Faculty of Philosophy

Formal Logic

Lecture 8

Peter Smith

- Tautological Entailment
  - The basic idea
  - The truth-table test for tautological validity
  - Speeding things up

- Truth-functionality

The idea of tautological entailment

Compare:

- A proposition ψ is necessary if ψ is true in every possible situation.
- An argument ϕ_1, ϕ_1, $\ldots\phi_n$, so ψ is valid if ψ is true in every possible situation where all of ϕ_1, ϕ_1, $\ldots\phi_n$ are true.

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Similarly compare:

- A PL wff ψ is a tautology if ψ is true on every possible valuation of the relevant atoms.
- A PL argument $\phi_1, \phi_1, \ldots \phi_n$, so ψ is tautologically valid if ψ is true on every possible valuation of the relevant atoms where all of $\phi_1, \phi_1, \ldots \phi_n$ are true.

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We determine whether a wff is a tautology, or whether an argument is tautologically valid, by a truth-table test.

An example

Take the argument $(P \vee Q)$, $\neg(P \wedge R)$, $\neg(Q \wedge R)$ so $\neg R$

An example

Take the argument $(P \lor Q)$, $\neg(P \land R)$, $\neg(Q \land R)$ so $\neg R$

We construct a truth-table as follows:

P	Q	R	$(P \lor Q)$	$\neg(P \land R)$	$\neg(Q \land R)$	$\neg R$
T	T	T				
T	T	F				
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F	T	F	T	T	T	T
F	F	T	F	T	T	F
F	F	F	F	T	T	T

There are no lines with true premisses and false conclusion – so the argument is tautologically valid.

Another example

Take the argument $\neg(R \land \neg\neg Q)$, $(P \lor Q)$ so $\neg(P \lor R)$

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T	F	T	T	T	
T	F	F	T	T	
F	T	T	F	T	
F	T	F	T	T	
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T	F	F	T	T	F
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F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	T	F	T

There are lines with true premisses and false conclusion – so the argument is NOT tautologically valid.

Speeding up the test

- The idea of the test is to look for 'bad lines' where the premisses are true and conclusion false. A bad line means the argument is invalid. No bad lines and it is valid.

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- Since we are just searching for bad lines (with premisses true and conclusion false) we can ignore lines with a false premiss as they can't be bad lines.
- So typically we can start a table by evaluating the conclusion and then ignoring lines on which it is true. Then evaluate the premisses in order of complexity, ignoring lines once a premiss turns out to be false.

Reworking our first example

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T	F	F				T
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F	T	F				T
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F	F	F				T

The second, fourth, sixth, eighth lines can't be bad lines as the conclusion are true. Next evaluate the simplest premiss.

Reworking our first example

Take the argument $(P \lor Q)$, $\neg(P \land R)$, $\neg(Q \land R)$ so $\neg R$

We look at the conclusion first –

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T	T	T	T			F
T	T	F				T
T	F	T	T			F
T	F	F				T
F	T	T	T			F
F	T	F				T
F	F	T	F			F
F	F	F				T

We can now ignore the seventh line as that can't turn out to be a bad line. Next we evaluate the second premiss.

Reworking our first example

Take the argument $(P \vee Q)$, $\neg(P \wedge R)$, $\neg(Q \wedge R)$ so $\neg R$

We look at the conclusion first –

P	Q	R	$(P \vee Q)$	$\neg(P \wedge R)$	$\neg(Q \wedge R)$	$\neg R$
T	T	T	T	F		F
T	T	F				T
T	F	T	T	F		F
T	F	F				T
F	T	T	T	T		F
F	T	F				T
F	F	T	F			F
F	F	F				T

We can now ignore the first and third line as they can't turn out to be a bad line. Next we evaluate the last premiss.

Reworking our first example

Take the argument $(P \vee Q)$, $\neg(P \wedge R)$, $\neg(Q \wedge R)$ so $\neg R$

We look at the conclusion first –

P	Q	R	$(P \vee Q)$	$\neg(P \wedge R)$	$\neg(Q \wedge R)$	$\neg R$
T	T	T	T	F		F
T	T	F				T
T	F	T	T	F		F
T	F	F				T
F	T	T	T	T	F	F
F	T	F				T
F	F	T	F			F
F	F	F				T

And we are done! There are no bad lines so the argument is indeed valid as we showed before.

Another example

Is the following argument tautologically valid?

$$(P \lor (\neg Q \lor R)),\ (R \lor \neg P) \text{ so } \neg(Q \land R)$$

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T	T	F			T
T	F	T			T
T	F	F			T
F	T	T			F
F	T	F			T
F	F	T			T
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So only lines 1 and 5 are potentially ‘bad’.

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P	Q	R	$(P \lor (\neg Q \lor R))$	$(R \lor \neg P)$	$\neg(Q \land R)$
T	T	T		T	F
T	T	F			T
T	F	T			T
T	F	F			T
F	T	T		T	F
F	T	F			T
F	F	T			T
F	F	F			T

Lines 1 and 5 are still potentially ‘bad’.

Another example

Is the following argument tautologically valid?

$$(P \vee (\neg Q \vee R)),\ (R \vee \neg P) \text{ so } \neg(Q \wedge R)$$

P	Q	R	$(P \vee (\neg Q \vee R))$	$(R \vee \neg P)$	$\neg(Q \wedge R)$
T	T	T	T	T	F
T	T	F			T
T	F	T			T
T	F	F			T
F	T	T		T	F
F	T	F			T
F	F	T			T
F	F	F			T

One 'bad' line is enough to establish invalidity

To repeat – how to speed up the test

- The idea of the test is to look for 'bad lines' where the premisses are true and conclusion false. A bad line means the argument is invalid. No bad lines and it is valid.
- Since we are just searching for bad lines we can ignore lines with a true conclusion as they can't be bad lines.
- Since we are just searching for bad lines we can ignore lines with a a false premiss as they can't be bad lines.
- So typically we can start a table by evaluating the conclusion and then ignoring lines on which it is true. Then evaluate the premisses in order of complexity, ignoring lines once a premiss turns out to be false.

Validity and tautological validity

- If a PL argument is tautologically valid, it is valid in virtue of the distribution of the connectives '$\wedge$', '$\vee$' and '$\neg$' in premisses and conclusion. Fixing the sense of those connectives suffices to ensure that, necessarily, if the premisses are true then the conclusion is too.

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- If an argument is tautologically valid, it is plain valid.
- But not conversely. A PL argument can be valid without being tautologically valid (e.g. the PL translation of our old friend 'Socrates is a man. All men are mortal. So Socrates is mortal' will expose no more structure than '$P, Q \therefore R$').

- Tautological Entailment
  - The basic idea
  - The truth-table test for tautological validity
  - Speeding things up
- Truth-functionality

Why does the truth-table test work?

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2. This requires that fixing the truth-values of atoms in a PL wff fixes the truth-value of the wff.
3. That requires the wff-building connectives to be truth-functional, each connective maps the truth-values of the wffs it operates on to a determinate value.

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2. This requires that fixing the truth-values of atoms in a PL wff fixes the truth-value of the wff.
3. That requires the wff-building connectives to be truth-functional, each connective maps the truth-values of the wffs it operates on to a determinate value.
4. Equivalently, the wff-building connectives must be definable by truth-tables.

Many ordinary-language connectives are not truth-functional

Because is only a partial truth-function.

ϕ	ψ	ϕ because ψ
T	T	?
T	F	F
F	T	F
F	F	F

It is improbable that is not even a partial truth-function

ϕ	It is improbable that ϕ
T	?
F	?

Extending the truth-table test

- Where a table for a connective has 'gaps', we can't always apply the truth table test to test arguments involving the relevant connective – because an assignment of values to the atomic sentences won't always settle the values of the premisses and conclusion of the argument to be tested.

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- But where a connective is (fully) truth-functional, i.e. is defined by a truth-table without any gaps, we can apply the truth-table method.

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- But where a connective is (fully) truth-functional, i.e. is defined by a truth-table without any gaps, we can apply the truth-table method.
- For example, we could use the test on arguments involving the connective $\otimes$ defined by the following table:

ϕ	ψ	$(\phi \otimes \psi)$
T	T	F
T	F	T
F	T	T
F	F	F

Exclusive disjunction – 1

- The table, to repeat, was

ϕ	ψ	$(\phi \otimes \psi)$
T	T	F
T	F	T
F	T	T
F	F	F

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- The table, to repeat, was

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Exclusive disjunction – 1

- The table, to repeat, was

ϕ	ψ	$(\phi \otimes \psi)$
T	T	F
T	F	T
F	T	T
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- This is of course the table for exclusive disjunction.
- Should we add $\otimes$ as a new connective?? Well consider

ϕ	ψ	$(\phi \otimes \psi)$	$((\phi \lor \psi)$	$\land$	$\neg(\phi \land \psi))$
T	T	F	t	F	f
T	F	T	t	T	t
F	T	T	t	T	t
F	F	F	f	F	t

Exclusive disjunction – 2

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- Hence we don't need to augment PL to cope with exclusive disjunction. And we can go on to apply the truth-table test to PL arguments involving exclusive disjunction.

Exclusive disjunction – 2

- So we can express exclusive disjunction using just the already available three connectives, $\wedge$, $\vee$, $\neg$.
- Hence we don't need to augment PL to cope with exclusive disjunction. And we can go on to apply the truth-table test to PL arguments involving exclusive disjunction.
- Question for later investigation: can every truth-function be expressed in PL using $\wedge$, $\vee$, $\neg$?