Solutions: Equinumerosity

Notation: $\Delta \subseteq \Gamma$ says that Δ is a subset of Γ in the sense that every member of Δ is a member of Γ, which allows $\Delta = \Gamma$. We use $\Delta \subset \Gamma$ when Δ is a proper subset of Γ, so $\Delta \subseteq \Gamma \wedge \Delta \neq \Gamma$

1. Two sets Δ and Γ are said to be *equinumerous* iff there is a one-one correspondence between them, i.e. there is some bijection $f : \Delta \to \Gamma$. For a simple reality check, show that equinumerosity is an equivalence relation. In other words, writing '$\Delta \approx \Gamma$' for 'Δ is equinumerous to Γ', show that
   (a) $\Delta \approx \Delta$.
   (b) If $\Delta \approx \Gamma$ then $\Gamma \approx \Delta$.
   (c) If $\Delta \approx \Gamma$ and $\Gamma \approx \Theta$, then $\Delta \approx \Theta$.

(a) The identity function that maps an element in Δ to the very same element in Δ is a bijection!

(b) If there is a bijection $f\colon \Delta \to \Gamma$, then it has an inverse $f^{-1}\colon \Gamma \to \Delta$ which is also a bijection.

(c) We just need to recall that if $f\colon \Delta \to \Gamma$ and $g\colon \Gamma \to \Theta$ are bijections, so is their composition $g \circ f\colon \Delta \to \Theta$. [See the exercise set on Functions.]

2. Show:
   (a) A finite set cannot be equinumerous with one of its proper subsets (i.e. with some some subset strictly contained in it). [Hint: argue for the contrapositive, i.e. if equinumerous, then not finite.]
   (b) An infinite set *can* be equinumerous with one of its proper subsets.
   (c) The set of natural numbers is equinumerous with the set of ordered pairs of natural numbers.
   (d) The set of natural numbers is equinumerous with the set of positive rational numbers.
   (e) The set of natural numbers is equinumerous with the set of ordered triples of natural numbers.
   (f) The set of natural numbers is equinumerous with the set containing all singletons of numbers, ordered pairs of natural numbers, ordered triples of natural numbers, quadruples, quintuples, ..., including ordered n-tuples (for any finite n).
   (g) The set of natural numbers is equinumerous with the set of all finite sets of numbers.

(a) Suppose we are given some $\Delta \subset \Gamma$ (that's strict containment) – so, trivially, there is an element e such that $e \in \Gamma$ but $e \notin \Delta$.

Assume there *is* a bijection $f\colon \Gamma \to \Delta$. Then consider the elements

$$f(e),\ f(f(e)),\ f(f(f(e))),\ f(f(f(f(e)))), \ldots$$

or in a simpler notation for repeated function-application:

$$f^1e,\ f^2e,\ f^3e,\ f^4e, \ldots$$

These are all distinct, for suppose $f^me = f^ne$ where $m < n$. Then since f is a bijection, it has an inverse f^{-1}. So we can apply that inverse m times to each side of the equation to get $e = f^{n-m}e$. But that's impossible because $e \notin \Delta$ and $f^{n-m}e \in \Delta$ (as is any value of f). Since $f^1e,\ f^2e,\ f^3e,\ f^4e, \ldots$ are all distinct and in Δ and hence in Γ, Γ must be infinite.

Contraposing. If Γ is finite and $\Delta \subset \Gamma$, there isn't a bijection $f\colon \Gamma \to \Delta$.

(b) The Galilean example will do as well as any. $\mathbb{N}$ is equinumerous with the set of even numbers $\mathbb{E} \subset \mathbb{N}$, as witnessed by the bijection $f\colon \mathbb{N} \to \mathbb{E}$ where $f(n) = 2n$.

(c) We saw in *IGT2* §2.4 how to set up a bijection using a zig-zag construction.

(d) A positive rational number can be uniquely represented in the form m/n by taking the fraction in 'lowest terms', i.e. where m and n have no common factors. So there is a bijection f between the the pairs of natural numbers $\langle m, n\rangle$ (where m and n have no common factors) and the positive rationals.

Now imagine running zig-zag through the pairs of numbers $\langle m, n\rangle$ jumping over cases where m and n have a common factor. That defines a bijective function g which maps k to the k-th such pair along the zig-zag path with jumps. Then the compound function $f \circ g$ is a bijection between the naturals and the rational numbers.

(e) Take the (original, unjumpy) zig-zag mapping in the direction that gives us a function which maps the ordered pair $\langle m, n\rangle$ to some number $k = p(m, n)$. Then consider the mappings

$$\langle m, n, o\rangle \longleftrightarrow \langle\langle m, n\rangle, o\rangle \longleftrightarrow \langle p(m, n), o\rangle\ [\textit{i.e. } \langle k, o\rangle] \longleftrightarrow p(k, o)\ [\textit{i.e. } \text{some } l]$$

Each mapping is a bijection, hence so is their composition.

Evidently, we can do the same trick again to get a bijection between numbers and ordered quadruples of numbers. And so on.

(f) We can use another zig-zag construction. Suppose $f_1\colon \mathbb{N} \to \mathbb{N}$ is the bijection from the set of numbers to the set of their singletons that sends n to $\{n\}$, $f_2\colon \mathbb{N} \to \mathbb{N}^2$ is our bijection from the numbers to the pairs of numbers, $f_3\colon \mathbb{N} \to \mathbb{N}^3$ is our bijection from the numbers to the triples of numbers, etc. Then imagine zig-zagging through

$$\begin{array}{ccccccccc}
f_1(0) & & f_1(1) & & f_1(2) & & f_1(3) & \ldots & \\
 & \swarrow & & \swarrow & & \swarrow & & & \swarrow \\
f_2(0) & & f_2(1) & & f_2(2) & & f_2(3) & \ldots & \\
 & \swarrow & & \swarrow & & \swarrow & & & \\
f_3(0) & & f_3(1) & & f_3(2) & & f_3(3) & \ldots & \\
 & \swarrow & & \swarrow & & & & & \\
f_4(0) & & f_4(1) & & f_4(2) & & f_4(3) & \ldots & \\
\vdots & \swarrow & \vdots & & \vdots & & \vdots & &
\end{array}$$

and define the function which maps n to the $n+1$-th tuple of numbers encountered on the path. That gives us the desired bijection.

(g) Imagining travelling the same zig-zag path again, but this time consider not the *ordered* tuples it generates but the corresponding *unordered* sets: and as we go along, we ignore any unordered set we've encountered before. Define the function which maps 0 to the empty set and then for $n > 0$ maps n to the n-th new set of numbers encountered on the path. That gives us the desired bijection.

3. Write '$\Delta \prec \Gamma$' for 'Δ is equinumerous with a subset of Γ but not equinumerous with Γ', and '$\Delta \preceq \Gamma$' for 'Δ is equinumerous with a subset of Γ'.

   Write '$\mathcal{P}\Delta$' for the powerset of Δ, i.e. the set of all subsets of Δ. Show

   (a) $\Delta \preceq \Delta$.

   (b) If $\Delta \preceq \Gamma$ and $\Gamma \preceq \Theta$, then $\Delta \preceq \Theta$.

   (c) $\mathbb{N} \prec \mathbb{R}$.

   (d) $\mathcal{P}\mathbb{N} \preceq \mathbb{R}$.

   (e) For any Δ, $\Delta \prec \mathcal{P}\Delta$. [Hint, generalize the first version of the diagonal argument of *IGT2*, §2.5(a).]

(a) Immediate.

(b) If $\Delta \preceq \Gamma$ and $\Gamma \preceq \Theta$, there's a bijection f from Δ to some or all of Γ and there is another bijection g from Γ to some or all of Θ. Put the two maps together and we'll get a map $g \circ f$. Consider the range $Z \subset \Theta$ of that map, i.e. the set of elements $g \circ f(d)$ for $d \in \Delta$. Then $g \circ f \colon \Delta \to Z$ is a bijection.

(c) $\mathbb{N}$ is equinumerous with a *subset* of $\mathbb{R}$ (just map the natural number 0 to the real number 0, and the each natural $1 + 1 + 1 + \ldots + 1$ to the corresponding positive real number $1_{\mathbb{R}} + 1_{\mathbb{R}} + 1_{\mathbb{R}} + \ldots + 1_{\mathbb{R}}$). And we showed that $\mathbb{N}$ isn't equinumerous with the whole of $\mathbb{R}$ in *IGT2*, §2.5(b).

(d) Again in *IGT2*, §2.5(b) we showed that the power set of $\mathbb{N}$ is equinumerous with the set of infinite binary strings which is equinumerous with the reals between 0 and 1, i.e. equinumerous to a subset of the reals.

(e) Consider the powerset of Δ, in other words the collection $\mathcal{P}(\Delta)$ whose members are all the subsets of Δ.

Suppose for reductio that there is a bijective function $f \colon \Delta \to \mathcal{P}(\Delta)$, and consider what we'll call the diagonal set $D \subseteq \Delta$ such that $x \in D$ iff $x \notin f(x)$.

Since $D \in \mathcal{P}(\Delta)$ and f by hypothesis is a bijection onto $\mathcal{P}(\Delta)$, there must be some number d such that $f(d) = D$.

So we have, for all x, $x \in f(d)$ iff $x \notin f(x)$. Hence in particular $d \in f(d)$ iff $d \notin f(d)$. Contradiction! There therefore cannot be such a bijective function $f \colon \Delta \to \mathcal{P}(\Delta)$.

However, the map that takes x to $\{x\}$ is a bijective map from Δ to the set of S of singletons of elements of Δ, and $S \subset \mathcal{P}(\Delta)$.

So putting everything together, $\Delta \prec \mathcal{P}(\Delta)$.

4. We say that a set is *countable* iff it is either empty or equinumerous with some set of natural numbers (maybe all of them!). It is *countably infinite* iff it is equinumerous with $\mathbb{N}$.

   Recalling the definition of enumerability in the sense of *IGT2*, p. 10, show that

   (a) If Δ is countable, it is enumerable.

   (b) If Δ is enumerable, it is countable.

   Also show

   (c) If Δ is countably infinite, then the set of finite subsets of Δ is countably infinite.

   (d) If Δ is countably infinite, then $\mathcal{P}\Delta$ is uncountably infinite.

(a) Suppose Δ is countable. If it is empty it is trivially enumerable. If it is not empty then there is a set of numbers $\Gamma \subseteq \mathbb{N}$ (maybe finite, maybe all the numbers) such that there is a bijection $f\colon \Gamma \to \Delta$. Now take an element $d \in \Delta$, and consider the new function $g\colon \mathbb{N} \to \Delta$ defined by $g(n) = f(n)$ for $n \in \Gamma$, and $g(n) = d$ when $n \in \mathbb{N} \setminus \Gamma$. (In other words we just extend our definition of f to give us a function defined for every number, by giving it a default value for all the new arguments.) By construction, g is still a surjection.

So we have a surjection $g\colon \mathbb{N} \to \Delta$ which makes Δ enumerable by our definition in *IGT2*, p. 10.

(b) Suppose conversely that Δ is enumerable, i.e. there is a surjective function $g\colon \mathbb{N} \to \Delta$. Now take Γ to the the set of numbers $\{n \mid m < n \to g(m) \neq g(n)\}$. In other words, when some numbers $n, n', n'', \ldots$ get mapped to the same value in Δ, we keep only the smallest one and discard the rest. Then by brute-force construction, the function $f\colon \Gamma \to \Delta$ which agrees with g for any number in Γ is a bijection. So Δ is countable.

(c) If Δ is countably infinite, then there is a bijection f from Δ to $\mathbb{N}$. Consider the action of f on subsets of Δ. It creates a corresponding map F from subsets of Δ to subsets of $\mathbb{N}$ defined as follows: when $X \subset \Delta$, $F(X) = \{f(x) \mid x \in X\}$. It is easy to check that this is a bijection that takes finite subsets of Δ to finite subsets of $\mathbb{N}$. So there will be as many finite subsets of Δ as there are finite subsets of $\mathbb{N}$, i.e. countably many.

(d) This is an immediate consequence of Question 3(e).

5. A trickier question. Is an infinite family of nested subsets of a countable set necessarily countable?

   To explain: We say that Σ is a *nested* family of sets if for any two sets A and B in the family, either $A \subset B$ or $B \subset A$ (where $\subset$ is strict containment). Suppose then that the members of the nested family Σ are all subsets of some *countable* set Δ. Then our question is: must Σ itself have a countable number of members?

Where r is a positive real number, take Q_r to be the set of rational numbers $\{\, q \mid 0 \leq q < r\}$.

Then take the family Σ of sets Q_r for $0 \leq r \leq 1$. Then Σ is a nested family of sets because of course given Q_r and Q_s, $Q_r \subset Q_s$ if $r < s$ and $Q_s \subset Q_r$ otherwise. (Here we rely on the familiar fact that between any two real numbers there is a rational one.) Every Q_r in the family Σ is a subset of $\mathbb{Q}$ and hence countable. But there are as many different Q_r as there are reals in the interval $[0, 1]$ which is uncountably many.

For more on this cute little problem, see the discussion of 'Problem 9' in the wonderfully entertaining Béla Bollobás, *The Art of Mathematics: Coffee Time in Memphis* (CUP, 2006).