Exercises: Formal theories

These exercises start to explore (informally, and in a very introductory way) a few aspects of the ideas of a formalized language and of a formalized theory.

Reading

1. *IGT2*, Ch. 4.
2. S. C. Kleene, *Introduction to Metamathematics* (1952): §15 'Formalization of a theory'.
3. J. R. Shoenfield, *Mathematical Logic* (1967): §1.1 'Axiom systems', §1.2 'Formal systems'.

Exercises

1. Consider a first-order language L for the theory of addition, whose logical apparatus comprises a suitable set of classical connectives, quantifiers, and identity. Its non-logical apparatus is to comprise the constants '$\mathsf{0}$' and '$\mathsf{1}$', the two-place function '$+$', and the two-place relation '$<$'.
   (a) Define the terms of L.
   (b) Show that it is algorithmically decidable which expressions are L-terms.
   (c) Define the atomic wffs of L.
   (d) Show that it is algorithmically decidable which expressions are atomic L-wffs.
   (e) Define the wffs of L [allowing wffs with free variables].
   (f) Show that it is algorithmically decidable which expressions are L-wffs.
   (g) Show that it is algorithmically decidable what is an L-wff which contains the variable 'x' free.
   (h) Show that it is algorithmically decidable which expressions are L-sentences (i.e. closed wffs without free variables).

2. Consider the following (uninterpreted) theory H. The alphabet of H's language consists of the symbols $\mathsf{M}, \mathsf{U}, \mathsf{I}$, and any finite string of symbols is a wff. The theory has one axiom: MI. H also has five rules of inference (σ indicates a string of symbols, possibly empty).
   1. Given a wff of the form $\sigma\mathsf{I}$, you can infer the wff $\sigma\mathsf{IU}$. (For example, from MUI infer MUIU.)
   2. Given a wff of the form $\mathsf{M}\sigma$, infer $\mathsf{M}\sigma\sigma$. (For example, from MIIU infer $\mathsf{MIIUIIU}$.)
   3. Given a wff which includes the string UI, infer the wff that results from replacing that string with IU. (For example, from MIUIU infer MIIUU.)
   4. Given a wff which includes the string UU, infer the wff that results from deleting that string. (For example, from MIIUU infer MII.)
   5. Given a wff which includes the string III, infer the wff that results from replacing that string with a U. (For example, from $\mathsf{MUIIIIU}$ infer MUIUU.)

   (a) Does H count as an effectively axiomatized (uninterpreted) formal theory?

(b) Prove every H-theorem starts with the symbol M, and contains no other occurrence of M.

Don't get bogged down on these next three little brain-teasers, but have a go before moving on:

(c) Can you derive MIIU as a theorem?

(d) Can you derive MUUIU?

(e) Can you derive MU?

Now three more questions about H:

(f) Show that, for each rule, if it is applied to a wff whose number of contained 'I's is not a multiple of 3, the result is a wff whose number of 'I's is also not a multiple of 3.

(g) Can you derive $\mathsf{MIUIIUIII}$?

(h) Now revisit question (e) again.

3. [This is really for philosophers: mathematicians will have seen this all before, in one guise or another.]

   Consider the formal first-order theory G whose non-logical vocabulary comprises just a two-place function expression '$\cdot$' and a constant 'e'. We'll in fact write the function 'infix' like ordinary multiplication, so we put e.g. '$(x \cdot y)$' rather than $\cdot(x, y)$. We will also allow the dropping of outer brackets. The axioms of G are:

   1. $\forall x \forall y \, x \cdot (y \cdot z) = (x \cdot y) \cdot z$
   2. $\forall x \, x \cdot e = x$
   3. $\forall x \exists y \, x \cdot y = e$

   (a) Is G an effectively formalized theory?

   (b) Prove $\forall x \exists y \, y \cdot x = e$.

   (c) Prove $\forall x \, e \cdot x = x$.

   (d) Prove that the 'unit' e is unique: in other words, if e and e' both satisfy (2) and (3), then $e = e'$.

   (e) Prove that 'inverses' are unique: i.e., $\forall x \forall y \forall y'((x \cdot y = e \wedge x \cdot y' = e) \to y = y')$.

   (f) Prove G is consistent by giving three interestingly different interpretations for the language of G on which G's axioms are all true.

   (g) Is $\forall x \forall y \; x \cdot y = y \cdot x$ a G-theorem?

   (h) Is G negation complete?

4. A reality check. Suppose T is an effectively axiomatized formal theory. Can T be

   (a) Inconsistent and negation-complete?

   (b) Consistent, negation-incomplete and decidable?

   (c) Inconsistent and undecidable?

   (d) Consistent and undecidable?

   (e) Consistent, negation-complete and undecidable?