IFL: Logicbite 14

Our first interesting 'metatheorem'

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We are in the middle of developing the formal apparatus we are going to be using over the next few chapters for investigating arguments which turn on the connectives 'and', 'or', 'not'. The first step has been introducing suitable formal languages – PL languages – which we can use for tidily regimenting arguments in clear unambiguous ways. Then, starting in Chapter 15, we will see how to test these regimented arguments for validity.

We are going to be discovering two sorts of result as we go along:

(1) We can *use* the formal apparatus. In other words, we can derive results about particular arguments once framed in suitable PL languages, and there will be generalizations about such arguments too.

   Here's a trivial sort of example. Informally, if we are told that *either P or Q*, but also that we can rule out the first, we can conclude *Q*. Formally, we'll show that the argument $(\mathsf{P} \lor \mathsf{Q}), \neg\mathsf{P} \therefore \mathsf{Q}$ is indeed valid. Moreover, for any wffs α and β, a corresponding one-step argument of the shape $(\alpha \lor \beta), \neg\alpha \therefore \beta$ is valid.

(2) Occasionally, however, we'll also want to stand back to theorize *about* our formal apparatus. We'll want to confirm that, in various respects, it behaves as advertised.

   Here's a not-so-trivial sort of example. We noticed that in ordinary language, a claim of the form *P or Q and R* can be ambiguous. So in constructing our formal PL languages we instituted a bracketing regime which is advertised as preventing that sort of ambiguity from arising. But does it do the job? We claimed that it does – and defending this claim requires proving that every wff has a unique parse tree.

The second sort of result is standardly called a *metatheorem*. Compare: our metalanguage is the language in which we talk about our formal apparatus; similarly a metatheorem is a theorem about our formal apparatus.

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The result that PL wffs have unique parse trees is therefore a metatheorem. It isn't quite trivial to prove, but it's not exactly an *exciting* metatheorem either (which is why I relegated the details of a proof to the Exercises). It's just a matter of checking that I haven't fouled up in the specification for PL's backeting regime.

In Chapter 13, however, we get to meet a more exciting result, our first interesting metatheorem. The question at issue is raised by Quine very early on in his *Methods of Logic*. He has just noted that alongside negation, conjunction, and inclusive disjunction (which he calls 'alternation') we can similarly define other truth-functional constructions; we could, if we want, introduce a new connective 'excl-or' to express exclusive disjunction. But do we need to do this? More generally:

> This question now arises: do our negation, conjunction, and alternation constitute a sufficient language for all truth-functional purposes? Given an explanation of a new truth-functional symbol (e.g., 'excl-or'), can we always be sure that the new symbol will be translatable into our existing notation?

It is pretty obvious that since 'P excl-or Q' is, by definition, true just when either P or Q is true but not both, we could express the same thing by using our existing three connectives, thus: $((\mathsf{P} \lor \mathsf{Q}) \land \neg(\mathsf{P} \land \mathsf{Q}))$. *But does this sort of thing always work?*

For example, in §12.2(e), I cooked up the three-place 'dollar connective' giving a truth-table which defines e.g. $\$(\mathsf{P},\mathsf{Q},\mathsf{R})$: could we again express the same thing by using our existing three connectives?

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Quine of course goes on to give the answer. So do most authors in their texts (though e.g. neither Hurley in his enormous book nor Simpson in his shorter one seem to raise the question).

In *IFL*, I show how to define $\$(\mathsf{P},\mathsf{Q},\mathsf{R})$ using the basic three connectives in §13.3. And then, having done this one case, it is easy to generalize, to give a positive answer to Quine's general question; I do that in §13.4.

As a mildly diverting but ultimately unimportant supplementary question, we can then ask – can we in fact define all the truth-functions using even less than our three basic connectives? That's the topic of §13.5.

All hopefully straightforward (metatheory doesn't have to be scary!). So now read Chapter 13!