Exercises: Equinumerosity

These exercises explore a theme in the background of Chapter 2, bringing to the fore the notion of one set being equinumerous with another, and giving a slightly different take on the idea of enumerability/countability.

Reading

1. *IGT2*, §§2.3–2.5.
2. Of course, you can rather spoil the fun and get most of the answers just by dipping into any standard elementary book on set theory such as *Introduction to Set Theory* by Karel Hrbacek and Thomas Jech. Or for some brisker hints, you could look at http://plato.stanford.edu/entries/set-theory/primer.html. But it is instructive to try the exercises using just what you learn from *IGT2*.

Reminder

$\mathbb{N}$ is the set of natural numbers, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{R}$ is the set of real numbers.

Exercises

1. Two sets Δ and Γ are said to be *equinumerous* iff there is a one-one correspondence between them, i.e. there is some bijection $f : \Delta \to \Gamma$. For a simple reality check, show that equinumerosity is an equivalence relation. In other words, writing '$\Delta \approx \Gamma$' for 'Δ is equinumerous to Γ', show that
   (a) $\Delta \approx \Delta$.
   (b) If $\Delta \approx \Gamma$ then $\Gamma \approx \Delta$.
   (c) If $\Delta \approx \Gamma$ and $\Gamma \approx \Theta$, then $\Delta \approx \Theta$.
2. Show:
   (a) A finite set cannot be equinumerous with one of its proper subsets (i.e. with some some subset strictly contained in it). [Hint: argue for the contrapositive, i.e. if equinumerous, then not finite.]
   (b) An infinite set *can* be equinumerous with one of its proper subsets.
   (c) The set of natural numbers is equinumerous with the set of ordered pairs of natural numbers.
   (d) The set of natural numbers is equinumerous with the set of positive rational numbers.
   (e) The set of natural numbers is equinumerous with the set of ordered triples of natural numbers.
   (f) The set of natural numbers is equinumerous with the set containing all singletons of numbers, ordered pairs of natural numbers, ordered triples of natural numbers, quadruples, quintuples, . . . , including ordered n-tuples (for any finite n).

(g) The set of natural numbers is equinumerous with the set of all finite sets of numbers.

3. Write '$\Delta \prec \Gamma$' for 'Δ is equinumerous with a subset of Γ but not equinumerous with Γ', and '$\Delta \preceq \Gamma$' for 'Δ is equinumerous with a subset of Γ'.

   Write '$\mathcal{P}\Delta$' for the powerset of Δ, i.e. the set of all subsets of Δ. Show

   (a) $\Delta \preceq \Delta$.

   (b) If $\Delta \preceq \Gamma$ and $\Gamma \preceq \Theta$, then $\Delta \preceq \Theta$.

   (c) $\mathbb{N} \prec \mathbb{R}$.

   (d) $\mathcal{P}\mathbb{N} \preceq \mathbb{R}$.

   (e) For any Δ, $\Delta \prec \mathcal{P}\Delta$. [Hint, generalize the first version of the diagonal argument of *IGT2*, §2.5(a).]

4. We say that a set is *countable* iff it is either empty or equinumerous with some set of natural numbers (maybe all of them!). It is *countably infinite* iff it is equinumerous with $\mathbb{N}$.

   Recalling the definition of enumerability in the sense of *IGT2*, p. 10, show that

   (a) If Δ is countable, it is enumerable.

   (b) If Δ is enumerable, it is countable.

   Also show

   (c) If Δ is countably infinite, then the set of finite subsets of Δ is countably infinite.

   (d) If Δ is countably infinite, then $\mathcal{P}\Delta$ is uncountably infinite.

5. A trickier question. Is an infinite family of nested subsets of a countable set necessarily countable?

   To explain: We say that Σ is a *nested* family of sets if for any two sets A and B in the family, either $A \subset B$ or $B \subset A$ (where $\subset$ is strict containment). Suppose then that the members of the nested family Σ are all subsets of some *countable* set Δ. Then our question is: must Σ itself have a countable number of members?