Exercises: Sufficiently expressive/strong

Some introductory exercises related to the informal arguments for incompleteness in Chapters 6 and 7 of *IGT2*.

Reading

1. *IGT2*, Chs 6 & 7.

Exercises

1. Suppose T is an effectively axiomatized sound theory. Which of the following questions are you currently placed to settle?
   (a) Suppose G is a sentence of T's language which is true iff G is not provable in T: can T decide G?
   (b) Suppose H is a sentence of T's language which is true iff H is provable in T: can T decide H?
   (c) (Looking ahead, but try thinking about it!) Suppose M is a sentence of T's language which is true iff M is not provable in T in less than a million inference steps: can T decide M?
2. In this exercise, take 'theory' to mean any set of sentences equipped with deductive rules, whether or not effectively axiomatizable:
   (a) If a theory is effectively decidable, must it be negation complete?
   (b) If a theory is effectively decidable, must it be effectively axiomatizable?
   (c) If a theory is negation complete, must it be effectively decidable?
   (d) Say a first-order theory Q is finitely axiomatizable iff there is a finite set of axioms which together entail every Q-theorem. Much such a theory Q be effectively axiomatizable?
   (e) First-order logic is compact: so if $\Gamma \vdash \varphi$ then $\Gamma^* \vdash \varphi$, where Γ^* is a finite subset of Γ. Must an effectively axiomatizable first-order theory therefore be finitely axiomatizable?
3. (a) What is it for a logical theory (a deductive proof system) to be effectively decidable?
   (b) Is your favourite proof system for classical propositional logic effectively decidable?
   (c) Suppose Q is a finitely axiomatizable theory with a standard first-order logic; then show that there is a single sentence $\hat{Q}$ such that $Q \vdash \varphi$ if and only if $\vdash \hat{Q} \to \varphi$ (where $\vdash$ is deducibility in your favourite first-order logic).
   (d) Prove that if there is a consistent, finitely axiomatizable, sufficiently strong theory with a first-order logic, then first-order logic is undecidable.
4. Let *True* be the set of all true sentences of a sufficiently expressive language L with classical negation. We can treat *True* as a theory (with just the trivial rule of inference 'from φ infer φ).
   (a) Show *True* is consistent.

   (b) Show *True* is negation complete.
   (c) Show *True* is sufficiently strong.
   (d) Use Theorem 7.2 to conclude that the set of sentences *True* is not effectively axiomatizable by any theory framed in language L.

5. Suppose T is an effectively axiomatized, consistent, sufficiently strong theory. And suppose we augment the language of T and add new axioms to get a new consistent, effectively axiomatized, theory U. Now let U^* be all the theorems of U which are expressed in T's original, unaugmented, language.
   (a) Show U^* is consistent, and sufficiently strong.
   (b) Show that if U^* is negation complete then it is decidable.
   (c) Show that U^* therefore cannot be negation complete.

   Why is the last result interesting?