IFL: Logicbite 16

Tautological entailment

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We now come to Chapters 15 and 16 of *IFL*, where I define the all-important notion of tautological entailment, and put it to work.

I hope that the individual sections of these two chapters are, taken separately, sufficiently clear. But looking at them again, I'm not at all sure that everything is organized as well as it should be. So this Logicbite will guide you through the chapters while taking their sections in a somewhat different order. I hope this helps!

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Let's start with the sections where the twin notions of tautological validity/tautological entailment are first defined and illustrated. Here's Paul Teller introducing the idea we need (just slightly changing his way of displaying the truth table):

> Consider the following argument:
>
> | | |
> |---|---|
> | $A \vee B$ | Adam loves Eve or Adam loves Bertha. |
> | $\sim A$ | Adam does not love Eve. |
> | B | Adam loves Bertha. |
>
> If you know, first of all, that either 'A' or 'B' is true, and in addition you know that 'A' itself is false; then clearly, 'B' has to be true. So from '$A \vee B$' and '$\sim A$' we can conclude 'B'. We say that this argument is *Valid*, by which we mean that, without fail, if the premises are true, then the conclusion is going to turn out to be true also.
>
> Can we make this idea of validity more precise? Yes, by using some of the ideas we have developed in the last three chapters [in which Teller has introduced what we are calling PL languages and truth tables]. (Indeed one of the main reasons these ideas are important is that they will help us in making the notion of validity very precise.) Let us write out a truth table for all the sentences appearing in our argument:
>
> | | A | B | $A \vee B$ | $\neg A$ | B |
> |---|---|---|---|---|---|
> | case 1 | T | T | T | F | T |
> | case 2 | T | F | T | F | F |
> | case 3 | F | T | T | T | T |
> | case 4 | F | F | F | T | F |
>
> We know that cases 1 through 4 constitute all the ways in which any of the sentences in the argument may turn out to be true or false. This enables us to explain very exactly what we mean by saying that, without fail, if the premises are true, then the conclusion is going to turn out to be true also. We interpret this to mean that in each possible case (in each of the cases 1 through 4), if the premises are true in that case, then the conclusion is true in that case. In other words, in all cases in which the premises are true, the conclusion is also true. In yet other words:
>
> > To say that an argument (expressed with sentences of sentence logic) is *Valid* is to say that any assignment of truth values to sentence letters which makes all of the premises true also makes the conclusion true.

The reservation I have about this is using the same word – an unqualified 'valid' – both for (i) the familiar informal notion which gets deployed in Teller's opening paragraph and then also for (ii) the notion officially defined in terms of valuations or assignment of truth values which is introduced at the end of this passage, a notion that applies specifically to arguments expressed in what Teller calls sentence logic. This double usage isn't an uncommon practice, but I deprecate it: for a start, as you'll see in a moment, there can be arguments 'expressed with sentences of sentence logic' which not valid in the second technical sense, but are valid in the first sense.

That's why I prefer e.g. Tim Button's more explicit terminology when he writes (using different schematic letters)

> The sentences $\alpha_1, \alpha_2, \ldots, \alpha_n$ *tautologically entail* the sentence γ if and only if no valuation of the [relevant] atomic sentences makes all of $\alpha_1, \alpha_2, \ldots, \alpha_n$ true and γ false.

Equivalently, in the same circumstances, the argument $\alpha_1, \alpha_2, \ldots, \alpha_n \therefore \gamma$ is said to be *tautologically valid.* (These defined terms should be universally understood, though they are far from universally used. The authors of *The Logic Book* instead use 'truth-functionally entail', Simpson has 'truth-functionally valid', while the authors of *Language, Proof and Logic*[1] prefer '[is a] tautological consequence'.)

Two key points about our pair of defined notions:

(1) The intuitive informal notion of validity was defined in terms of necessity (or, in Teller's words, in terms of what happens without fail). In the one-premiss case, the inference from α to β is valid just so long as *necessarily*, if α is true, β is true. But that account raises the tricky issue of explicating the relevant notion of necessity in play here. By contrast the inference from α to β is said to be tautologically valid just so long as, on every valuation of the relevant atoms, if α is true, β is true: and *that* definition doesn't invoke a potentially puzzling notion of necessity.

(2) However, connecting the old and new, if an argument is tautologically valid, it is valid – i.e. being tautologically valid is indeed one way of being valid in the intuitive sense. Can you see why?

For more explanations – in particular of the key point (2) – now read *IFL* §15.1 and §15.2.

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Suppose we want to know whether the premisses $\alpha_1, \alpha_2, \ldots, \alpha_n$ tautologically entail a conclusion γ. Of course, *one* way of finding the answer is to use *brute force*, and trudge through the hack work of examining every possible valuation of the relevant atoms and in each case checking whether it makes the premisses true and conclusion false!

That's not very elegant, but the brute force method must (eventually) give the answer to our question. You've met a few simple cases of this method being used in §15.1: I say rather more about it as we continue Chapter 15 – and I give some tactics for cutting down the amount of work involved (essentially: approach the stages of the task in a sensible order).

Now, as you will later discover, there do exist other ways of determining whether $\alpha_1, \alpha_2, \ldots, \alpha_n$ indeed tautologically entail γ. The brute force use of truth tables is just one method. (You may sometimes hear it said of an argument that it is 'valid by the truth-table test'. But that's an unfortunate mis-speak. It confuses the *relation* of tautological validity with just one *technique* we can use to determine whether the relation holds.) A question then arises: is one of the other methods guaranteed to be more efficient than the use of brute-force truth-tables? I *briefly* say something about this in Chapter 16.

So now read *IFL* §15.3, §15.4 and §16.2.

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[1] This book by Dave Barker-Plummer, Jon Barwise and John Etchemendy is a descendent of The *Language of First-Order Logic* by the second two authors, which famously first introduced a software package *Tarski's World*. The later *LPL* is one of my top recommendations for reading in parallel with, or instead of, *IFL*. You can find out more about the book and the associated software here.

You have now met basic accounts about (i) what tautological validity is and (ii) one way of determining whether an argument is tautologically valid. I next want to clarify three general points.

(A) The first can be introduced by taking up Teller's presentation again. Immediately after the passage I quoted before, he writes this:

> Let's look at an example of an *invalid* argument (an argument which is not valid):
>
> $$\begin{array}{l} A \vee B \\ A \\ \hline B \end{array}$$
>
> | | A | B | $A \vee B$ | $\neg A$ | B |
> |---|---|---|---|---|---|
> | case 1* | T | T | T | T | T |
> | case 2* | T | F | T | T | F |
> | case 3 | F | T | T | F | T |
> | case 4 | F | F | F | F | F |
>
> I have set up a truth table which shows the argument to be invalid. First I use a '*' to mark each case in which the premises are all true. In one of these cases (the second) the conclusion is false. This is what can't happen in a valid argument. So the argument is invalid. I will use the term Counterexample for a case which in this way shows an argument to be invalid. A counterexample to an argument is a case in which the premises are true and the conclusion is false.
>
> In fact, we can use this idea of a counterexample to reword the definition of validity. To say that an argument is valid is to say that any assignment of truth values to sentence letters which makes all of the premises true also makes the conclusion true. We reword this by saying: An argument is valid just in case there is no possible case, no assignment of truth values to sentence letters, in which all of the premises are true and the conclusion is false. To be valid is to rule out any such possibility. We can break up this way of explaining validity into two parts:
>
> > A *Counterexample* to a sentence logic argument is an assignment of truth values to sentence letters which makes all of the premises true and the conclusion false.
> > An argument is *Valid* just in case there are no counterexamples to it.

Again, read Teller to be defining tautological validity. And the point we want to think about now is this: what is the connection between Teller's talk about a 'counterexample' here and our earlier use in Chapter 5 of the idea of a 'counterexample' (a counterexample to the validity of a pattern of argument). Obviously there is *some* connection. But what is it?

(B) I need to say just a bit more about what we can learn about an ordinary-language argument *Arg* by translating it into the PL argument Arg, and running a truth-table test on *that*. The story is simple enough, though.

If Arg is tautologically valid, then – as you now know – it is plain valid. So assuming the translation is good enough for *Arg* and Arg to stand together as far as validity is concerned, then *Arg* is valid too.

If an argument is tautologically invalid, however, we can't straight off infer that it is plain invalid. Here's Tim Button's example we can borrow to make the needed point:

> Consider the argument
>
> > Daisy has four legs. So Daisy has more than two legs.
>
> To symbolise this argument in TFL, we would have to use two different atomic sentences – perhaps 'P' and 'Q' – for the premise and the conclusion respectively. Now,

> it is obvious that 'P' does not tautologically entail 'Q'. But the English argument seems clearly valid.

And the formal translation into a language where 'P' and 'Q' are given those meanings is therefore valid too – valid for some other reason than the distribution of connectives.

Generalizing, suppose Arg is tautologically invalid. Then at least we know that it can't be valid in virtue of the way the connectives explicitly feature in it. And assuming this translation reflects all the connectives in *Arg*, we can conclude at least that *Arg* too is not valid in virtue of the way the connectives explicitly feature in it. To get an outright verdict on *Arg*, however, we need to be confident that it isn't valid for some other reason.

(C) Let's return to Teller's example of a tautologically valid argument, using *IFL* notation: $(\mathsf{P} \lor \mathsf{Q}), \neg\mathsf{P} \therefore \mathsf{Q}$. This is valid because of the way the connectives are distributed – i.e. valid because of what is shared with any argument for the form $(\alpha \lor \beta), \neg\alpha \therefore \beta$. It is natural then to say that the argument is tautologically valid *because of its form*.

Some authors take a strong line here. Goldfarb, for example, writes

> If one [propositional logic] schema implies another, and a pair of statements can be schematized by those schemata, then we may say that the one statement truth-functionally implies the second.

As this suggests, for Goldfarb, implication or entailment relations hold primarily between schemas representing forms of argument, and only derivatively can we say that one statement truth-functionally entails another. Indeed he goes as far as saying

> Some logic textbooks call an argument "valid" if and only if its premises imply its conclusion. We do not use this terminology here, in order to avoid confusion with the notion of validity as applied to schemata ...

That strikes me as a perverse choice! Anyway, in *IFL* I take validity in general and tautological validity in particular to be primarily a property of particular arguments, albeit a property which can be shared by other arguments which share the same structural form. To be sure, an argument's (shareable) property of tautological validity, for example, is due to its (shareable) property of having its connectives distributed in a certain way: but it doesn't follow from *that* that validity isn't primarily a property of actual arguments!

Now read *IFL* §15.5 (on point B), §16.3 (on point A), and §16.6 (on point C).

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Two more points:

(D) We can usefully say a bit more, a bit more carefully, about how particular claims about tautological validity can be generalized to apply to other arguments of the same form (we basically recycle moves we made when generalizing particular claims about tautologies).

(E) Another sort of generalization. Note that our definition of tautological validity only requires that we are dealing with truth-functional connectives, not that we are dealing with the particular bunch of three connectives that we have officially built into our PL languages. Whatever truth-functional connectives we have available, it makes sense to talk about tautologically valid arguments involving those connectives.

So now read *IFL* §16.4 (on an important bit of notation), §16.5 (on point D), and §16.1 (on point E).

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Back in §2.2 I introduced two concepts related to the notion of validity – the ideas of *consistency* and *equivalence*.

As you'll recall, I hope, some propositions are jointly consistent just if it is possible for them to be true together. So an argument is valid just if its premisses and the negation of its conclusion are inconsistent.

Two propositions are equivalent just if they are true in exactly the same possible situations. So they are equivalent if each one entails the other.

In our two current chapters, we now meet two new concepts similarly related to the notion of tautological validity – namely the ideas of *tautological consistency* and *tautological equivalence.*

The details are entirely predictable given what we said in §2.2. But do check them by finishing the two chapters, reading §15.6 and §16.7.