true (as we want) just when exactly two of $\alpha, \beta, \gamma, \delta$ are true. So we can define the connective $\bigsqcup$ in the sense of producing a wff which has the corresponding truth table.

(b) *More on expressive adequacy:*

(1) *Compare the truth tables for the down-arrow '$\downarrow$' and '$\lor$': one is formed from the other by swapping 'T's and 'F's in the last column. Define an up-arrow connective '$\uparrow$' (also symbolized '$|$', and then known as the 'Sheffer stroke') whose table stands in the same relation to the table for '$\land$'. Show that, like the down-arrow, this up-arrow connective ('NAND') taken just by itself is expressively adequate.*

The truth-table is of course

α	β	$(\alpha \uparrow \beta)$
T	T	F
T	F	T
F	T	T
F	F	T

$(\alpha \uparrow \beta)$ says that it isn't the case that α and β.

To see that $\uparrow$ can be used to define negation, consider what happens when we give it the same wff twice: $(\alpha \uparrow \alpha)$ is true just when $\neg\alpha$ is true. And $(\alpha \lor \beta)$ is equivalent to *it isn't the case that both* $\neg\alpha$ *and* $\neg\beta$, so is true just when $(\neg\alpha \uparrow \neg\beta)$, and hence will have the same truth table as $((\alpha \uparrow \alpha) \uparrow (\beta \uparrow \beta))$, as you can quickly check.

Which gives us the following result:

> The up-arrow connective $\uparrow$ can be used to define negation and disjunction and so is expressively adequate just by itself.

(2) *Show that the up-arrow and down-arrow connectives are the only* binary *connectives that, taken by themselves, are expressively adequate.*

There are sixteen different binary truth-functional connectives (four lines of the truth table, each line can be T or F, so 2^4 possible tables!). We could hack through all sixteen. But let's be smarter.

Suppose we are looking at the table for a binary connective $(\alpha * \beta)$. If $(\alpha * \beta)$ were always true when the values of α and β are true, then we won't be able to construct from $*$ a wff which is false when its atom(s) are true – so in particular won't get a wff which expresses negation. If $(\alpha * \beta)$ were always false when the values of α and β are false, then we won't be able to construct from $*$ a wff which is true when its atom(s) are false – so in particular won't get a wff which expresses negation.

So we know that if $*$ is to be expressively adequate, then it's table must come out F on the first line and T on the fourth line. Which leaves just four possibilities:

α	β	$(\alpha \uparrow \beta)$	$(\alpha \downarrow \beta)$	$\neg\alpha$	$\neg\beta$
T	T	F	F	F	F
T	F	T	F	F	T
F	T	T	F	T	F
F	F	T	T	T	T

We know that first two are expressive adequate and the last two obviously aren't. So we are done.

(3) *Define a* ternary *connective which, taken by itself, is expressively adequate.*

You are supposed to spot that the answer is trivial, given what you already know. Just define e.g. $\Downarrow(\alpha, \beta, \gamma)$ so that on any line of its truth-table it's value depends on just α, β and equals $(\alpha \downarrow \beta)$ (so the third input is an idle wheel). Then $\Downarrow(\alpha, \beta, \beta)$ will do as well as $(\alpha \downarrow \beta)$ to define any truth function!

(4) *Are '$\oplus$' and '$\neg$' taken together expressively adequate? What about '\$' and '$\neg$'? The tables for $\oplus$ and \$ are*

α	β	$(\alpha \oplus \beta)$
T	T	F
T	F	T
F	T	T
F	F	F

α	β	γ	$\$(\alpha, \beta, \gamma)$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Note that the truth-tables for $\neg$ and $\oplus$ are 'balanced': in the final column there is the same number of Ts and Fs. And is easy to check that as we build up more wffs using $\neg$ and $\oplus$ we'll get further expressions whose truth tables are 'balanced', i.e. contain the same number of Ts and Fs in the final column. So we'll never get an expression with the truth table for $\wedge$ or $\vee$ which have unbalanced truth tables. Hence $\neg$ and $\oplus$ do not form an expressively adequate set of connectives.

On the other hand, $\neg$ and \$ do form an expressively adequate set of connectives. For look at the table for $\$(\alpha, \beta, \beta)$

α	β	$\$(\alpha, \beta, \beta)$
T	T	F
T	F	T
F	T	F
F	F	F

This is the truth-table for $\wedge$ if we can swap the lines for β T and β F. So consider

α	β	$\$(\alpha, \neg\beta, \neg\beta)$
T	T	T
T	F	F
F	T	F
F	F	F

Hence with $\neg$ and \$ we can define $\wedge$; and we know that $\neg$ and $\wedge$ are expressively complete. So we are done!

(c*) *Assume that we are working in some* PL *language. Then:*

(1) *Show that pairs of wffs of the forms* $(\alpha \wedge (\beta \vee \gamma))$ *and* $((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$ *have the same truth table.*

Just check a truth-table!

(2) *Show that pairs of wffs of the forms* $((\alpha \wedge \beta) \wedge \gamma)$ *and* $(\alpha \wedge (\beta \wedge \gamma))$ *have the same truth table. Generalize to show that any way you bracket an unmixed conjunction* $\alpha \wedge \beta \wedge \gamma \wedge \ldots \wedge \lambda$ *to give a properly bracketed wff expresses the same truth function. Check the comparable results for disjunctions.*

The first part involves a trivial truth-table.

How could we officially prove the general claim that an unmixed conjunction $\alpha \wedge \beta \wedge \gamma \wedge \ldots \wedge \lambda$, however we bracket it up, is true in one and only one case, namely when each ultimate conjunct wff $\alpha, \beta, \gamma \ldots, \lambda$ is true? Well, any of the bracketed up versions has a parse tree, with a conjunction introduced at each fork of the tree as we go up. Suppose one of the wffs at the bottom of a branch is false. Then the conjunctive wff at the node above will be be false, and hence the conjunctive wff at the node above that will be be false, until we get to the top of the tree. So if any ultimate conjunct is false so is the whole conjunction. And similarly if all the ultimate conjuncts are true, so is is the whole conjunction.

Similarly, however we bracket up an unmixed conjunction $\alpha \vee \beta \vee \gamma \vee \ldots \vee \lambda$ to give a properly bracketed wff, we get a truth function which is true when at least one of the ultimate disjuncts is true, and is otherwise false – again, think about walking up any relevant parse trees!

(3) *Show that pairs of wffs of the forms* $\neg(\alpha \wedge \beta)$ *and* $(\neg\alpha \vee \neg\beta)$ *also have the same truth tables. Generalize to show that a negated unmixed conjunction* $\neg(\alpha \wedge \beta \wedge \ldots \wedge \lambda)$ *has the same truth table as* $(\neg\alpha \vee \neg\beta \vee \ldots \neg\lambda)$, *however we insert brackets to get wffs. What are the comparable results for negated disjunctions?*

The first part again involves a trivial truth-table.

And evidently $\neg(\alpha \wedge \beta \wedge \ldots \wedge \lambda)$ is true only when at least one of $\alpha, \beta, \gamma \ldots, \lambda$ is false, i.e. when $(\neg\alpha \vee \neg\beta \vee \ldots \vee \neg\lambda)$ is true.

Similarly $\neg(\alpha \vee \beta \vee \ldots \vee \lambda)$ is true only when all of $\alpha, \beta, \gamma \ldots, \lambda$ are false, i.e. when $(\neg\alpha \wedge \neg\beta \wedge \ldots \wedge \neg\lambda)$ is true.

(4) *Say that an atom or the negation of an atom is a* basic *wff. A wff is in* disjunctive normal form *if it is, ignoring bracketing, of the form* $\alpha \vee \beta \vee \ldots \vee \lambda$ *for one or more disjuncts, where each disjunct is a conjunction of one or more basic wffs. Show that any wff has the same truth table as a wff in disjunctive normal form.*

We've in effect proved this in §§13.3–13.4. Our construction there takes us from any wff to its truth-table to a wff with the same truth-table which is (when you look at it) in DNF.

But we can also argue using the equivalences we've just been noting:

i Use (3) – instances of De Morgan's Laws – to push negation signs to the right until they attach to atoms (eliminating double negation signs as you go). So for example,

$\neg((\mathsf{P} \wedge \neg\mathsf{Q}) \vee \neg\neg(\mathsf{R} \wedge \neg\mathsf{S})) \quad \approx$
$\neg((\mathsf{P} \wedge \neg\mathsf{Q}) \vee (\mathsf{R} \wedge \neg\mathsf{S})) \quad \approx$
$(\neg(\mathsf{P} \wedge \neg\mathsf{Q}) \wedge \neg(\mathsf{R} \wedge \neg\mathsf{S})) \quad \approx$
$((\neg\mathsf{P} \vee \neg\neg\mathsf{Q}) \wedge (\neg\mathsf{R} \vee \neg\neg\mathsf{S})) \quad \approx$
$((\neg\mathsf{P} \vee \mathsf{Q}) \wedge (\neg\mathsf{R} \vee \mathsf{S}))$

ii At this stage any remaining negation signs attach to atoms, and we have a wff built up from basic wffs using conjunction and disjunction. Now use (1)-type equivalence to push signs conjunctions inside disjunctions. Thus $((\alpha\vee\beta)\wedge\gamma)$ is equivalent to $((\alpha\wedge\gamma)\vee(\beta\wedge\gamma))$; and similarly $(\alpha \wedge (\beta \vee \gamma))$ is equivalent to $((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$. So we have

$((\neg\mathsf{P} \vee \mathsf{Q}) \wedge (\neg\mathsf{R} \vee \mathsf{S})) \quad \approx$
$([\neg\mathsf{P} \wedge (\neg\mathsf{R} \vee \mathsf{S})] \vee [\mathsf{Q} \wedge (\neg\mathsf{R} \vee \mathsf{S})]) \quad \approx$
$([(\neg\mathsf{P} \wedge \neg\mathsf{R}) \vee (\neg\mathsf{P} \wedge \mathsf{S})] \vee [(\mathsf{Q} \wedge \neg\mathsf{R}) \vee (\mathsf{Q} \wedge \mathsf{S})])$

So we have ended up with a wff in DNF. Evidently the procedure generalizes.

(5) *Define an analogous notion of being in* conjunctive normal form. *Show that any wff α has the same truth table as a wff in conjunctive normal form. (Hint: consider a wff in disjunctive normal form which is equivalent to $\neg\alpha$ and take negations.)*

A wff is in CNF if it is a conjunction of one or more disjunctions of one or more basic wffs.

We could can construct a wff in CNF equivalent to a given wff by the same procedure just sketched for producing equivalents in DNF, except that at step (ii) instead of driving conjunctions inside disjunctions, we drive disjunctions inside conjunctions!

Alternatively follow the hint. Given a wff α find an equivalent in DNF to $\neg\alpha$. Then α will be equivalent to the negation of that wff in DNF. But the negation of a wff in DNF is equivalent to one in CNF. Why? Just push negation signs to the right, flipping conjunctions with disjunctions as you go by De Morgan's Laws!