IFL: Logicbite 3

Proofs

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What is an argument? Here's Paul Teller in his *Primer*:

> An *argument* is a collection of declarative sentences one of which is called the conclusion and the rest of which are called the premises.

And here is an author we haven't met before, Irving Copi in his *Symbolic Logic*:[1]

> Every argument has a structure, in the analysis of which the terms 'premiss' and 'conclusion' are usually employed. The conclusion of an argument is that proposition which is affirmed on the basis of the other propositions of the argument, and these other propositions which are affirmed as providing grounds or reasons for accepting the conclusion are the premisses of that argument.

So say the many (as Aristotle might put it!). But repetition doesn't make for truth. For of course, arguments – in the sense that logicians are going to be interested in – are not *typically* (and certainly not *always*) truncated affairs with some premisses, a conclusion, and nothing else in between. Very often, arguments involve extended passages of reasoning, where we move from initial assumptions via intermediate steps to an eventual conclusion.

Now of course our two authors could have said something like this: "We are in charge of our terminology! And just as we logicians use 'valid' and 'sound' in a special restricted way, we are going to use 'argument' too in a restricted way, specifically to refer to one-step reasonings." But they *don't* explicitly announce this up front. Perhaps they should have done! And you need to look around a bit to find textbook authors being more explicit from the outset. But Lemmon in his *Beginning Logic* remarks

> A feature of ordinary argumentation is that it proceeds in stages.

And he describes how the output (intermediate conclusion) from one stage can be used as an input (a premiss) for another stage. And here are Daniel Bonevac, Nicholas Asher and Robert Koons distinguishing one-stage from multi-stage arguments in the opening pages of their *Logic, Sets, and Functions* (1999):[2]

> Many arguments in natural language are complicated. A lawyer arguing for the innocence of a client, for instance, offers many more specific arguments in presenting the case. The lawyer may argue that a piece of evidence is inadmissible, that results from a lab test are ambiguous, that the client could not have reached the scene of the crime by the time it was committed, and so on. All these smaller arguments form part of the larger argument for the client's innocence.
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> We can divide arguments, then, into two groups: *extended* arguments, which contain other arguments, and *simple* arguments, which do not. Extended arguments may have several conclusions. Such arguments may consist of several simple arguments in sequence. They may contain other extended arguments. . . .
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> Mathematical proofs are extended arguments. A mathematician may begin a proof by stating some assumptions. The mathematician then draws out consequences of the assumptions, perhaps making other assumptions along the way. Finally, the proof

[1] Copi was taught by Bertrand Russell at the University of Chicago in the 1930s, and later wrote two widely used logic textbooks – his *Introduction to Logic*, originally published in 1954, has just been updated to its fifteenth edition, which must be some sort of record. Longevity isn't always a sign of excellence . . .

[2] Despite its more mathematical-sounding title, the book is still introductory and the authors were at the time of writing philosophers at the University of Texas.

> ends with a conclusion, the theorem it proves. A mathematical proof is thus a series of simple arguments.
>
> A simple argument, like an extended argument, starts with premises justifying a conclusion. We will be so often concerned with simple arguments that we will drop the adjective simple and speak of *arguments*. (Later, when we examine proofs, we will just call them *proofs*.)

There's a good terminological proposal here for using (one-step) 'argument' vs (possibly multi-step) 'proof'. Perhaps it is already the de facto usage among logicians. John MacFarlane[3] in his *Philosophical Logic* (2021) explicitly says

> An *argument*, in the logician's sense, is just a pair consisting of a set of premises and a conclusion. This is a departure from the ordinary sense of 'argument', which is usually used either for a dispute or for the reasoning that connects the premises with the conclusion. The logician's notion of *proof* is related to this latter sense.

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I too want to draw a distinction like the one made by Bonevac, Asher and Koons. But I'm not sure I like their use of 'simple' vs 'extended' to label it. Why not?

Suppose we have an argument of the shape $A, B \therefore C$. This will count as 'simple' in the sense of our last authors (because it doesn't contain any sub-arguments as parts). But it needn't be at all 'simple' in the sense of basic, elementary, or obvious. Of course, logic books typically start with lots of illustrative examples which *are* simple in both senses – one step arguments whose validity (or invalidity) is basic and more or less obvious. Still, an argument can be a one-step valid argument (so simple, in our authors' sense) without being in the slightest bit basic, elementary or obvious (so not simple in other natural senses).

Indeed, it is because the inferential leap in the one-step $A, B \therefore C$ may not simple in the ordinary sense that we go in for extended, multi-step, arguments. If you are puzzled about whether we can indeed derive that conclusion from the two premisses, then an extended argument may give you just what you want: if starting with the premisses A, B you can go through a series of safe and relatively simple intermediate steps to eventually derive the conclusion C then that settles it – the argument $A, B \therefore C$ is valid.

For clarity, then, rather than our authors' 'simple argument' (which has too many of the wrong everyday connotations) I'd rather go for the more explicit 'one-step argument'.

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To repeat, challenged to defend an inferential leap from the premisses A, B to the conclusion C, we can aim to come up with an extended, multi-step, argument – a proof, or derivation – that gets us from the premisses to the conclusion by smaller, more obviously correct, steps. But in what ways can we legitimately structure a multi-step argument out of simpler one-step inferences?

That's a non-trivial question. Bonevac, Asher and Koons remark that "[extended] arguments may consist of several simple arguments in sequence". But you already at some level know that (a) not any old sequence will do (e.g. you mustn't go round in circles). And you also know that (b) a proof may not be a straightforwardly linear sequence of simple steps (e.g. we can make temporary assumptions for the sake of argument, assumptions which are later dropped, as when we assume A for the sake of argument, derive an absurdity, and drop the assumption in concluding *not-A*).

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[3]MacFarlane has written extensively on topics in the history and philosophy of logic in particular. This book could well be of interest in retrospect, after you have worked through *IFL*.

How do you think that the term 'valid' might most usefully be extended to apply to multi-step arguments? Note the following possible situation: we could have a multi-step argument which starts from A and makes a horribly fallacious inference to B; from that, it makes a horribly fallacious inference to C; and from *that*, it makes another horribly fallacious inference to D. So the argument starting from the premiss A and ending at D is invalid at every step. But it could be that – by luck, and no credit at all to the horrible argument – that the final conclusion D really does follow from the premiss A (e.g. by another, correct, proof). Should we call such multi-step argument $A - B - C - D$ valid?

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Later in *IFL* I will be saying quite a bit about ways of structuring proofs when I discuss formal derivations in a so-called natural deduction system. But before we go formal, I should say just a little more about extended, multi-step, arguments at our current informal level. So that's the business of Chapter 4.