Exercises: Computability, etc.

These exercises relate to the introductory material in *IGT2*, Ch. 3 on effectively computable functions, effectively decidable sets, and effective enumerable sets.

Reading

1. *IGT2*, §§3.1–3.3.
2. A. Shen and N. K. Vereschchagin, *Computable Functions* (American Mathematical Society: 2003), Ch. 1.

Exercises

1. (a) What is an algorithm?
   (b) Show that there are algorithms for
      i. deciding whether a natural number is prime;
      ii. finding the highest common factor of two natural numbers;
      iii. deciding whether a traditional syllogism (i.e. argument with two premisses and a conclusion, all of A,E, I or O, form) is valid;
      iv. deciding whether a string of symbols of your favourite system of the classical propositional calculus is a wff.
      v. deciding whether a given wff of your favourite system of the classical propositional calculus is a theorem (provable from no assumptions).
   (c) Is there an algorithm for deciding whether an arbitrary real number is greater than one?

2. A total function $f\colon \mathbb{N} \to \mathbb{N}$ is *(effectively) computable* iff there is an algorithmic procedure Π which computes it – i.e. Π applied to input n produces output $f(n)$ in a finite number of steps.

   A set $\Sigma \subseteq \mathbb{N}$ is *(effectively) decidable* iff there is an algorithmic procedure Π which applied to input n determines whether $n \in \Sigma$ by outputting '0' for a positive verdict and '1' for a negative verdict.[1]

   Some very easy 'reality checks':

   (a) Which of the following definitions characterize (total) computable functions $f\colon \mathbb{N} \to \mathbb{N}$?
      i. $f(n) = 3^n$.
      ii. $f(n) = n^3 + 6n^2 + 7$.
      iii. $f(n) = \sqrt[3]{n}$.
      iv. $f(n) = 0$ if n is prime, and $f(n) = 1$ otherwise.
      v. $f(n)$ is the $n + 1$-th prime number.
   (b) Suppose $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ are computable functions.

[1] A note on notation. Strictly speaking we should write 'A set Σ, where $\Sigma \subseteq \mathbb{N}$, is ...': but the slang compression into 'A set $\Sigma \subseteq \mathbb{N}$ is ...' is handy and pretty standard.

   i. Is their product $h(n) =_{\text{def}} f(n) \cdot g(n)$ computable?
   ii. Is their composition $f \circ g$ computable?
 (c) Suppose Δ and Γ are decidable subsets of $\mathbb{N}$. Show that so too are
   i. $\Delta \cup \Gamma$,
   ii. $\Delta \cap \Gamma$,
   iii. $\mathbb{N} \setminus \Delta$.

3. Requiring a little more thought:
 (a) Define $j\colon \mathbb{N} \to \mathbb{N}$ as follows: $j(n) = n + 1$ if Julius Caesar ate grapes on his fifth birthday, and $j(n) = 2n$ otherwise. Is j an effectively computable function? [Hint: look carefully at the summary definition of an effectively computable function.]
 (b) The function $k\colon \mathbb{N} \to \mathbb{N}$ is defined as follows: $k(n) = 0$ if there are at least n consecutive 7s somewhere in the decimal expansion of π, and $k(n) = 1$ otherwise. Is k effectively computable? [Hint: we don't know whether there is a maximum number m such that there are at least m consecutive 7s somewhere in the decimal expansion of π, so consider the cases where there is and there isn't separately.]
 (c) The function $h\colon \mathbb{N} \to \mathbb{N}$ is defined as follows: $h(n) = 0$ if there are *exactly* n consecutive 7s [bounded by some other digits] somewhere in the decimal expansion of π, and $h(n) = 1$ otherwise. Is h effectively computable?

4. Recall: a set Σ is *effectively enumerable* iff it is either empty or the range of a computable function $f\colon \mathbb{N} \to \Sigma$, i.e. $n \in \Sigma$ iff $\exists m\, f(m) = n$.
 (a) Define the function $f\colon \mathbb{N}^2 \to \mathbb{N}$ as follows: f 'codes' the ordered pair of numbers $\langle m, n\rangle$ by mapping it to $2^m(2n + 1)$.
   i. Prove f is an effectively computable bijection.
   ii. Show that the functions $\mathit{fst}(k)$ and $\mathit{snd}(k)$ are effectively computable, where these are the functions which 'decode' a number k by finding respectively the m and the n such that f maps $\langle m, n\rangle$ to k.
   iii. Prove f^{-1} is an effectively computable bijection.
   iv. Conclude that $\mathbb{N}^2$ is effectively enumerable.
 (b) Suppose g maps the pair $\langle m, n\rangle$ to the code number $\{(m + n)^2 + 3m + n\}/2$.
   i. Show g is an effectively computable bijection from $\mathbb{N}^2$ to $\mathbb{N}$. [Hint, put $j = m + n$, and express the function g in terms of j and m.]
   ii. Show informally that the functions $\mathit{fst}(k)$ and $\mathit{snd}(k)$ are effectively computable, where these are the functions which 'decode' a number k by finding respectively the m and the n such that g maps $\langle m, n\rangle$ to k.
   iii. Show that that g's inverse is the effectively computable function which maps k to the k-th pair counting from 0 along the zig-zag route graphically displayed in *IGT2*, §2.4.
   iv. Conclude again that $\mathbb{N}^2$ is effectively enumerable.

5. (a) Show that $\Sigma \subseteq \mathbb{N}$ is a decidable infinite set of numbers iff the members of Σ can be effectively enumerated in ascending order of size.
 (b) Show that a non-empty set $\Sigma \subseteq \mathbb{N}$ is decidable iff Σ is the range of a non-decreasing computable function $f\colon \mathbb{N} \to \mathbb{N}$. [A function is non-decreasing iff $m < n \to f(m) \leq f(n)$.]
 (c) Can your proof of (b) be adapted to show that any effectively enumerable set of numbers is decidable?