Exercises: The language L_A

Some introductory exercises on the language of basic arithmetic, dubbed L_A in Chapter 5, and introducing some variations and augmentations of this language.

Reading

1. *IGT2*, Ch. 5.

Exercises

1. Given the conventions adopted in §5.1 and the definition of L_A given in §5.2, which of the following are (i) wffs of L_A, (ii) abbreviations of wffs of L_A, (iii) expressions of mathematical English?
   (a) $(\mathsf{1} + \mathsf{2}) = \mathsf{3}$.
   (b) $\forall x\, x + 0 = x$.
   (c) $\forall \mathsf{x}\, (\mathsf{x} + \mathsf{0}) = \mathsf{x}$.
   (d) $\forall \mathsf{x}(\mathsf{x} + \mathsf{0} = \mathsf{x})$.
   (e) $\forall m \exists n\, n > m$.
   (f) $\forall \mathsf{m} \exists \mathsf{n}\, \mathsf{n} > \mathsf{m}$.
   (g) $\mathsf{x} + \mathsf{1} \neq \mathsf{x}$.
   (h) $\xi + \mathsf{S0} \neq \xi$.
2. Give L_A wffs that express the following properties, relations, and functions (x, y, z range over natural numbers):
   (a) The property of being an even number [give three alternatives].
   (b) The property of being divisible by seven.
   (c) The relation *being less than*.
   (d) The property of being a composite number [i.e. not a prime].
   (e) The function $x \mapsto x^2$.
   (f) The relation of that x has to y when x is the square root of y.
   (g) The relation that x has to y when x is a factor of y.
   (h) The relation that x has to y and z when x is strictly between y and z.
   (i) The function $x, y \mapsto |x - y|$.
   (j) The property of being the sum of two primes.

   Conclude that Goldbach's Conjecture can be stated in the language L_A.
3. The fundamental non-logical vocabulary of L_A is $\{\mathsf{S}, +, \times, \mathsf{0}\}$. Describe a variant language L'_A whose fundamental non-logical vocabulary is $\{+, \times, <, \mathsf{0}, \mathsf{1}\}$ (where the symbols have the obvious interpretations). Explain why it is a matter of indifference whether we choose to use the language L_A or L'_A.

4. This exercise is for enthusiasts, and gives a glimpse ahead. For convenience, we use (as part of our ordinary mathematical notation) p_j for the $j+1$-th prime: so $p_0, p_1, p_2, p_3, p_4, \ldots$ are the primes $2, 3, 5, 7, 11, \ldots$. By the Fundamental Theorem of Arithmetic, every natural number has a unique factorization into primes – and hence a unique representation in the form $p_0^{e_0} \cdot p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot \ldots$ where only a finite number of the exponents e_j are non-zero.

   Consider the language L_A^β whose built-in non-logical vocabulary is $\{\mathsf{S}, +, \times, \mathsf{B}, \mathsf{0}\}$. Here B is a two-place function expression [written prefix], which expresses the function β, where $\beta(n, j)$ is the exponent of the prime p_j in the factorization of n (so is zero if p_j is not a factor of n).

   Give L_A^β wffs that express the numerical following properties, relations, and functions:

   (a) The property of being a power of 2.

   (b) The property of being prime.

   (c) The property c has when the exponent of p_2 in its prime factorization is twice the exponent of p_1, and the exponent of p_3 in its prime factorization is 2 times the exponent of p_1.

   (d) The relation c has to n when (i) the exponent of p_0 in the factorization of c is 1, and (2) for all $j < n$, the exponent of p_{j+1} in the factorization of c is $(j+1)$ times the exponent of p_j.

   (e) The factorial function $x \mapsto x!$. [Hint: note that given a c which has that last relation to n, the exponent of p_n in its prime factorization must be $n!$.]

   (f) The exponential function $x, y \mapsto x^y$. [Use the same sort of trick you've just used to express the factorial.]

   What are the prospects for expressing the factorial and exponential functions in unaugmented L_A?