Faculty of Philosophy

Formal Logic

Lecture 10

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- The material conditional again
- 'Only if' and the biconditional
- Expressive adequacy
  - The basic 'adequacy theorem'
  - De Morgan's Laws
  - Other adequate sets of connectives

The material conditional '$\supset$' defined

We arrived at the following definition:

ϕ	ψ	$(\phi \supset \psi)$
T	T	T
T	F	F
F	T	T
F	F	T

This is standardly known as the material conditional or the truth-functional conditional: the best shot at an if-like truth-function rendering indicative conditionals.

More apparent differences between ‘if’ and ‘$\supset$’

- We noted yesterday some apparent divergencies between ordinary ‘if’ and the truth-functional ‘$\supset$’. Here are two more examples.
- Note that the following two arguments are trivially valid by the truth-table test

  A. $\neg P$ hence $(P \supset Q)$
  B. Q hence $(P \supset Q)$

  But compare

  A′. The bomb will not drop; hence if it drops, we survive.
  B′. I will ski tomorrow; hence if I break my leg today, I will ski tomorrow

  Can we deal with these by treating future indicatives as possible world conditionals??? What about ...

  A′′. It isn't cold; hence if it is cold then it's hot.
  B′′. It is hot; hence if it is cold then it is hot.

A logical health warning

- There are close connections but also apparent differences between 'if' and '$\supset$'.
- A suggestion: we use 'if P, then Q' to both (a) assert $(P \supset Q)$, and (b) signal that we are prepared to use modus ponens.
- That explains why the inference $\neg P$ hence if P then Q strikes us as unacceptable. For while $\neg P$ warrants $(P \supset Q)$, we wouldn't be prepared to use modus ponens with the material conditional if $\neg P$ is our reason for believing it. If we discover that in fact P, we wouldn't go on to use modus ponens to conclude Q, we'd retract $(P \supset Q)$!
- For more discussion of the relationship, see *IFL*, Ch. 15.
- Warning: translating 'if' by '$\supset$' arguably doesn't capture all the content of the ordinary language conditional (though maybe the translation gets the truth-relevant content right).

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'Only if'

- Compare If P, then Q and P only if Q.
- Suppose If P, then Q. Then P's truth gives us the truth of Q too – so that means we only get P if we get Q too, i.e. P only if Q.
- Suppose P only if Q. Then if we have P true then we get Q too, i.e. If P, then Q.
- In sum, (in many cases) If P, then Q is equivalent to P only if Q.
- For example: compare 'If Einstein is right, the velocity of light is constant' and 'Einstein is right only if the velocity of light is constant'.

'If' and 'only if'

- So *P*, if *Q* is equivalent to If *Q*, then *P* is equivalent (in many cases) to *Q* only if *P* is equivalent to Only if *P*, *Q*.
- But of course *P*, if *Q* is not equivalent to *P*, only if *Q*.
- The joint assertion of *P*, if *Q* and *P*, only if *Q* – i.e. the assertion *P* if and only if *Q* – is the biconditional.

The material biconditional – 1

- The only truth functional rendition of P, only if Q is $(P \supset Q)$ (or an equivalent).
- The only truth functional rendition of P, if Q is $(Q \supset P)$ (or an equivalent).
- The only truth functional rendition of P if and only if Q is therefore $((P \supset Q) \land (Q \supset P))$ (or an equivalent)
- Call that the material biconditional.

The material biconditional – 2

It is standard to add another symbol to abbreviate the material biconditional

ϕ	ψ	$((\phi \supset \psi) \wedge (\psi \supset \phi))$			$(\phi \equiv \psi)$
T	T	t	T	t	T
T	F	f	F	t	F
F	T	t	F	f	F
F	F	t	T	t	T

The usual basic repertoire of truth-functional connectives is thus $\neg, \wedge, \vee, \supset, \equiv$.

But why stop there?

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Examples so far

- We saw that exclusive disjunction $(\phi \otimes \psi)$ can be defined using the basic three connectives. $(\phi \otimes \psi)$ is equivalent to $((\phi \vee \psi) \wedge \neg((\phi \wedge \psi))$.
- Likewise, $(\phi \supset \psi)$ is equivalent to $(\neg\phi \vee \psi)$ and to $\neg(\phi \wedge \neg\psi)$. So we don't need to add a new connective to express the material conditional.
- Similarly, we don't need to add a new connective to express the material biconditional. We can express the truth-function $(\phi \equiv \psi)$ using '$\wedge$' and $\supset$, and hence '$\wedge$', '$\vee$' and '$\neg$'.

Generalizing

The point now generalizes:

Every truth-functional combination of atomic wffs can be expressed using just '$\wedge$', '$\vee$' and '$\neg$'.

- Definition: A truth-functional combination of atomic wffs is one whose truth-value is determined for each assignment of values to its atoms.
- Equivalently: A truth-functional combination of atomic wffs is one that can be defined by a truth-table.
- The result that every truth-functional combination of atomic wffs can be expressed using just '$\wedge$', '$\vee$' and '$\neg$' is often put this way: the set of connectives '$\wedge$', '$\vee$' and '$\neg$' is *expressively adequate*.
- The result is a metalogical theorem.

Proving the expressive adequacy theorem

- The result to be proved is essentially this. If we define a new connective by a truth-table, then we can mock-up an equivalent to any wff involving that connective using '$\wedge$', '$\vee$' and '$\neg$'.
- For example, suppose the connective ♣ is defined by

P	Q	R	♣(P, Q, R)
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	F

Proving the expressive adequacy theorem

Now, take the lines on which $\clubsuit(P, Q, R)$ is true and write down the conjunction which 'describes' that line (we'll be careless about brackets to promote readability):

P	Q	R	$\clubsuit(P, Q, R)$	
T	T	T	F	
T	T	F	T	$(P \land Q \land \neg R)$
T	F	T	F	
T	F	F	F	
F	T	T	F	
F	T	F	T	$(\neg P \land Q \land \neg R)$
F	F	T	T	$(\neg P \land \neg Q \land R)$
F	F	F	F	

Proving the expressive adequacy theorem

Now write down the disjunction of the conjunctions which 'describe' the lines where $\clubsuit(P, Q, R)$ is true (again being careless about brackets to promote readability):

P	Q	R	$\clubsuit(P, Q, R)$	
T	T	T	F	
T	T	F	T	$(P \wedge Q \wedge \neg R)$
T	F	T	F	
T	F	F	F	
F	T	T	F	
F	T	F	T	$(\neg P \wedge Q \wedge \neg R)$
F	F	T	T	$(\neg P \wedge \neg Q \wedge R)$
F	F	F	F	

$((P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R))$
is then true on exactly the same lines as $\clubsuit(P, Q, R)$.

Proving the expressive adequacy theorem

- The trick is to write down a suitable conjunction of atoms and negated atoms to describe each line with $\clubsuit(P, Q, R)$ is true, and then disjoin the results.
- That works for any table defining any connective $\clubsuit$ connecting any number of atoms – unless $\clubsuit$ generates a contradiction and is always false. But obviously we can write down an always-false wff using just '$\wedge$', '$\vee$' and '$\neg$'.
- So every truth-function can be expressed just using '$\wedge$', '$\vee$' and '$\neg$'. QED

De Morgan's Laws

Now note we have the following:

ϕ	ψ	$(\phi \wedge \psi)$	$\neg(\neg\phi \vee \neg\psi)$	
T	T	T	T	f
T	F	F	F	t
F	T	F	F	t
F	F	F	F	t

ϕ	ψ	$(\phi \vee \psi)$	$\neg(\neg\phi \wedge \neg\psi)$	
T	T	T	T	f
T	F	F	T	f
F	T	F	T	f
F	F	F	F	t

So '$\wedge$' can be defined in terms of '$\vee$' and '$\neg$'.
So '$\vee$' can be defined in terms of '$\wedge$' and '$\neg$'.

Expressive adequacy again

- The set of connectives $\{\wedge, \vee, \neg\}$ is expressively adequate – i.e. gives us enough to express every truth-function. [This is the key result.]
- Since we can define '$\wedge$' can be defined in terms of '$\vee$' and '$\neg$', the set of connectives $\{\vee, \neg\}$ is expressively adequate.
- Since we can define '$\vee$' can be defined in terms of '$\wedge$' and '$\neg$', the set of connectives $\{\wedge, \neg\}$ is expressively adequate.
- We can also define '$\wedge$' and '$\vee$' in terms of '$\supset$' and '$\neg$', so the set of connectives $\{\supset, \neg\}$ is expressively adequate.
- The set of connectives $\{\wedge, \vee\}$ is not expressively adequate.
- There's a two-place connective (the Sheffer stroke) in terms of which we can define '$\wedge$', '$\vee$' and '$\neg$'. So the Sheffer stroke is expressively adequate all by itself. (*IFL*, ch. §11.8)