Exercises: Functions

This set of exercises is for those who haven't much acquaintance with function notation, or with the idea of 'injective', 'surjective', and 'bijective' functions, and inverse functions, and who want to get a bit clearer about these things. Make what use of these exercises that you will: *don't* get bogged down here, when there is so much more exciting stuff to come! In fact, we'll only really be making use of the answer to Qn. 4 in *IGT2*.

Reading

1. *IGT2*, §§2.1–2.2, §14.3(a).
2. http://www.mathsisfun.com/sets/function.html, http://www.mathsisfun.com/sets/injective-surjective-bijective.html Very introductory pages, with lots of diagrams – very unscary!
3. Innumerable maths texts, early on, will have sections covering this material. Here's a very clear and quite gentle treatment, available in any university library: James R Munkres, *Topology* (Prentice Hall, 2nd edn., 2000), read pp. 1–30, which also covers/revises other stuff you should know anyway.
4. http://en.wikipedia.org/wiki/Function_(mathematics)

Reminders

1. The notation $f\colon A \to B$ tells us that f is a (total, single-valued) function with domain A and codomain B, so for every argument $x \in A$ there is some value $y \in B$ such that $f(x) = y$.
2. The functions $f\colon A \to B$ and $g\colon A \to B$ are identical iff, for all $x \in A$, $f(x) = g(x)$.
3. $f\colon A \to B$ is *surjective* iff for every $y \in B$ there is some $x \in A$ such that $f(x) = y$. f is *injective* iff whenever $x \neq x'$ then $f(x) \neq f(x')$. f is *bijective* iff it is both surjective and injective.
4. $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, 3, \ldots\}$, $\mathbb{E}$ is the set of even natural numbers $\{0, 2, 4, 6, \ldots\}$, $\mathbb{Z}$ is the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ is the set of positive reals r, where $0 \leq r$.

Exercises

1. First, some very elementary reality checks.
   (a) Which of these definitions is legitimate?
      i. f is a function from domain $\mathbb{N}$ to codomain $\mathbb{N}$ defined by $f(n) = n + 1$. Or, in an obvious shorthand: $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n + 1$.
      ii. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n + 1\frac{1}{2}$.
      iii. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n - 1$.
      iv. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = n - 1$.
      v. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \sqrt{x}$
      vi. $f : \mathbb{R}^+ \to \mathbb{R}$ where $f(x) = \sqrt{x}$

vii. $f : \mathbb{R}^+ \to \mathbb{R}^+$ where $(f(x))^2 = x$

(b) Which of these functions is injective, which surjective, which bijective?

i. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = 2n$.

ii. $f : \mathbb{N} \to \mathbb{E}$ where $f(n) = 2n$.

iii. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n/2$ if n is even, and $f(n) = (n + 1)/2$ otherwise.

iv. $f : \mathbb{N} \to \mathbb{N}$ where $f(n)$ is the number of grapes Julius Ceasar ate on his n-th birthday (assuming that is always a whole number)!

v. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x$.

vi. $f : \mathbb{R} \to \mathbb{R}^+$ where $f(x) = x^2$.

(c) Let A be the set $\{1, 2, 3\}$ and B be the set $\{1, 2\}$:

i. How many functions $f\colon A \to B$ are injective, how many surjective?

ii. How many functions $f\colon B \to A$ are injective, how many surjective?

iii. How many functions $f\colon A \to A$ are bijective?

(d) In which cases are the functions f and g mentioned one and the same?

i. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n(n + 1)/2$; $g : \mathbb{N} \to \mathbb{N}$ where $g(n)$ is the sum of the first n natural numbers.

ii. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n^2$; $g : \mathbb{Z} \to \mathbb{Z}$ where $g(n) = n^2$.

iii. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = 2n$; $g : \mathbb{N} \to \mathbb{E}$ where $g(n) = 2n$.

2. An inverse f^{-1} to a function f 'undoes' f's effect – in other words, applying f and then applying f^{-1} to the result takes us back to where we started. More carefully, if $f\colon A \to B$ is a function, then $f^{-1}\colon B \to A$ is an inverse to f iff for all $y \in B$, $f^{-1}(y) = x$ iff $f(x) = y$, which implies $f^{-1}(f(x)) = x$.

(a) Prove that if f has an inverse, it has exactly one.

(b) Prove that f has an inverse iff it is a bijection.

(c) Assuming $f\colon A \to B$ has an inverse, what function is defined by $I(x) = f(f^{-1}(x))$?

(d) Prove that $f\colon A \to B$ is a bijection, its inverse is a bijection.

3. Given functions $f\colon A \to B$ and $g\colon B \to C$ (with the domain of g being the same as the codomain of f) their *composition* is the function $(g \circ f)$ (read 'g following f'), where $(g \circ f)\colon A \to C$, and for any $x \in A$, $(g \circ f)(x) = g(f(x))$.

(a) When are $(h \circ j)$ and $(j \circ h)$ the same function?

(b) When are $(h \circ (j \circ k))$ and $((h \circ j) \circ k)$ the same functions?

Now dropping unnecessary brackets, given functions $f\colon A \to B$ and $g\colon B \to C$ show that

(c) If f and g are injective, so is $g \circ f$.

(d) If f and g are surjective, so is $g \circ f$.

(e) If f and g are bijective, so is $g \circ f$.

Finally,

(f) If f and g are bijective, what is the inverse function to $g \circ f$?

4. Suppose the characteristic functions of the numerical properties P and Q are respectively c_P and c_Q. What are the characteristic functions of

(a) The property n has when it isn't the case that Pn?

(b) The property n has when either Pn and/or Qn?

(c) The property n has when both Pn and Qn?

(d) The property n has when, if Pn then Qn?

Suppose a function $c : \mathbb{N} \to \{0, 1\}$ is given: must there always be a numerical property whose characteristic function is c?