IFL: Logicbite 17

Contradiction!

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After the action-packed Chapters 15 and 16 of *IFL*, we pause for a very short chapter. Indeed I'm not quite sure now why these brisk sections were put together here in a stand-alone chapter. No doubt it seemed a good idea at the time! Anyway, there are just two simple themes here.

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First, we note that a contradictory pair of PL wffs α and $\neg\alpha$ can *never* be true together n any valuation. So, whatever γ we choose, no valuation of the relevant atoms can make α true and make $\neg\alpha$ true and yet also make γ false. Which is just to say that, whatever γ we choose, α and $\neg\alpha$ together tautologically entail γ. (That's an immediate consequence of our definitions.)

From a contradiction, you can tautologically derive anything you like. In a summary Latin tag, *ex contradictione, quodlibet.*

Question: is that an acceptable feature of the notion of tautological entailment? Or is it a bug we ought to try to iron out? After all, tautological entailment is supposed to capture the idea of logically-following-in-virtue-of-the-meanings-of-the-connectives: does any irrelevant conclusion you like really logically follow from a contradiction? We might naturally suppose that it doesn't – shouldn't validly derived conclusions be related to their premisses?

However, it turns out that the costs of avoiding *ex contradictione, quodlibet* are pretty hight. Here's one set of considerations to think about:

(1) From P we can infer $P \lor Q$. (That's just because '$\lor$' means inclusive disjunction.)

(2) From $\neg P$ and $P \lor Q$ we can infer Q. (Given that at least one of P and Q is true, but also that we can rule out the first, then we can infer the second.)

(3) Using these two principles, we can now argue as follows (since we can chain valid inference moves into longer deductively cogent arguments):

(*i*)	P	premiss
(*ii*)	$\neg P$	premiss
(*iii*)	$P \lor Q$	(from *i* by 1)
(*iv*)	Q	(from *ii* and *iii* by 2)

So we have indeed derived an irrelevant conclusion from the contradictory premisses, just using the surely(?) secure principles (1) and (2), and the surely(?) essential proof-building principle that allows us to chain small inferential steps into a longer proof. The cost of avoiding *ex contradictione, quodlibet* thus seems to be unpalatably high – having to reject (1) or (2) or the chaining of inference steps. (Which isn't to deny that some logicians are prepare to pay the cost!)

But I'm already jumping ahead here, beyond what I say in §17.1. For the moment, I just want to highlight that point that, from a contradiction, you can tautologically derive anything you like – and to recommend that we learn to live with this!

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While still talking about contradictions, we are going to expand our PL languages by adding an expression '$\bot$' that can be thought of as a special new fixed-value wff which serves as an all-purpose-contradiction, so is always false on any valuation of the language.

Equivalently, '$\bot$' can be thought of as a zero-place connective. Really? How does that work? Well here is how Nick Smith introduces the idea. Two-place connectives are defined by $2^2 = 4$ line truth tables. Three place connectives like our dollar connective are defined by $2^3 = 8$ line truth tables. Similarly, a zero-place connective * will be defined by a $2^0 = 1$ line truth table. So the truth table of a zero-place connective * will either assign it T or F.

> So there are two possible zero-place connectives; they are usually called the *verum* and the *falsum* and are symbolized by $\top$ and $\bot$, respectively.
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> The idea of a zero-place connective seems rather odd at first, but it makes sense when we think about it. An *n*-place connective plus *n* propositions makes a proposition. So a zero-place connective all by itself – that is, with zero propositions added – forms a proposition. Being a proposition, this entity (i.e., the zero-place connective by itself) is either true or false. Note that $\top$ is always true, and $\bot$ is always false: these propositions have no component propositions, so their truth values cannot vary with the truth values of their components.

It is the second of these, the falsum or *absurdity constant*, that will play a role in what follows, when we come to natural deduction proofs. And that more or less sums up the content of §17.2 and §17.3.

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So now read *IFL* Chapter 17.